

High order parametrization of invariant manifolds for periodic orbits

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VU University Amsterdam



Joint work with J.P Lessard (Laval Univ.), J.D Mireles James (Rutgers Univ.)

BCAM, Bilbao, June 2nd, 2014

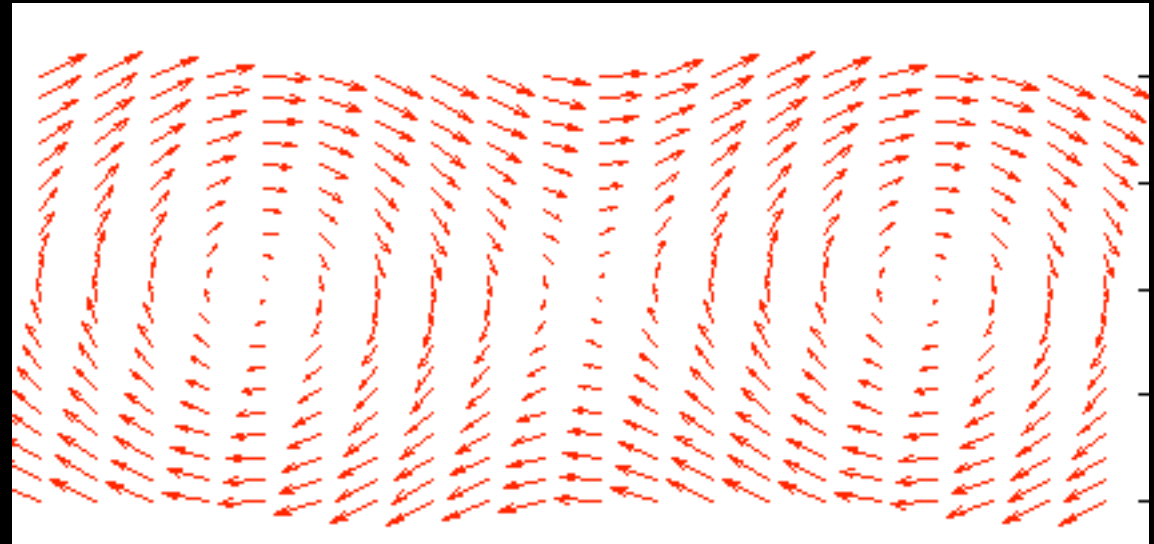
Introduction and Motivations

Vector field

$$f : X \rightarrow X \quad \text{real analytic}$$

$$\dot{x} = f(x)$$

$$\phi(x, t) : X \times \mathbb{R} \rightarrow X$$



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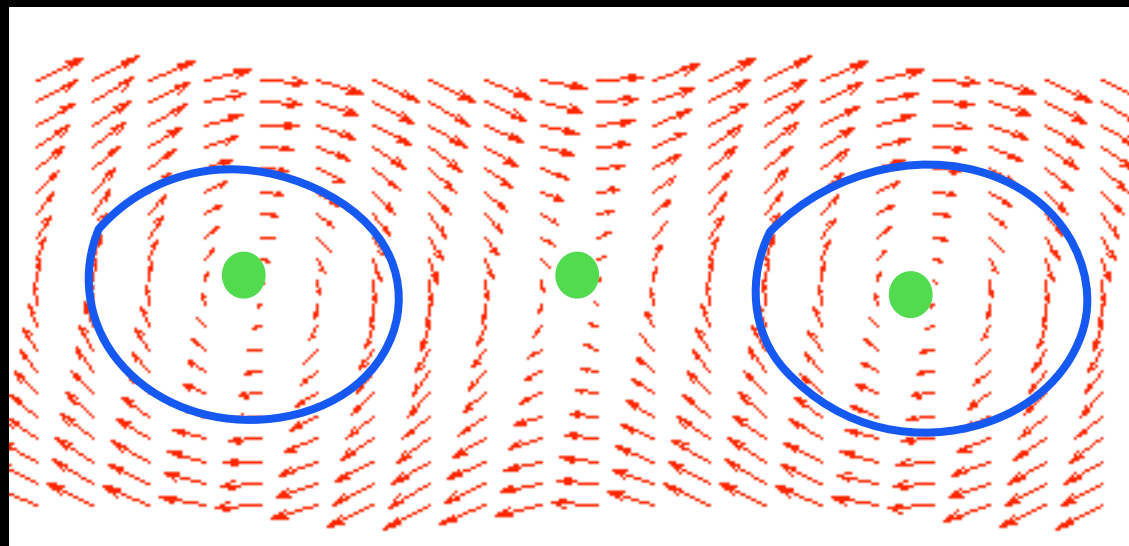
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Invariant sets

Fixed points
Periodic orbits
Invariant tori



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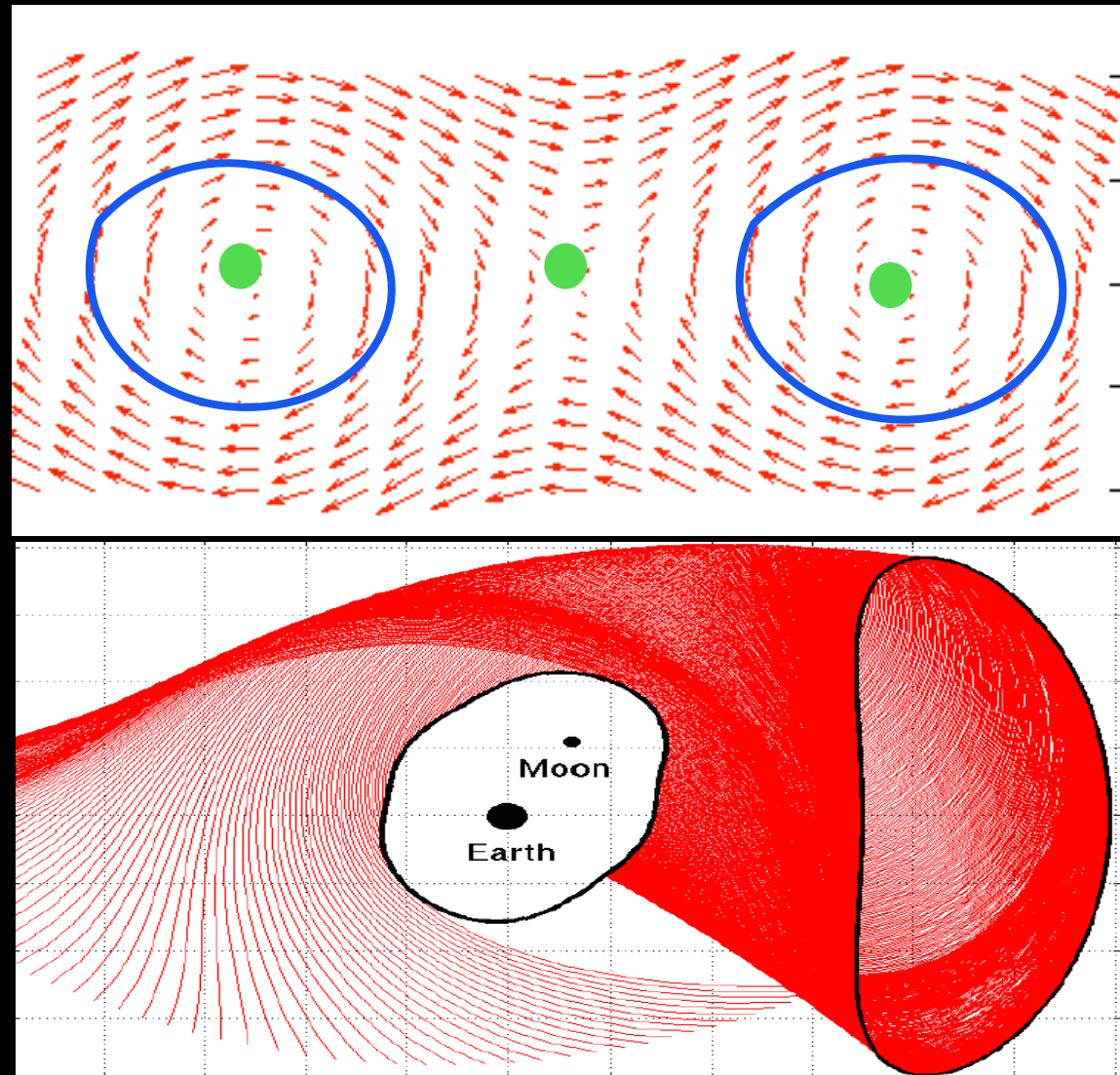
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Invariant Manifolds for fixed point



$$W^s(p) = \left\{ x \in X : \phi(x, t) \rightarrow p, t \rightarrow +\infty \right\}$$

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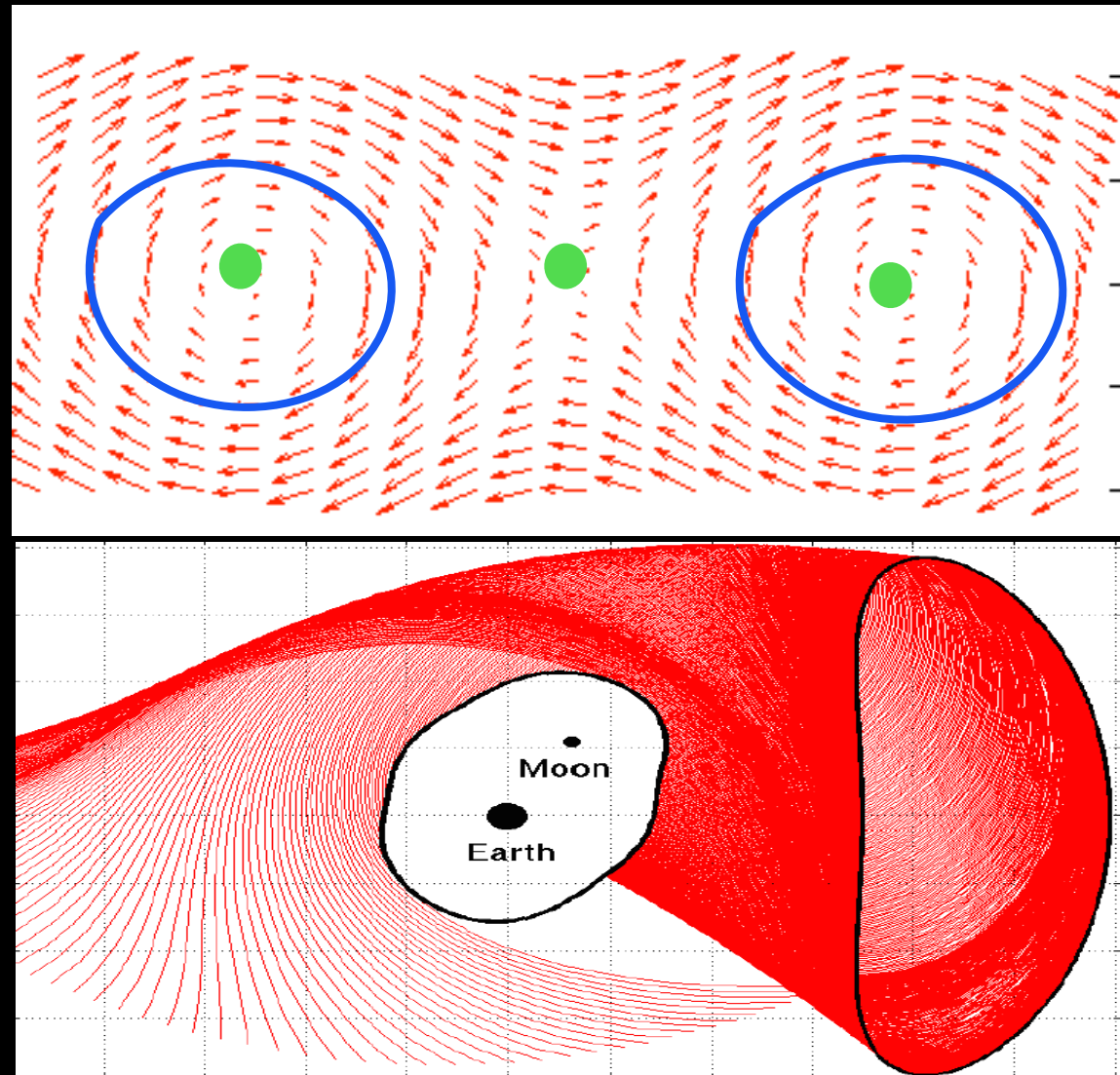
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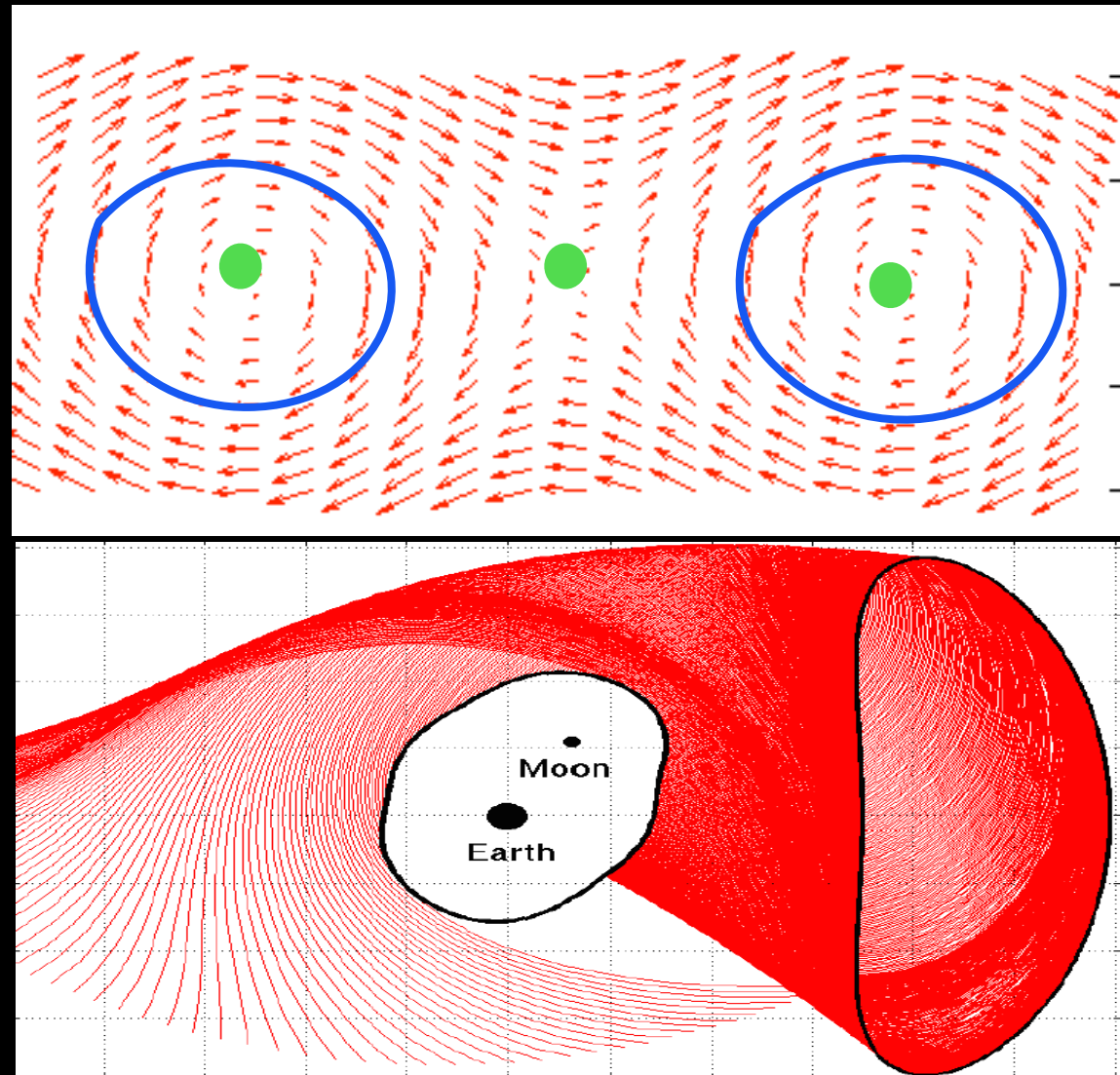
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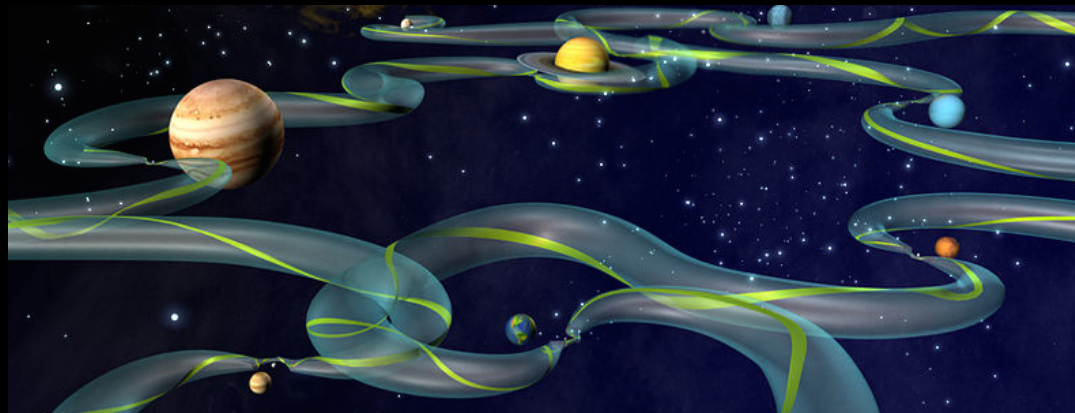
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Invariant manifolds for ...

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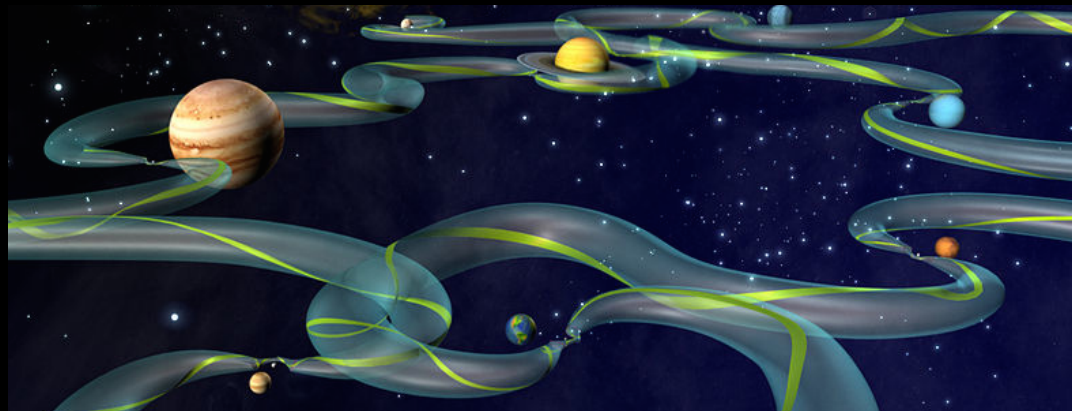
Provide dynamical skeleton:
dynamical *channels*,
separatrices of phase space



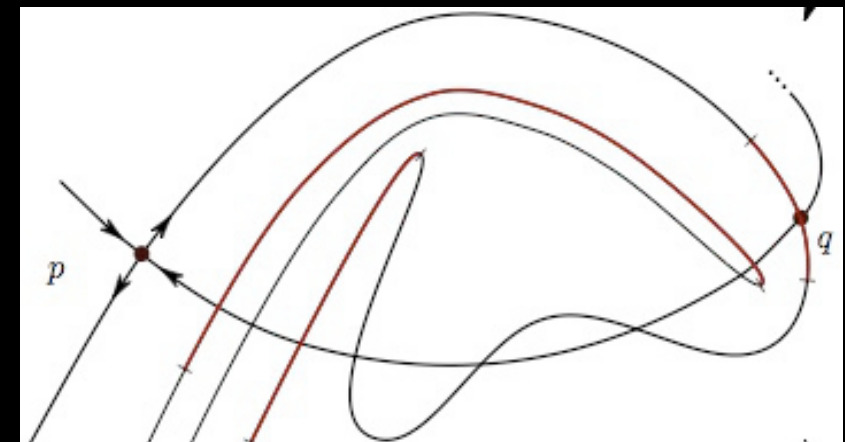
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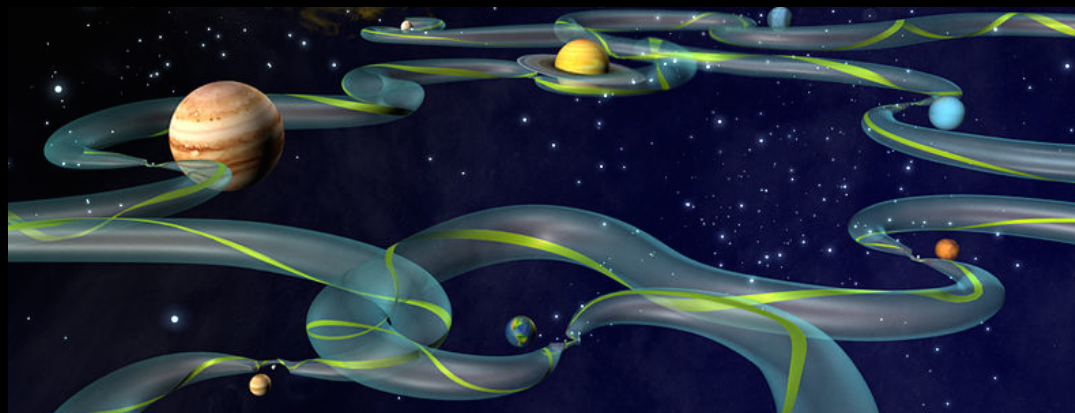
Provide information about
Chaotic Dynamics:
Smale, Conley-Moser, Melnikov theory
Topology of the attractors



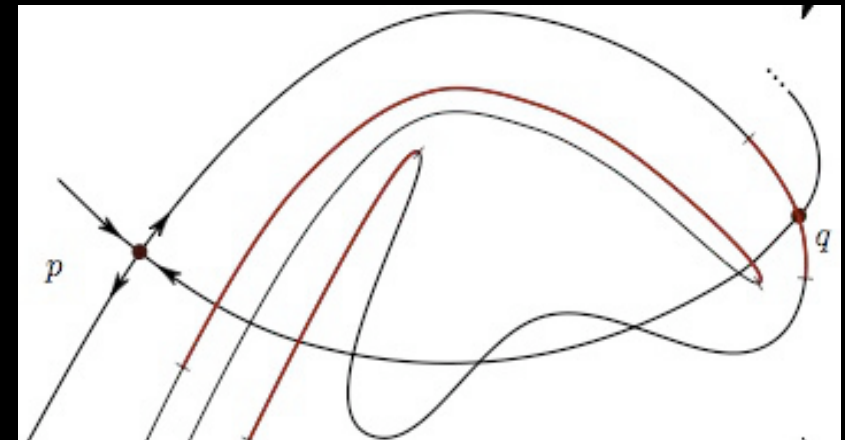
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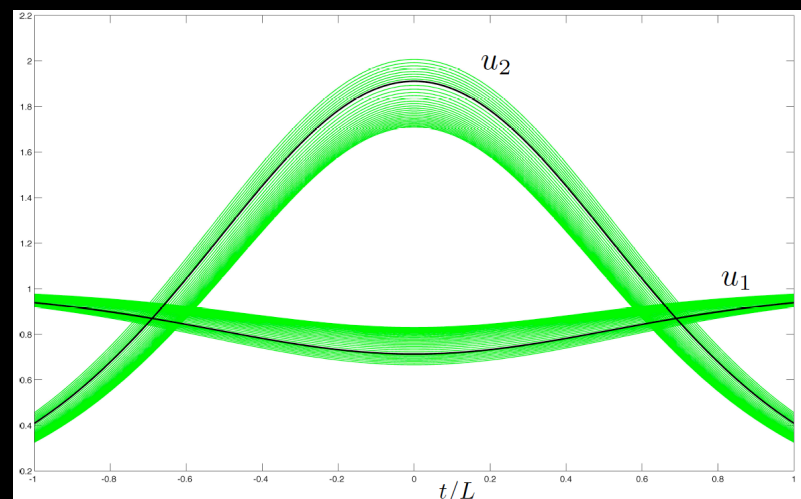
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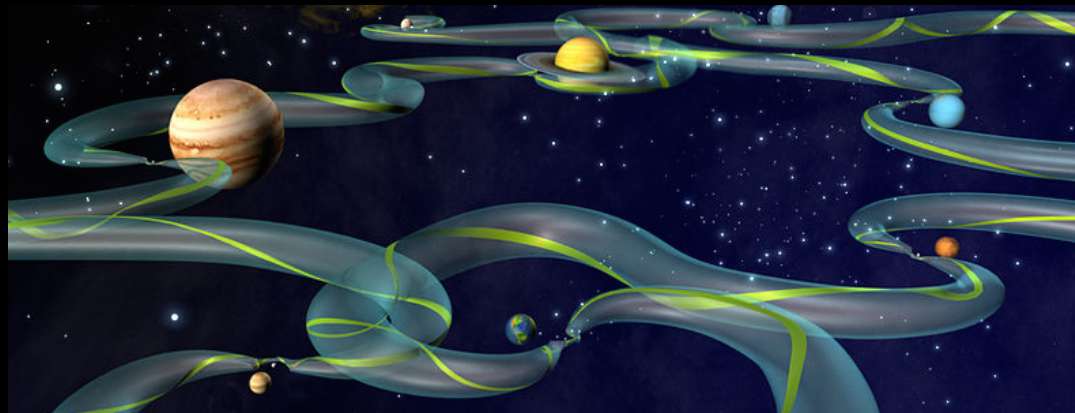
Traveling Waves
for PDEs



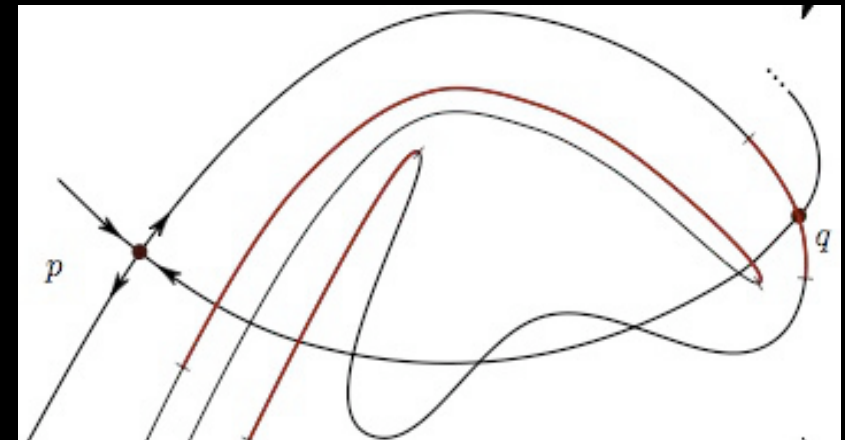
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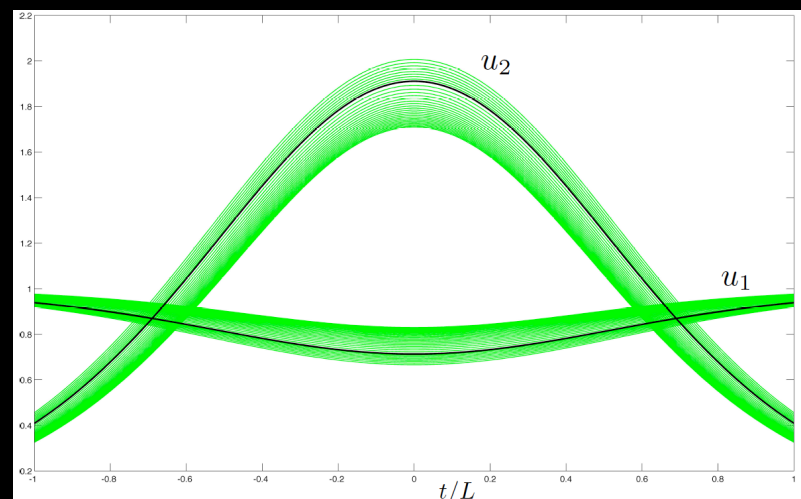
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Traveling Waves
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Can be used to compute
connecting orbits:
homoclinic, heteroclinic

Applications:
Fluid mechanics
Chemical reaction dynamics
Oceanography
Astrodynamics

...

Introduction and Motivations

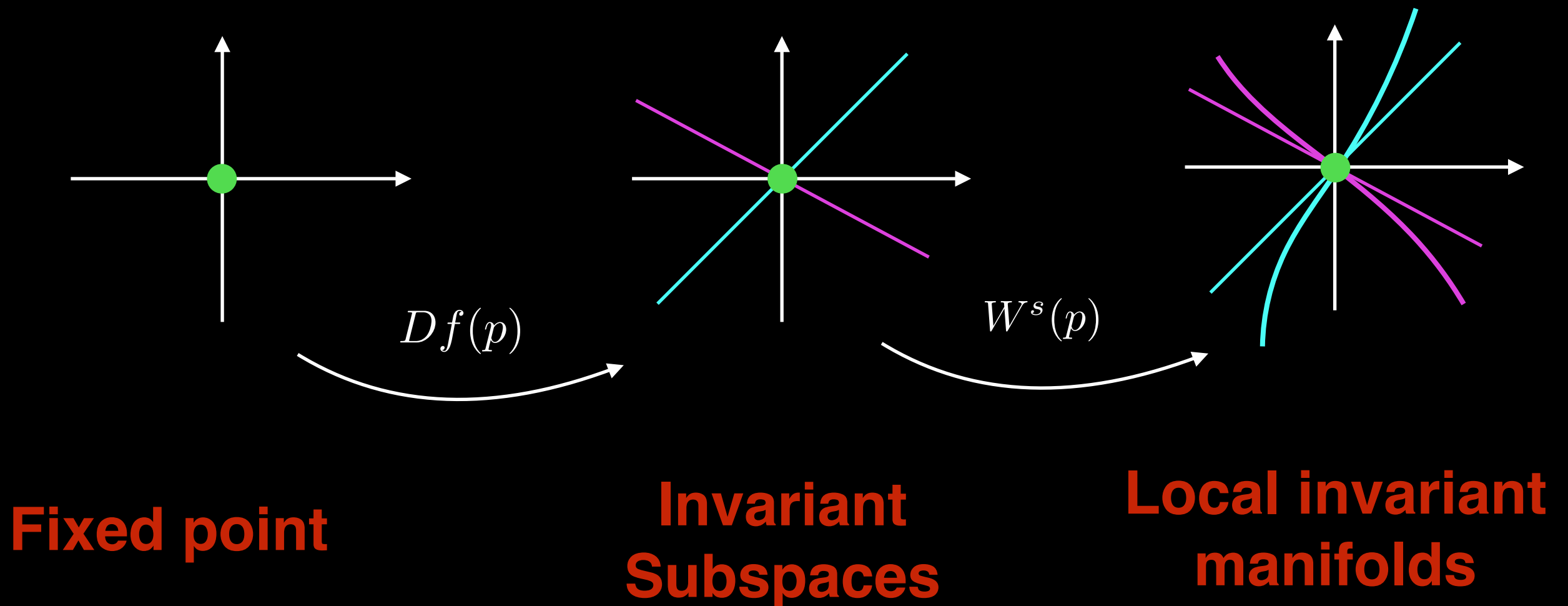
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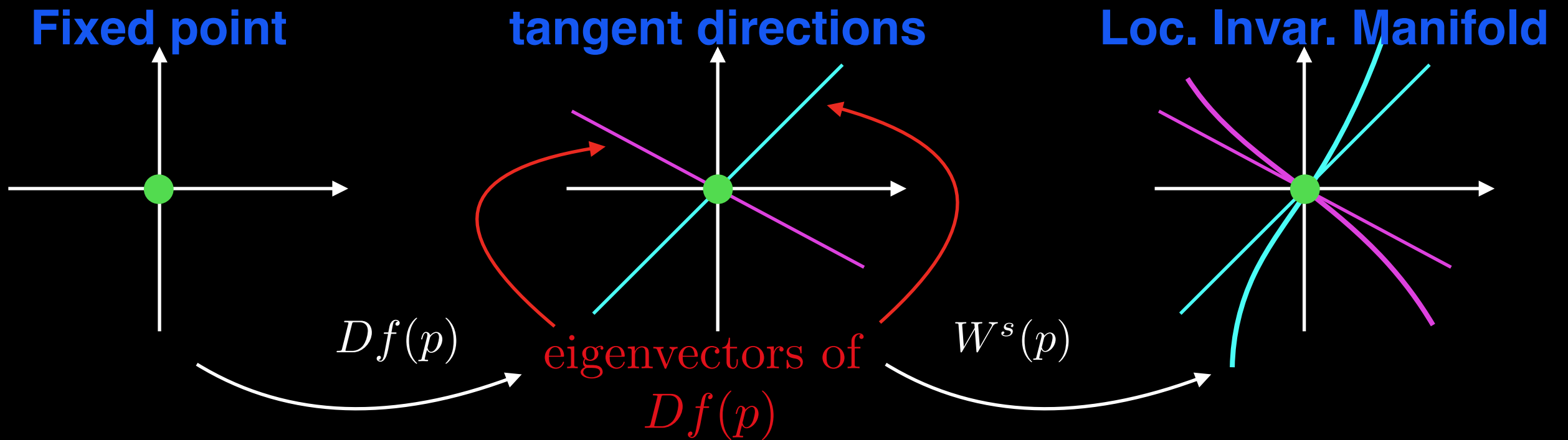
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Local Manifold theorem (for fixed point)

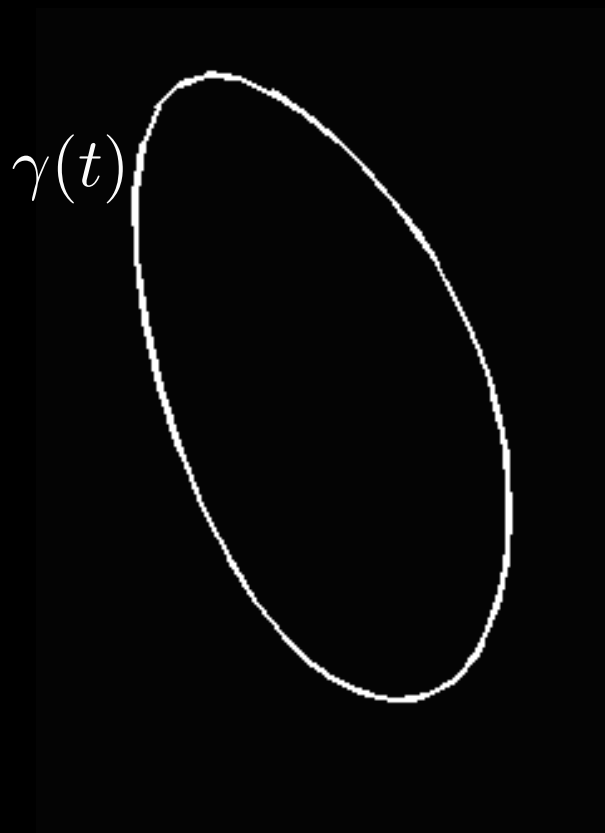
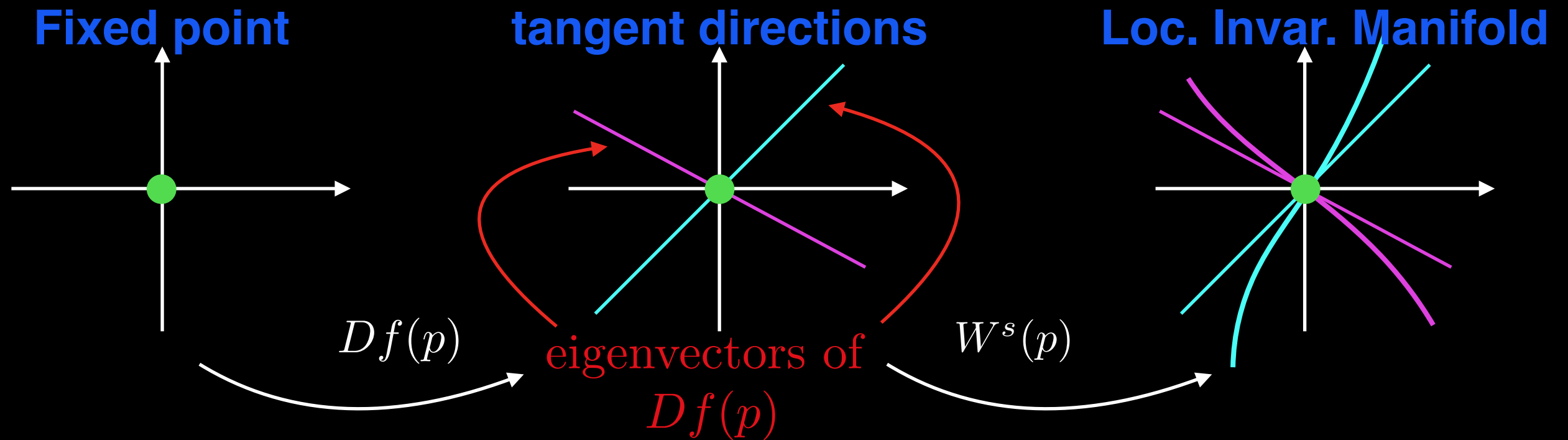
Let $f(x)$ be a C^r vector field and let p be a fixed point for $\dot{x} = f(x)$. Suppose that $Df(p)$ admits s eigenvalues with negative real part, u eigenvalues with positive real part. Then $W^s(p)$ (resp. $W^u(p)$) is a C^r s -dimensional (u -dim) manifold. The manifold is tangent to the respective invariant subspace at p of the linearized vector field.



Introduction and Motivations

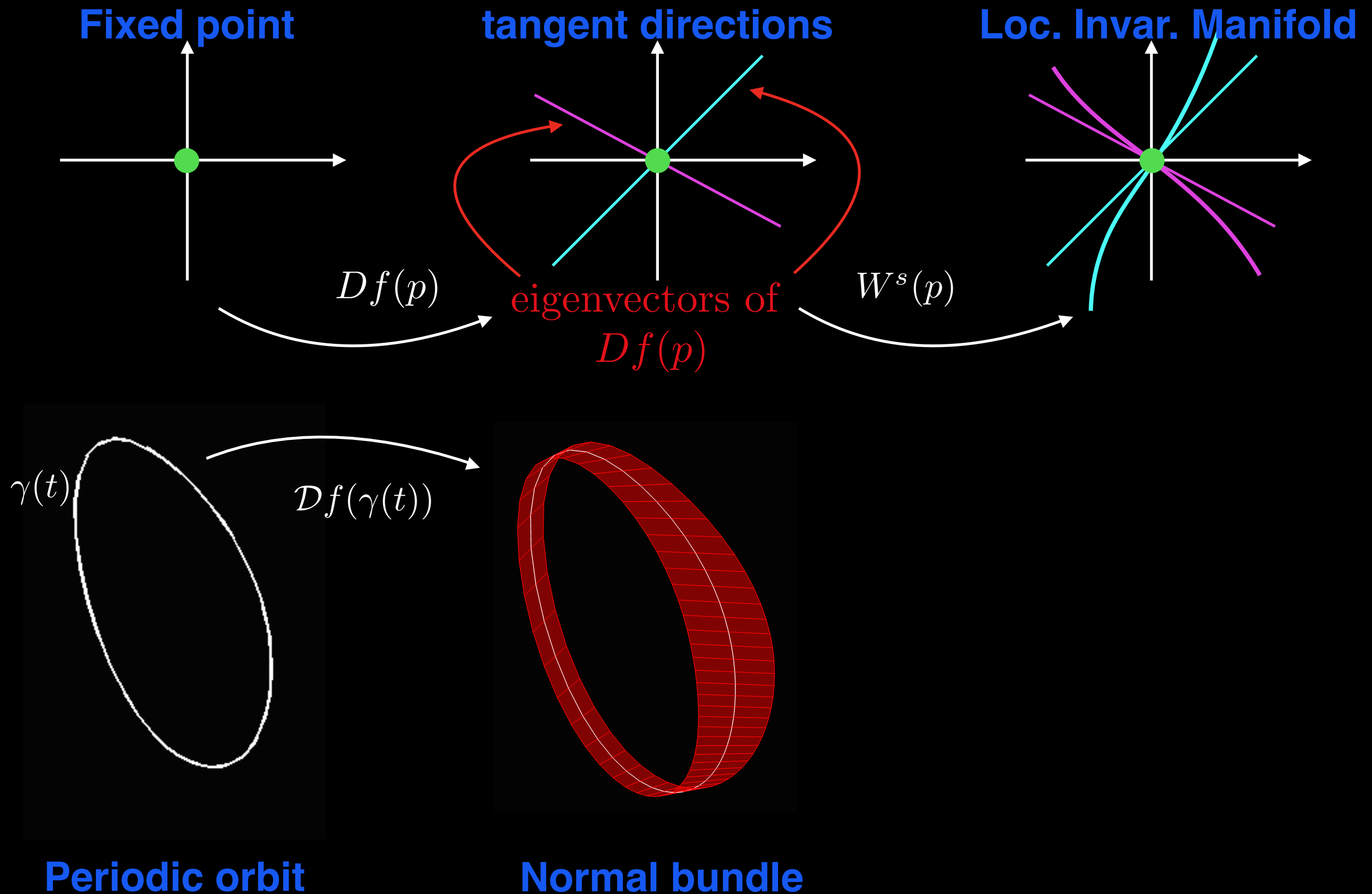


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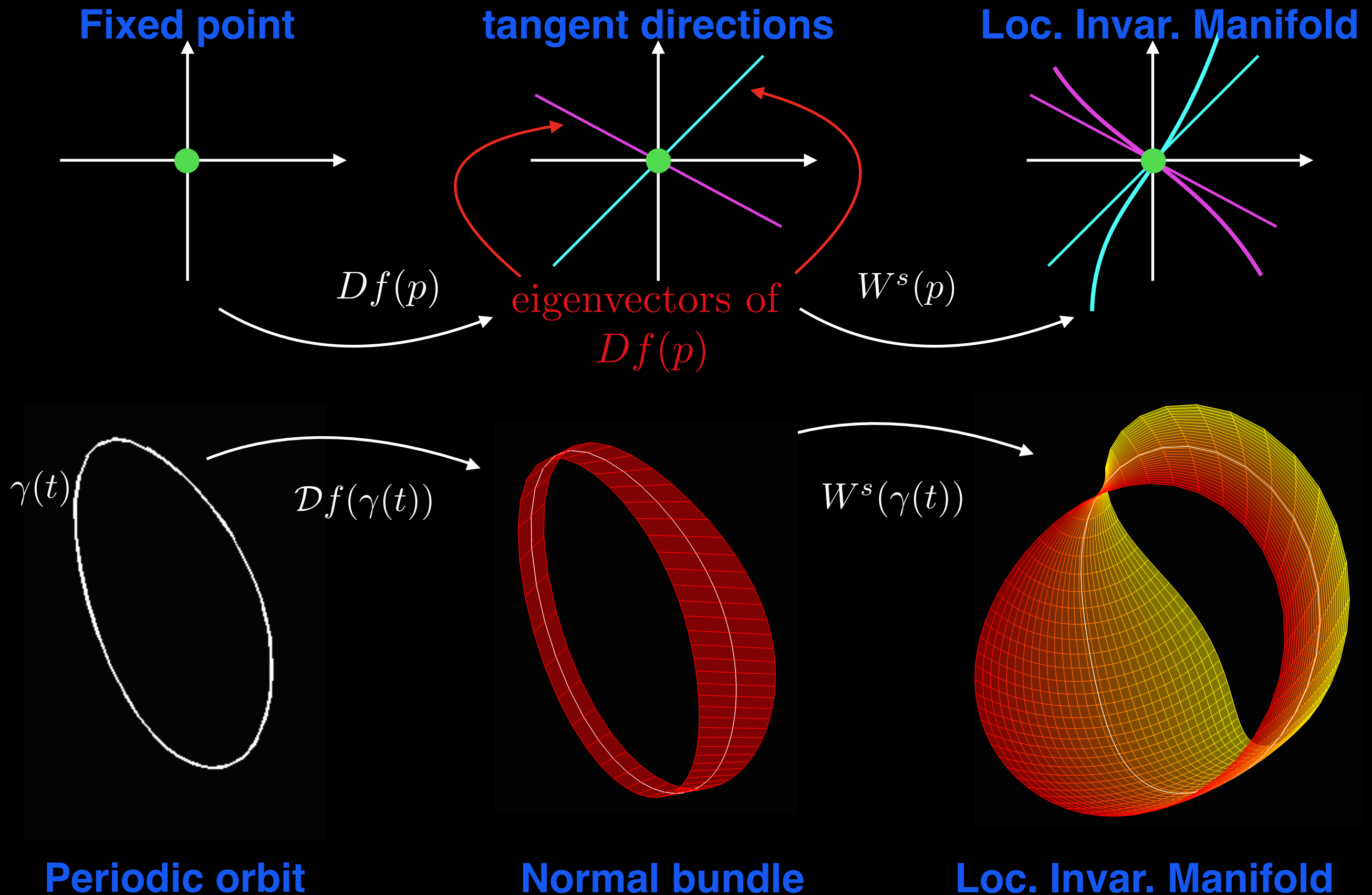


Periodic orbit

Introduction and Motivations



Introduction and Motivations



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Project: rigorous computation of cycle-to-cycle
connecting orbits

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Different Steps:

I- Computation of periodic orbits



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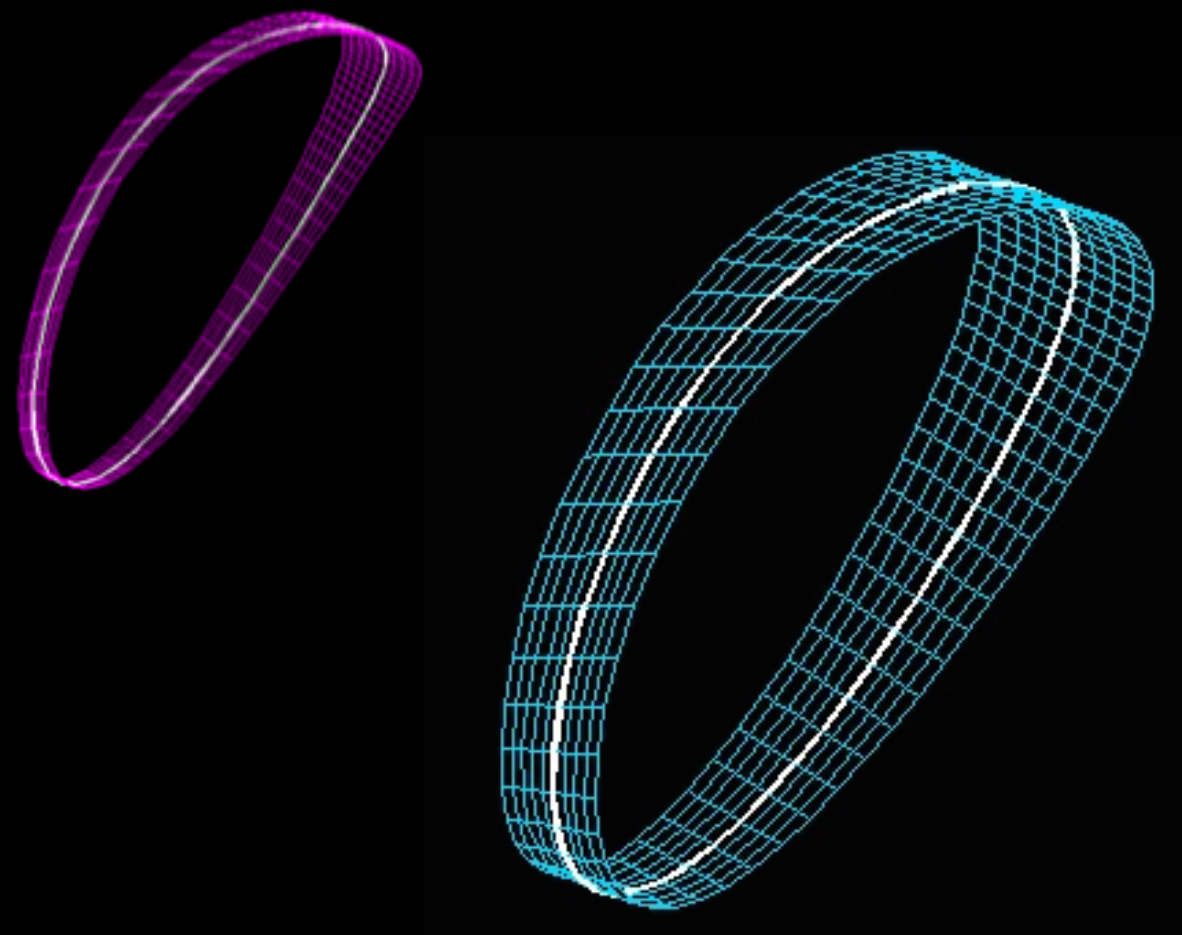
Project: rigorous computation of cycle-to-cycle connecting orbits

Different Steps:

1- Computation of periodic orbits

2- Computation of the tangent bundle

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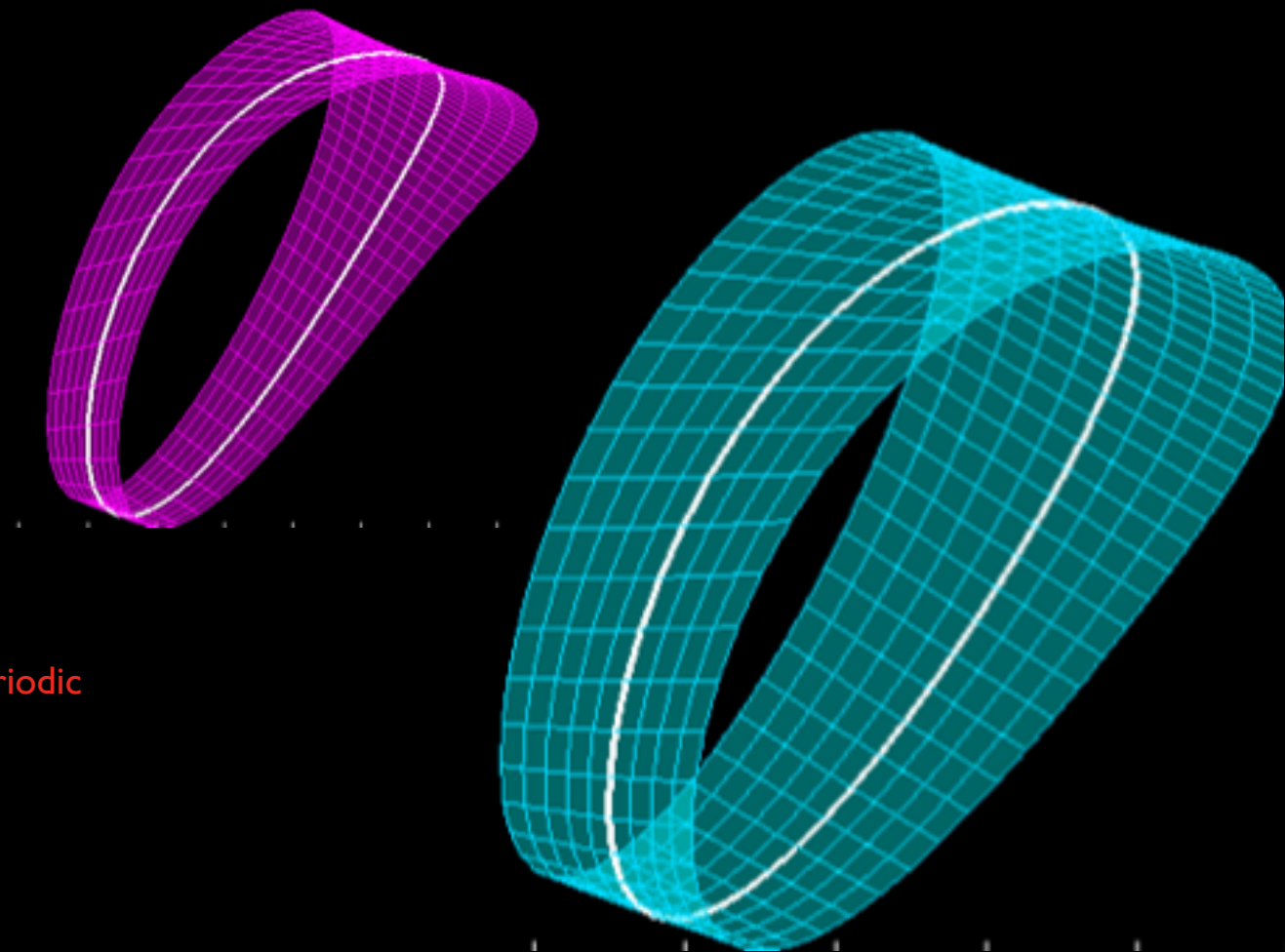
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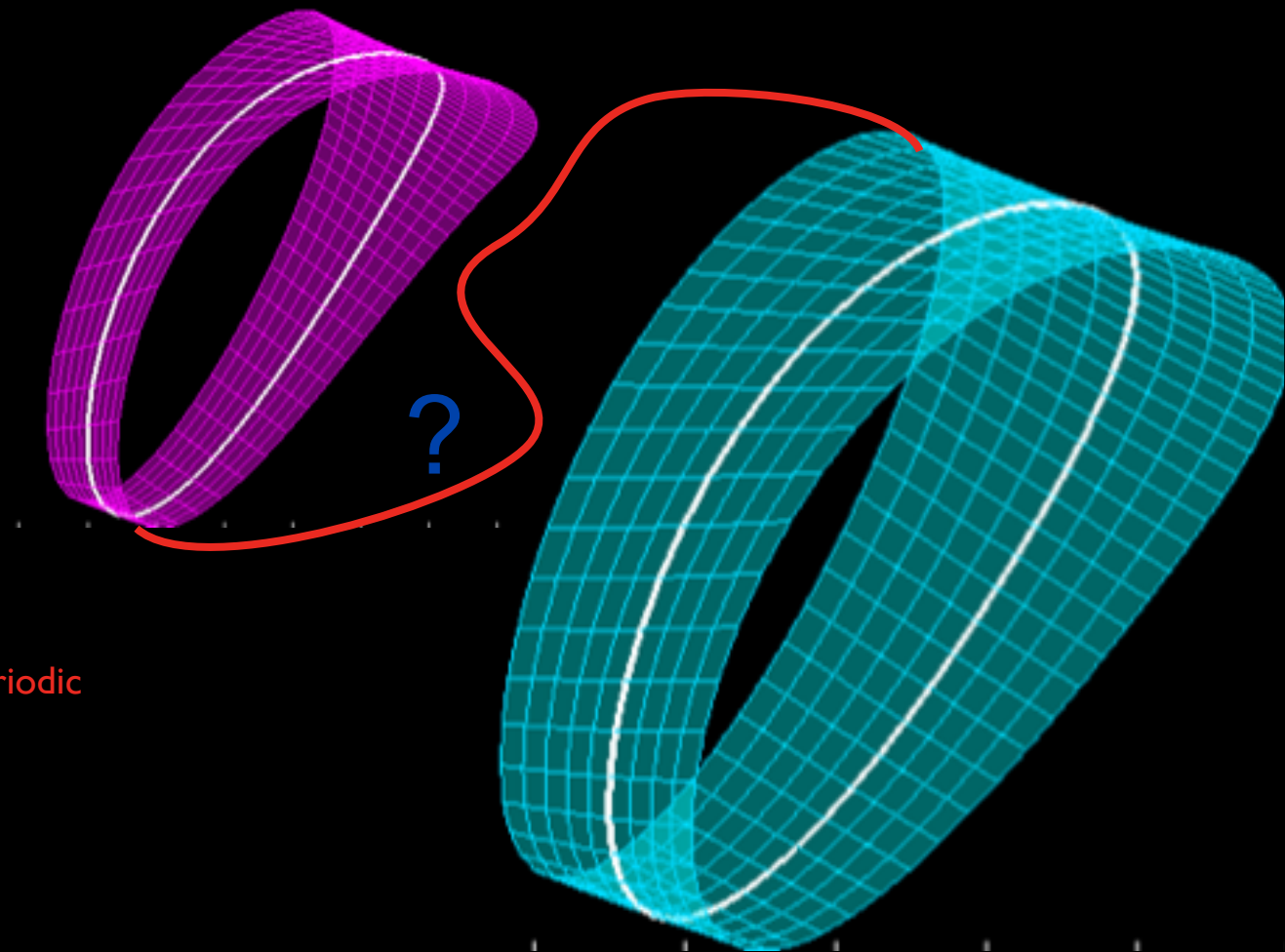
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Formal construction of the parametrisation

Linearization around periodic orbit: Normal bundle

In case of fixed point p one looks at the linearisation $Df(p)$ and computes eigenvalues and eigenvectors.

In case of periodic orbit the linearisation of the vector field around the orbit is given by the non-autonomous linear system

$$\begin{cases} Y' = Df(\gamma(t + \theta))Y \\ Y(0) = I \end{cases}$$

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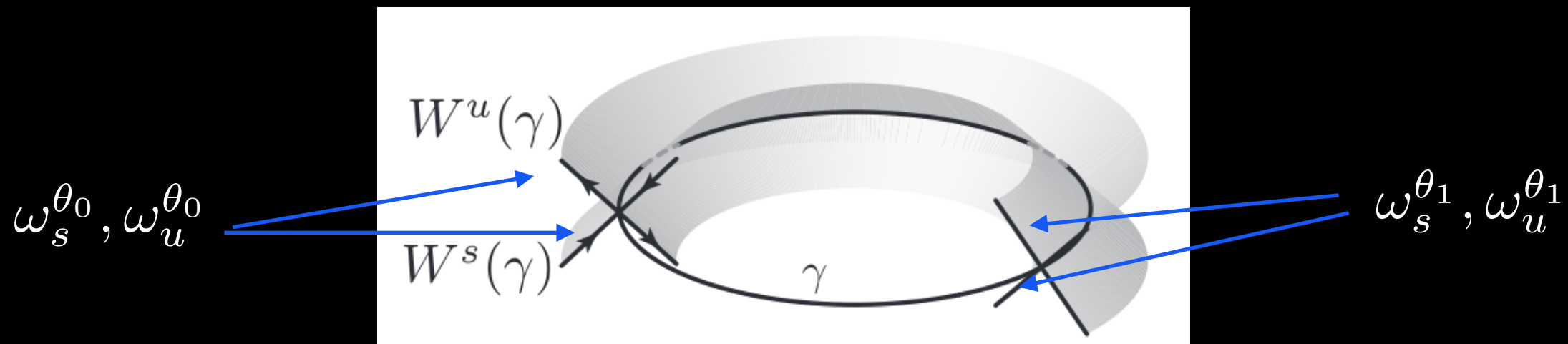
Denote $\Phi_\theta(\tau)$ the solution after a period τ , called **Monodromy matrix**.

Definition: The eigenvalues of $\Phi_\theta(\tau)$ are called the **Floquet multipliers**.

$\dim W^s(\gamma) =$ number of Floquet multipliers with modulus less than one.

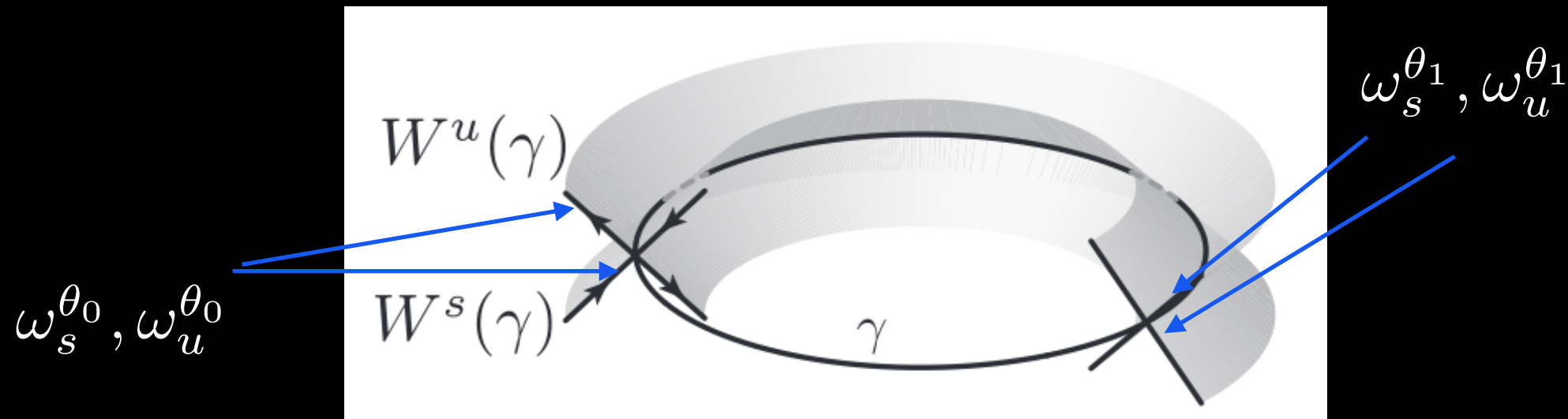
$\dim W^u(\gamma) =$ number of Floquet multipliers with modulus greater than one.

ω_j^θ : eigenvectors of $\Phi_\theta(\tau)$ are the **tangent directions** at $\gamma(\theta)$



Formal construction of the parametrisation

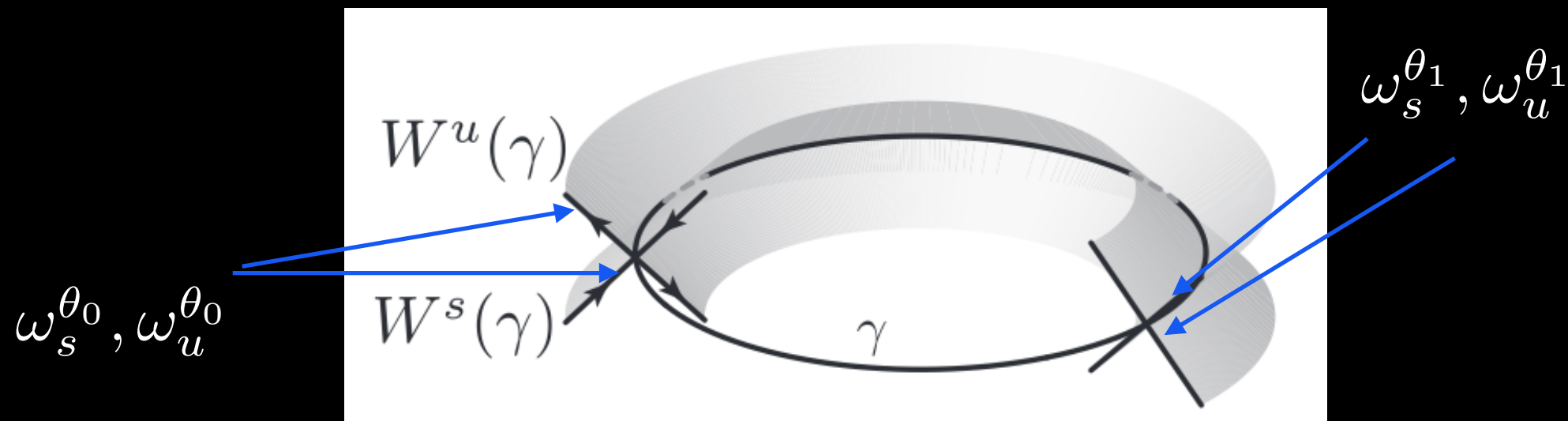
Linearization around periodic orbit: Normal bundle



Is it possible to obtain a **functional expression** of the normal bundle?
That is periodic functions $B_s([0, \tau])$ such that $B_s(\theta) = w_s^\theta$?

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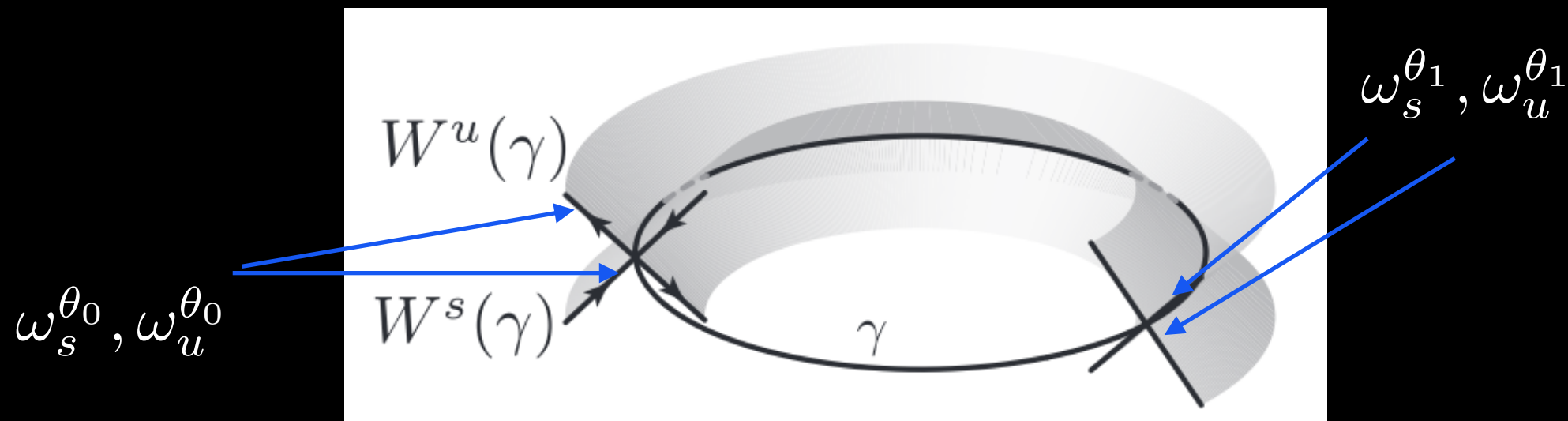
Theorem Floquet

The fundamental matrix solution $\Phi(t)$ can be written as $\Phi(t) = Q(t)e^{Rt}$ for a τ -periodic complex function $Q(t)$ and a complex matrix R .

$$\dot{Q}(\theta) = Df(\gamma(\theta))Q - QR$$

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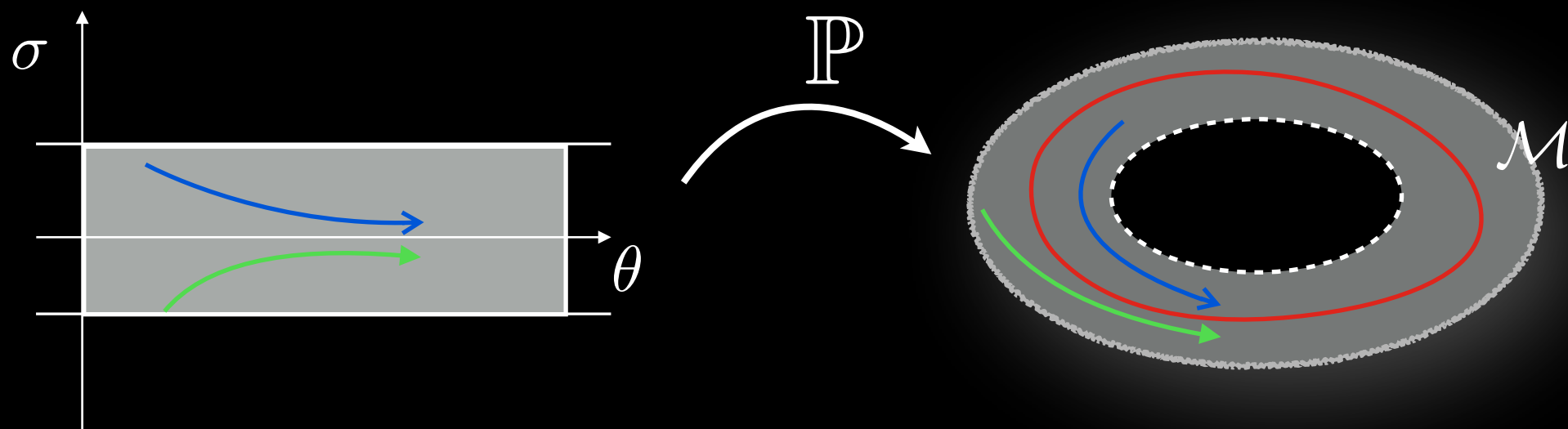
$B_s(\theta) = Q(\theta)\omega_s^0$ **Parametrization of normal bundles** $B_u(\theta) = Q(\theta)\omega_u^0$

Formal construction of the parametrisation

Goal:

Look for a function $\mathbb{P}(\theta, \sigma)$ and a vector field Σ on the (θ, σ) – space so that

1. $\mathbb{P}$ is a chart for $\mathcal{M}$
2. the dynamics on $\mathcal{M}$ is conjugated to the dynamics induced by Σ



$$\Sigma = \begin{cases} \dot{\theta} = 1 \\ \dot{\sigma} = \Lambda \cdot \sigma \end{cases}$$

$$\dot{x} = f(x)$$

$$(\theta, \sigma) \in T \times U$$



$\mathcal{M}$

Formal construction of the parametrisation

Let $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a real analytic vector field and consider system

$$\dot{x} = f(x)$$

$\phi(x, t) : \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}^n$ the flow

P fixed point

$$\mathbb{P}(\sigma)$$

$$\dot{\sigma} = \Lambda \cdot \sigma \rightarrow \sigma(t) = \sigma_0 e^{\Lambda t}$$

$$\psi(\sigma, t)$$

$\gamma(t)$ τ -periodic orbit

$$\mathbb{P}(\theta, \sigma)$$

$$\begin{cases} \dot{\theta} = 1 \\ \dot{\sigma} = \Lambda \cdot \sigma \end{cases} \rightarrow \begin{cases} \sigma(t) = \sigma_0 e^{\Lambda t} \\ \theta(t) = \theta_0 + t \end{cases}$$

flow in par. space

$$\psi((\theta, \sigma), t)$$

$$\phi(\mathbb{P}(\sigma), t) = \mathbb{P}(\psi(\sigma, t))$$

**Conjugacy
eq.**

$$\phi(\mathbb{P}(\theta, \sigma), t) = \mathbb{P}(\psi((\theta, \sigma), t))$$

$$D\mathbb{P}(\sigma)\Lambda\sigma = f(\mathbb{P}(\sigma))$$

**Invariance
eq.**

$$\begin{aligned} \frac{\partial}{\partial \theta} \mathbb{P}(\theta, \sigma) + \sum_i \lambda_i \sigma_i \frac{\partial}{\partial \sigma_i} \mathbb{P}(\theta, \sigma) \\ = f(\mathbb{P}(\theta, \sigma)) \end{aligned}$$

Formal construction of the parametrisation

P fixed point (λ_s, ξ_s)

$\gamma(t)$ τ -periodic orbit (λ_s, ω_s^0)

$$D\mathbb{P}(\sigma)\Lambda\sigma = f(\mathbb{P}(\sigma))$$

**Invariance
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$$\mathbb{P}(\sigma) = \sum_{|n| \geq 0} \alpha_n \sigma^n$$

$$\alpha_0 = P, \alpha_1 = \xi_s$$

$$(Df(p) - \lambda_s \cdot nI)\alpha_n = b_n$$

$$|n| = 0, |n| = 1$$

$$|n| \geq 2$$

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Homological eq.

$$\frac{d\alpha_n(\theta)}{d\theta} + n \cdot \lambda_s \alpha_n(\theta) = Df(\alpha_0(\theta))\alpha_n(\theta) + b_n(\theta)$$

Since b_n and $b_n(\theta)$ depend on lower order term, $b_n = b_n(\alpha_0, \dots, \alpha_{n-1})$, one can compute recursively the coefficients α_n and $\alpha_n(\theta)$.

Formal construction of the parametrisation: high order terms

We need to solve the homological equation

$$\frac{d\alpha_n}{d\theta} + n \cdot \lambda \alpha_n = Df(\alpha_0) \alpha_n + b_n$$

Non-autonomous linear ODE



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$$\frac{d}{d\theta}w_n + (n \cdot \lambda I - R)w_n = Q^{-1}(\theta)b_n(\theta)$$



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Some algebra and expansion in Fourier

$$w_n(\theta) = \sum_k w_k e^{i\frac{2\pi}{\tau}\theta} \quad Q^{-1}(\theta)b_n(\theta) = \sum_k A_k^n e^{i\frac{2\pi}{\tau}\theta} \quad w_k = Mv_k \quad R = MDM^{-1}$$



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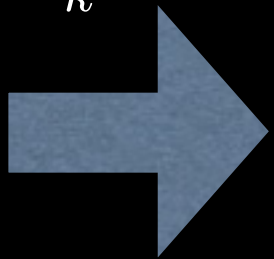
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$$(v_k)_j = \frac{1}{\frac{2\pi i k}{\tau} + n \cdot \lambda - (D)_j} (M^{-1} A_k^n)_j$$

Formal construction of the parametrisation

$$\mathbb{P}(\theta, \sigma) = \sum_{|n| \geq 0} \alpha_n(\theta) \sigma^n$$

REMARKS

- The method is integration free

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- The method does not depend on the dimension of the vector field neither the dimension of the manifolds

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- The construction of b_n is immediate for polynomial nonlinearities
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- Both (non)-orientable manifolds can be treated (by doubling the period)
- The method does not depend on the dimension of the vector field neither the dimension of the manifolds
- The computation is reasonable fast

Formal construction of the parametrisation

$$\mathbb{P}(\theta, \sigma) = \sum_{|n| \geq 0} \alpha_n(\theta) \sigma^n$$

REMARKS

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- Only few mild resonance conditions must be satisfied
- Both (non)-orientable manifolds can be treated (by doubling the period)
- The method does not depend on the dimension of the vector field neither the dimension of the manifolds
- The computation is reasonable fast
- The construction allows a posteriori error estimate

$$\mathbb{P}_{N,K}(\theta, \sigma) = \sum_{|n|=0}^N \sum_{k=-K}^K \alpha_{n,k} e^{\frac{2\pi i}{\tau} k \theta} \sigma^n$$

$$E(\theta, \sigma) = \frac{\partial}{\partial \theta} \mathbb{P}_{N,K}(\theta, \sigma) + \sum_i \lambda_i \sigma_i \partial_{\sigma_i} \mathbb{P}_{N,K}(\theta, \sigma) - f(\mathbb{P}_{N,K}(\theta, \sigma))$$

Example

$$f(x, y, z) = \begin{pmatrix} -sx + sy \\ \rho x - y - xz \\ -\beta z + xy \end{pmatrix}, \quad (s, \beta, \rho \in \mathbb{R}). \quad \text{Lorenz equation}$$

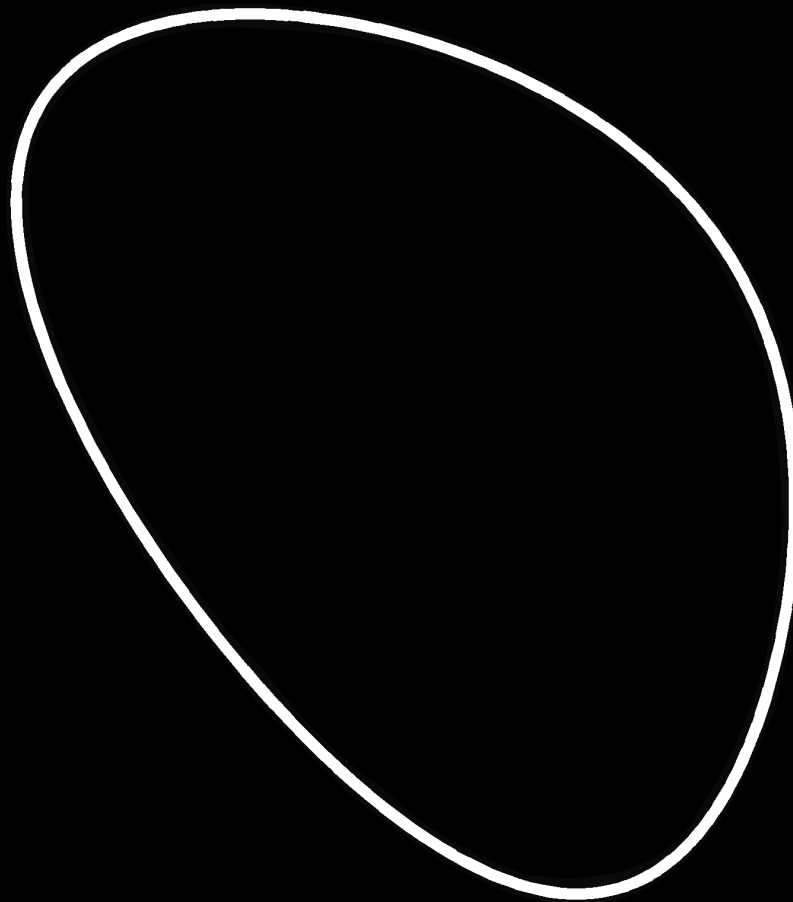
$$\sigma = 10, \beta = \frac{8}{3}, \rho = 22$$

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$$\sigma = 10, \beta = \frac{8}{3}, \rho = 22 \quad \text{unstable} \quad \text{stable}$$

Periodic orbit: Period $T=0.7644$. Floquet Multipliers: 1.1607, $2.5e-5$

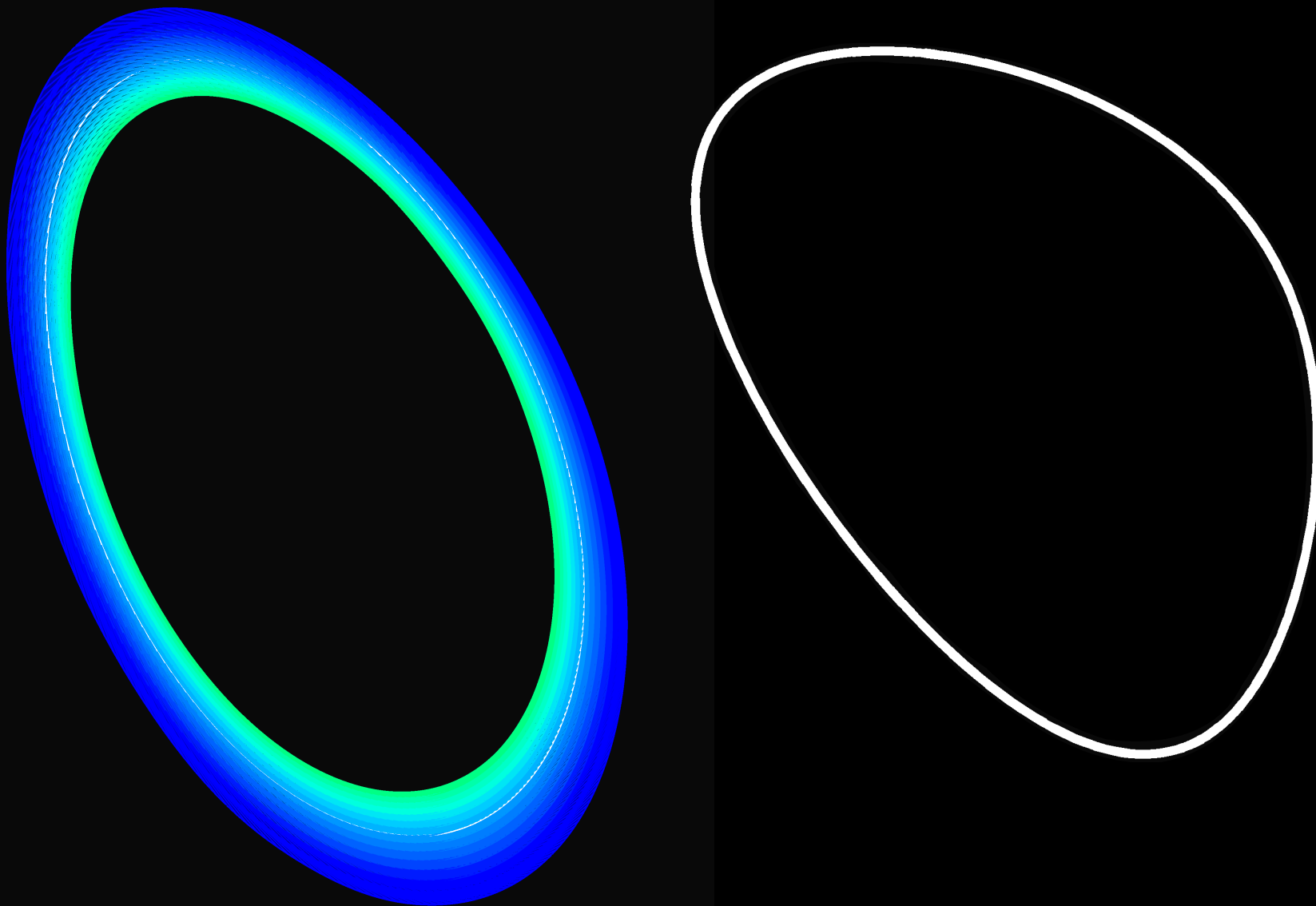


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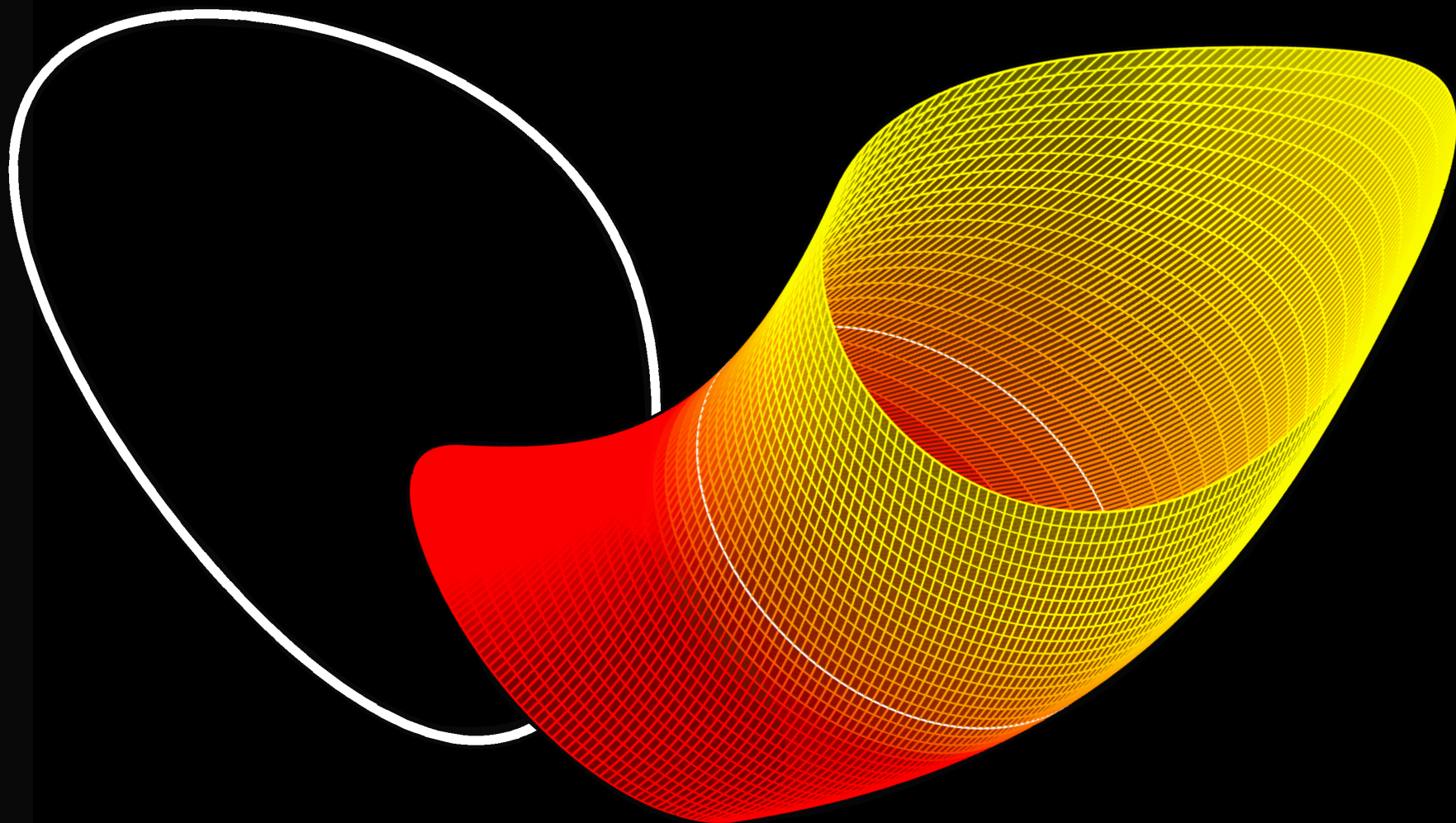
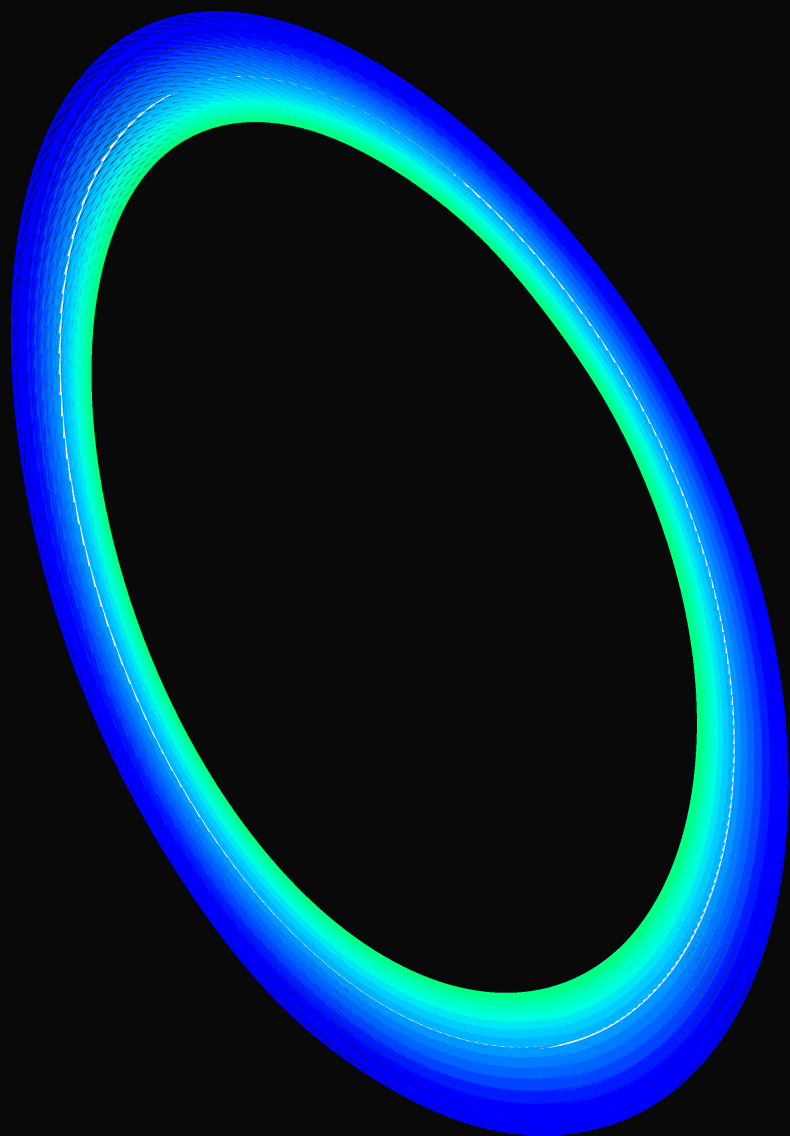


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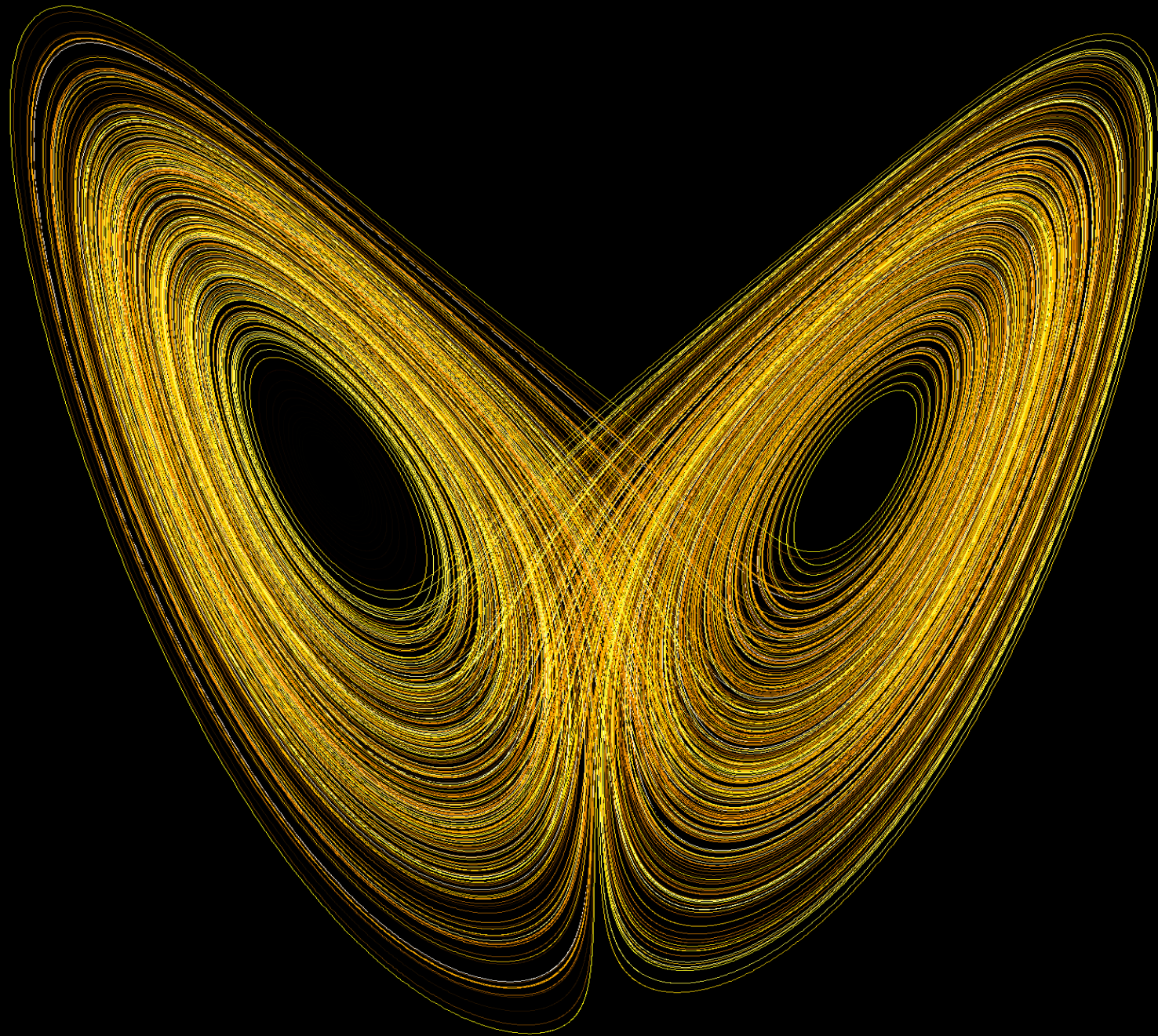
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Example

Lorenz equation

$$\sigma = 10, \beta = \frac{8}{3}, \rho = 28$$

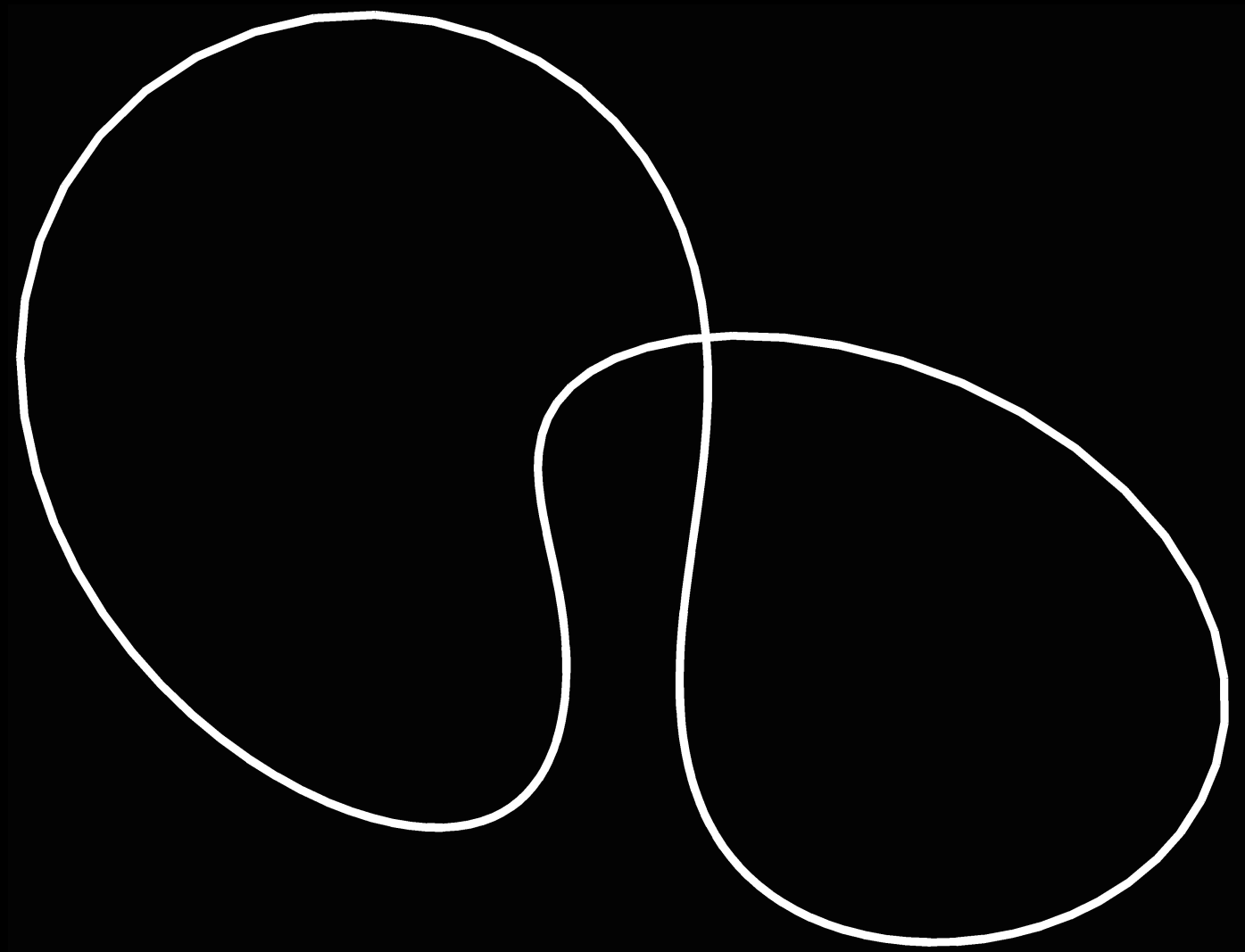


Example

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Periodic orbit: Period $T=0.7644$. Floquet Multipliers: 4.7129 , $1.2e-10$



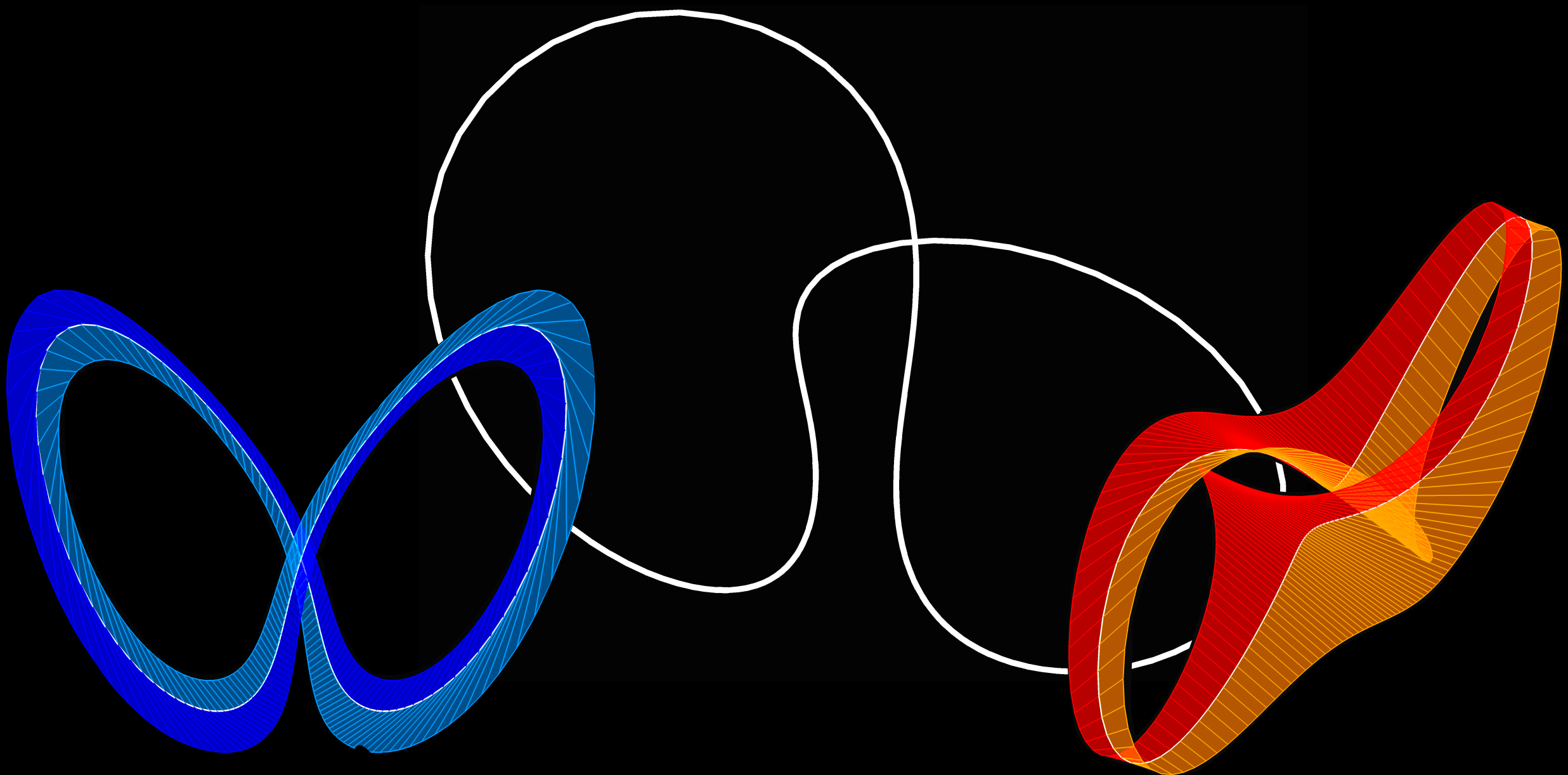
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Normal Bundles



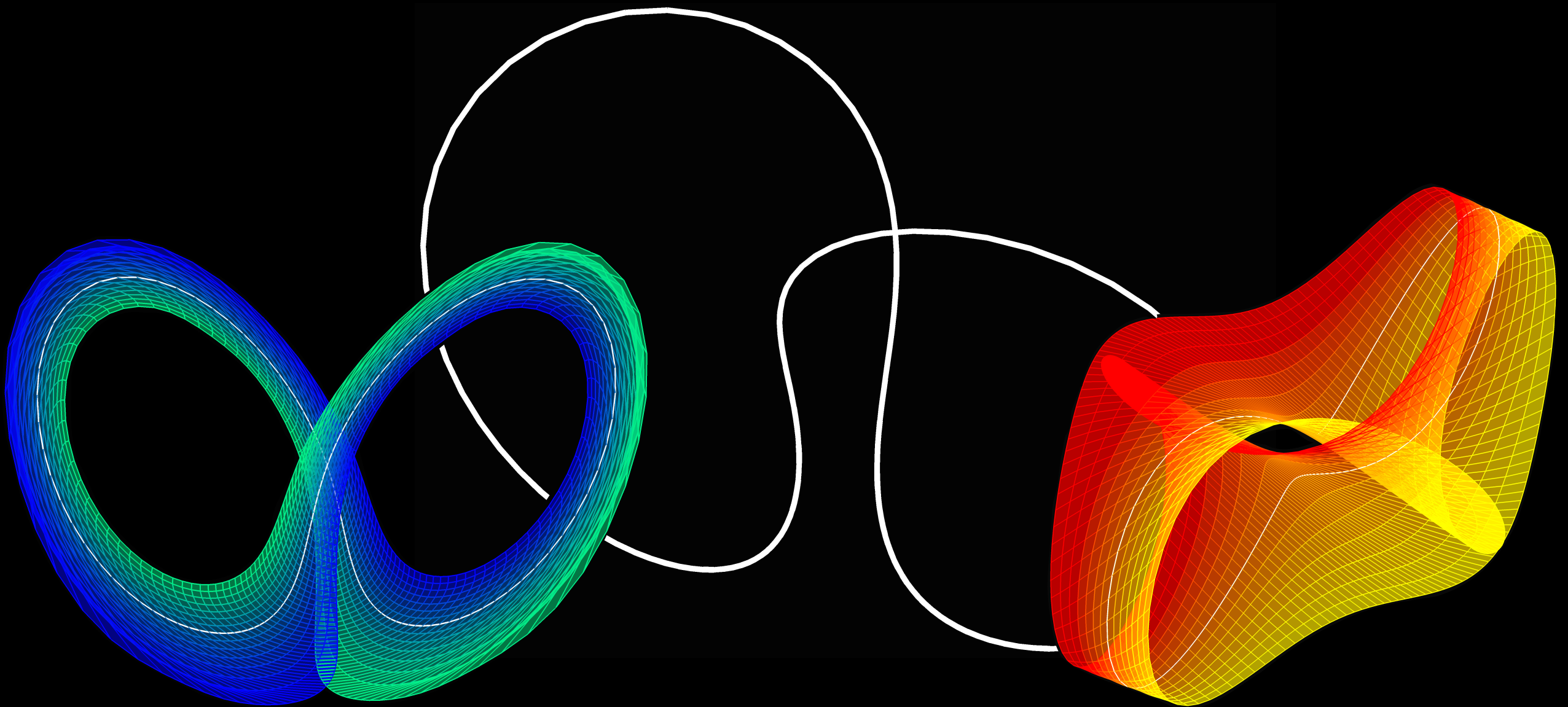
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Invariant Manifolds



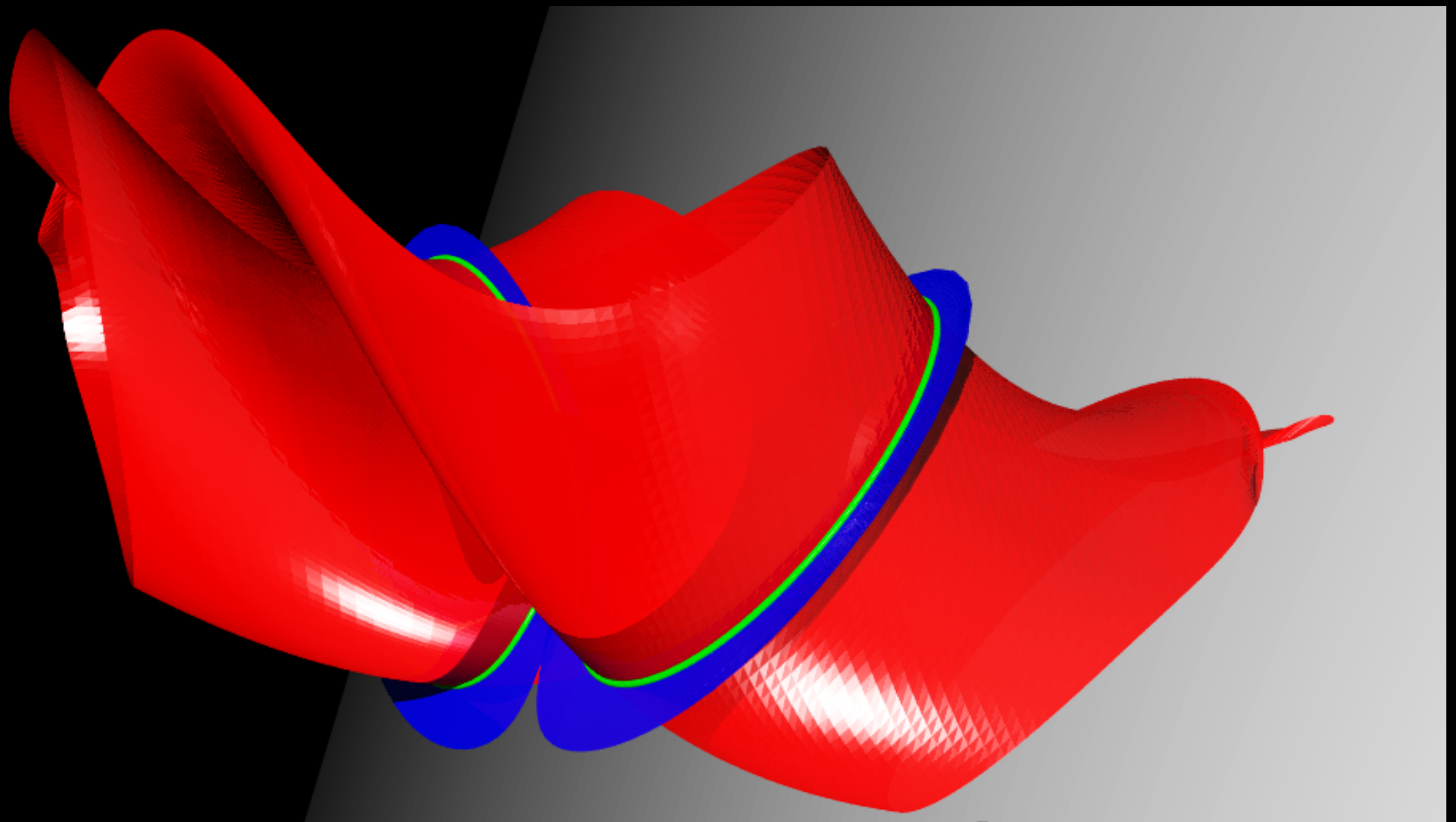
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Large range parametrisation



Example: error estimation

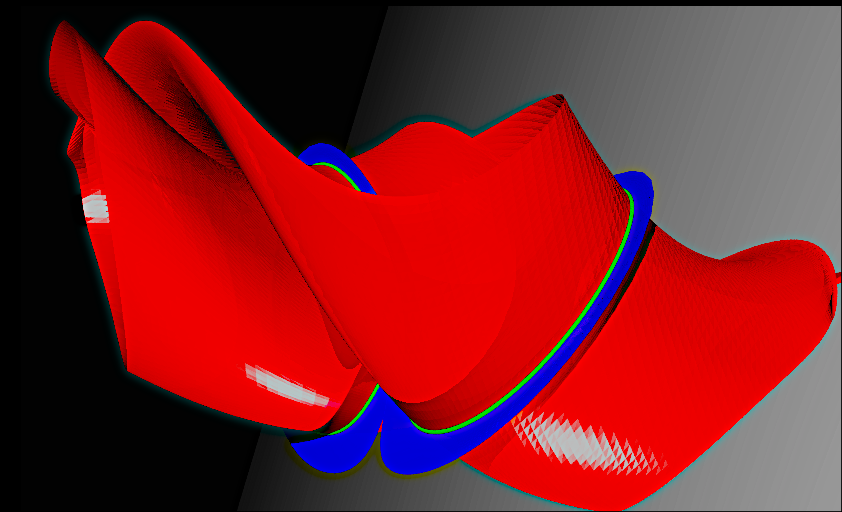
Lorenz equation

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Example: error estimation

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Example: error estimation

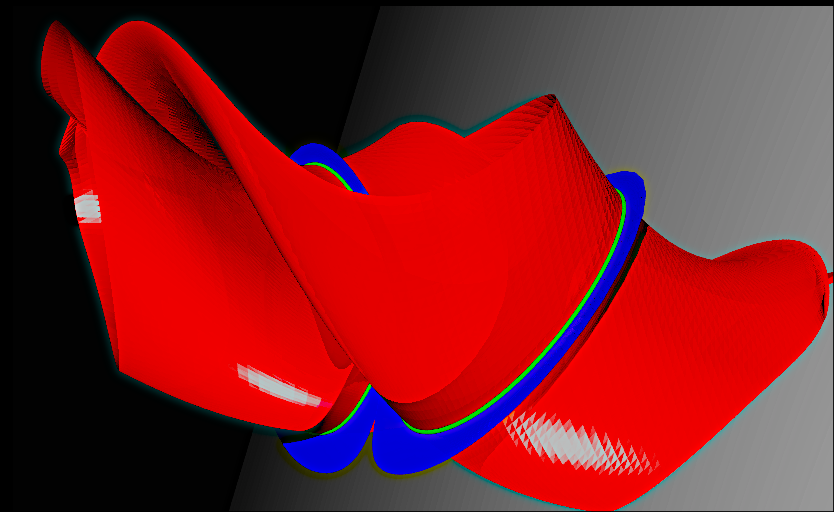
Lorenz equation

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Weighted Wiener-norm

$$\|E\|_{r,\nu}^W = \sum_{|n| \geq 0} \sum_{k \in \mathbb{Z}} \left\| \frac{2\pi i k}{2\tau} \alpha_{n,k} + n \cdot \lambda \alpha_{n,k} - b_{n,k} \right\| e^{\frac{2\pi |k|}{2\tau} r} \nu^n$$



Example: error estimation

Lorenz equation

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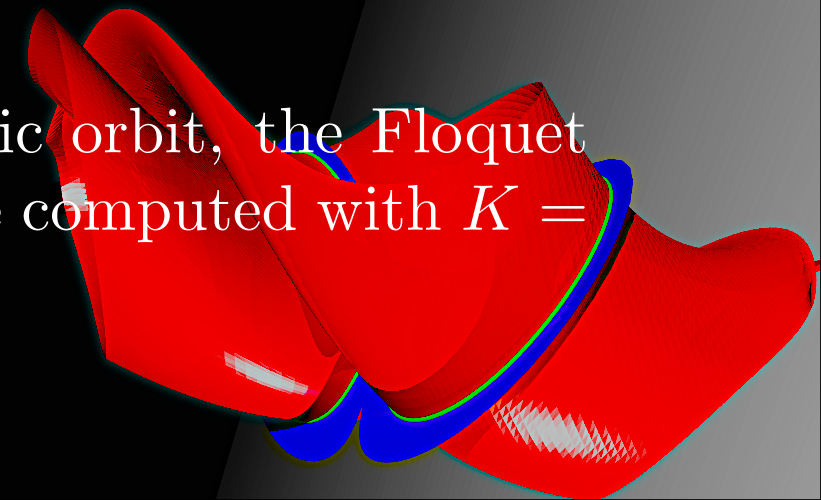
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N	ν	r	$\ E\ _{r,\nu}^W$	Computation Time	Error Computation Time
2	0.1	0.05	4.39×10^{-6}	0.47 sec	0.008 sec
2	0.1	0.01	7.96×10^{-10}	0.47 sec	0.008 sec
4	0.5	0.01	7.96×10^{-10}	0.82 sec	0.012 sec
10	6	0.01	8.19×10^{-10}	1.83 sec	0.029 sec
20	15	10^{-3}	6.47×10^{-9}	3.57 sec	0.071 sec
25	20	10^{-6}	1.77×10^{-6}	4.42 sec	0.096 sec

Stable manifold data. For each computation, the periodic orbit, the Floquet normal form, and all manifold Taylor coefficients $a_n(\theta)$ are computed with $K = 66$ Fourier modes.

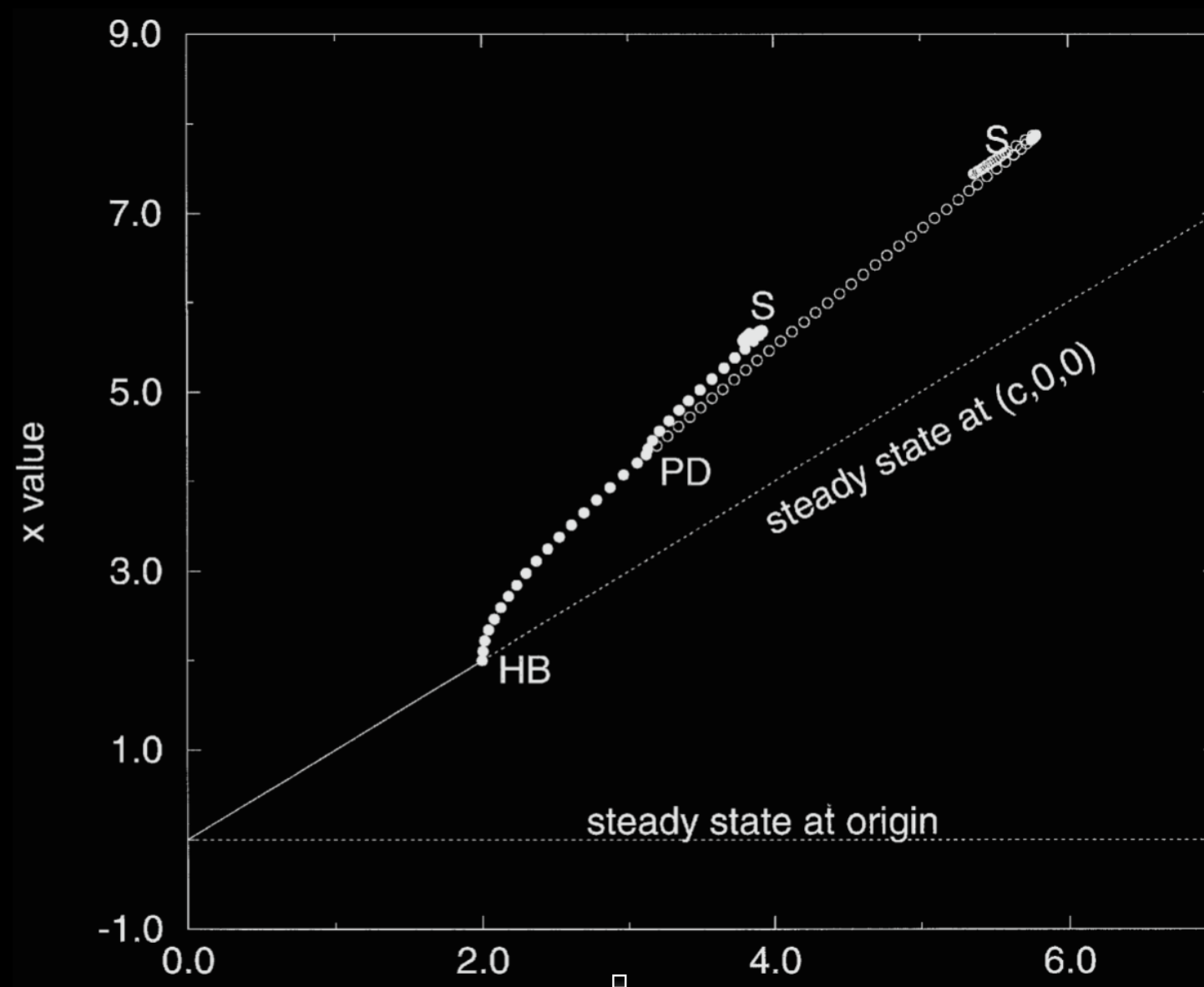


Example

Arneodo System

$$\begin{cases} \dot{x} = y \\ \dot{y} = z \\ \dot{z} = \alpha x - x^2 - \beta y - z, \end{cases}$$

$\beta = 2, \alpha = 3.372$



Example

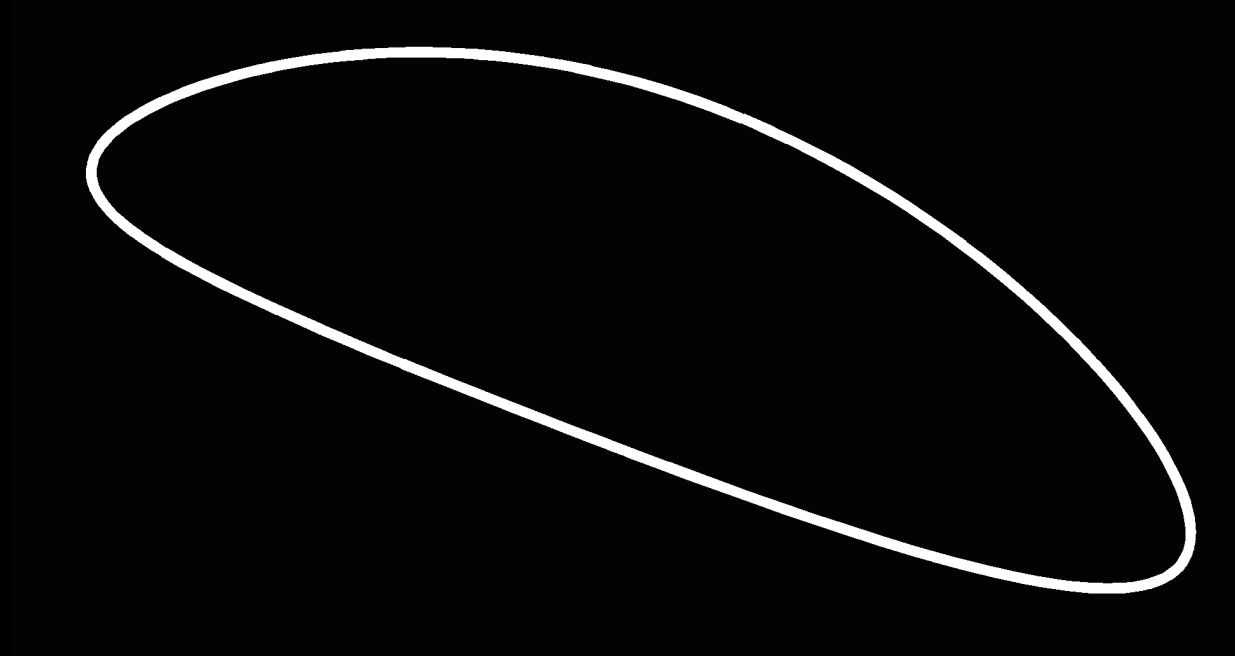
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Neg. Floquet mult =
non orientable manifold

Periodic orbit: Period $T = 4.53284$. Floquet Multipliers: -1.527 , -0.070



Example

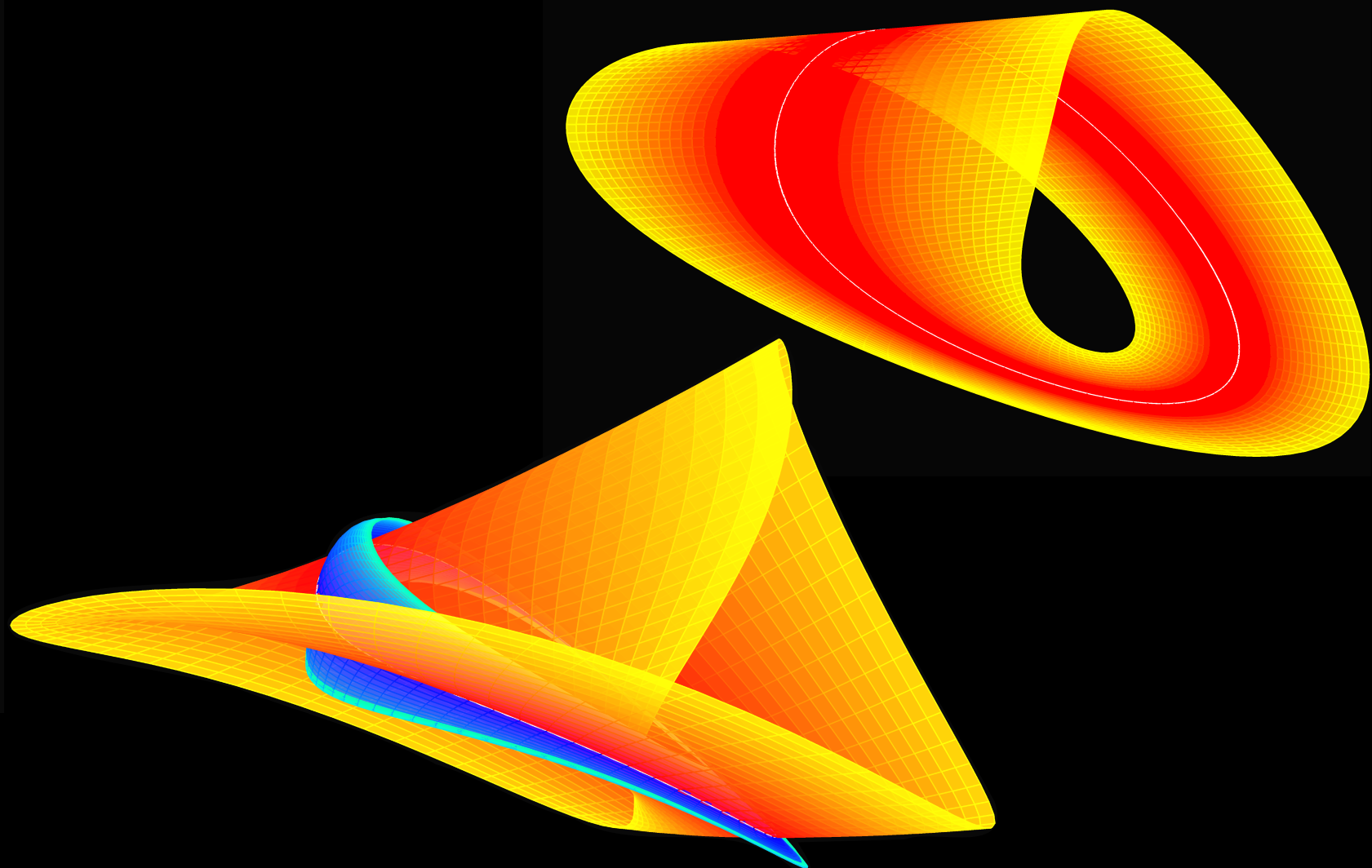
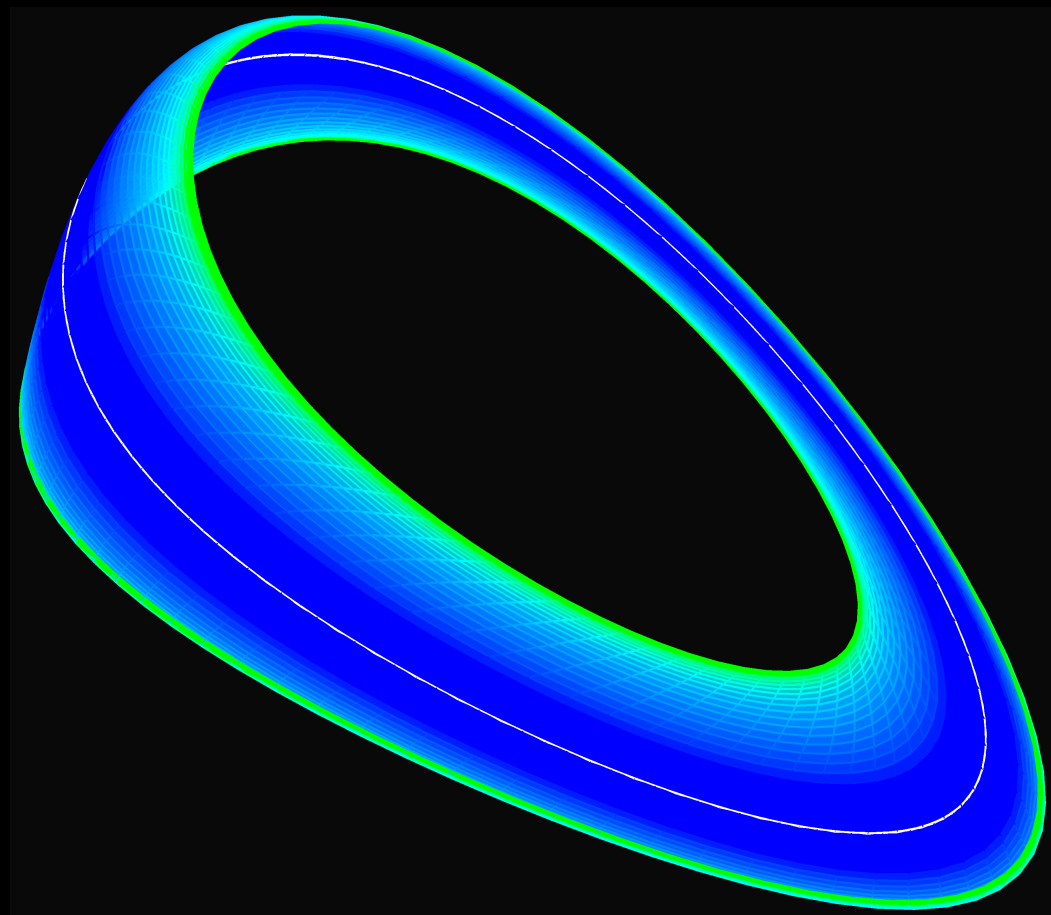
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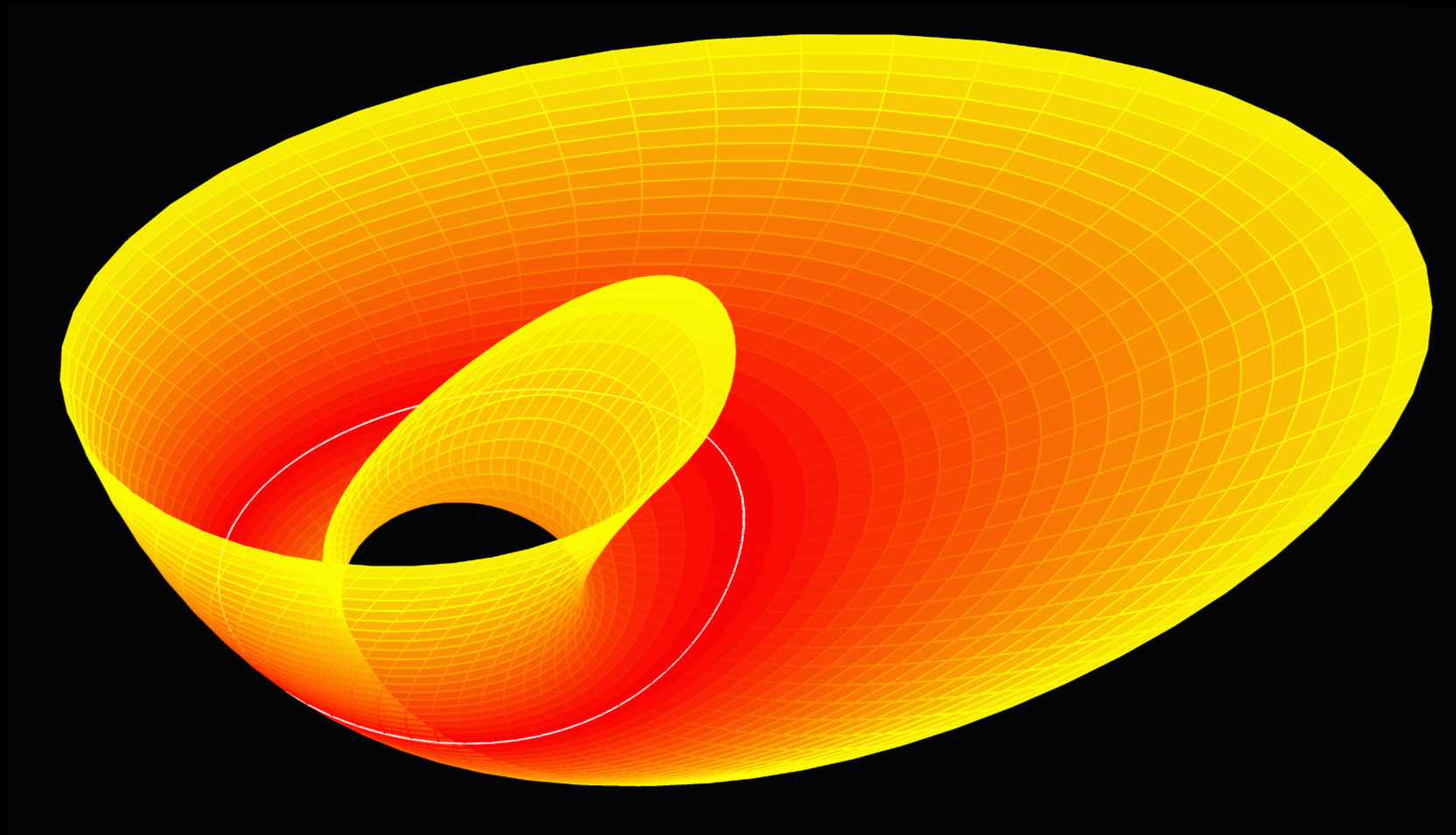
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Example

Kuramoto-Sivashinsky equation

Periodic and symmetric boundary conditions

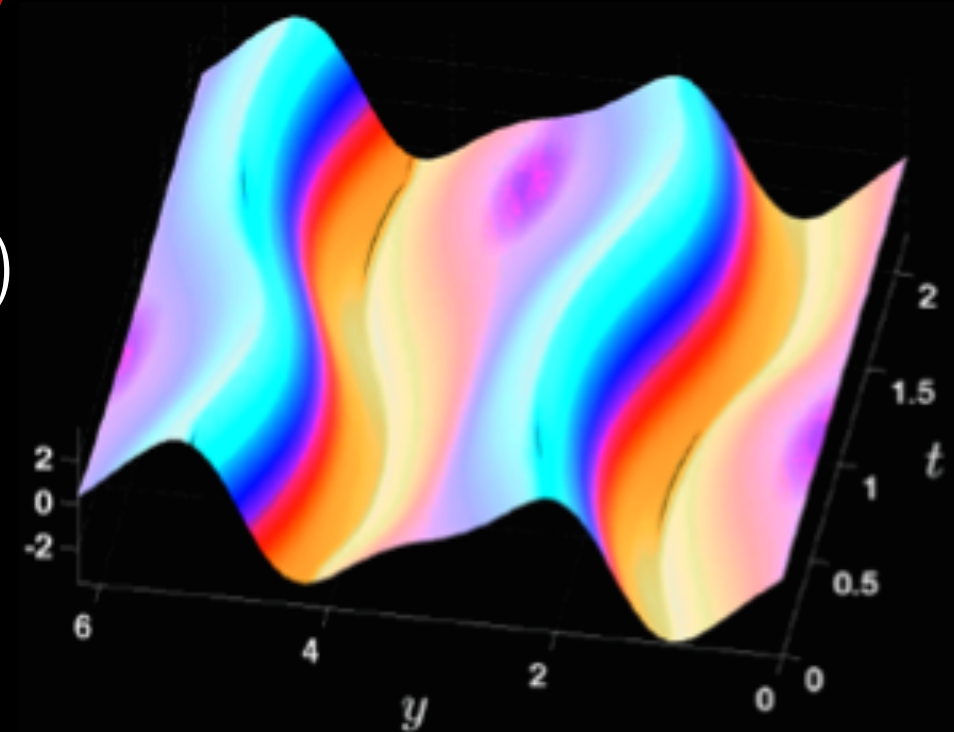
$$u_t = -u_{yy} - \lambda u_{yyyy} + 2uu_y$$
$$u(t, y) = u(t, y + 2\pi), u(t, -y) = -u(t, y)$$

Finite-dimensional projection in Fourier space

$$u(t, y) = \sum_{k \in \mathbb{Z}} x_k(t) e^{iky} \quad \longrightarrow \quad u(t, y) = \sum_{|k| \leq N} x_k(t) e^{iky}$$

Setting $x_k(t) = iw_k(t)$

$$\dot{w}_n = (n^2 - \nu n^4)w_n - n \sum_{k_1 + k_2 = n} w_{k_1} w_{k_2} \quad n = 1, \dots, N$$
$$w_{-n}(t) = -w_n(t)$$



Example

Kuramoto-Sivashinsky equation

10-dimensional Galerkin projection

$$\rho = 0.11878$$

Example

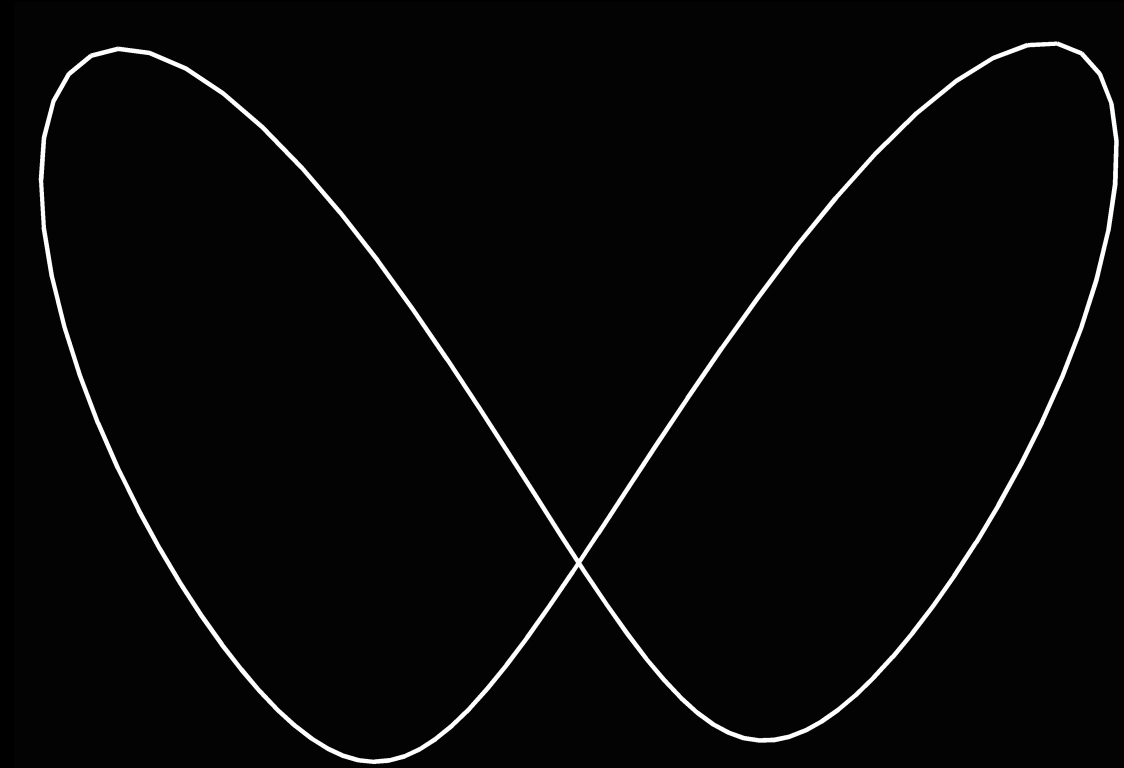
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Floquet Multipliers: 83.3, 0.026



Example

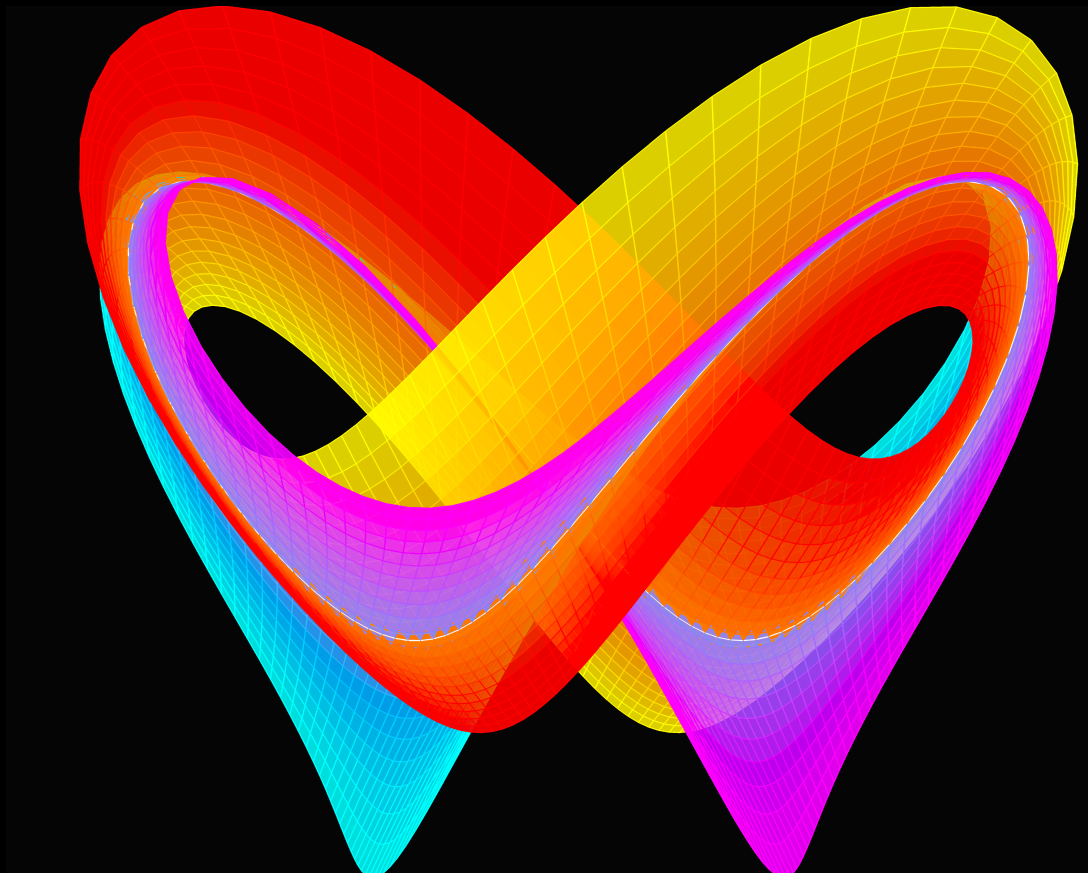
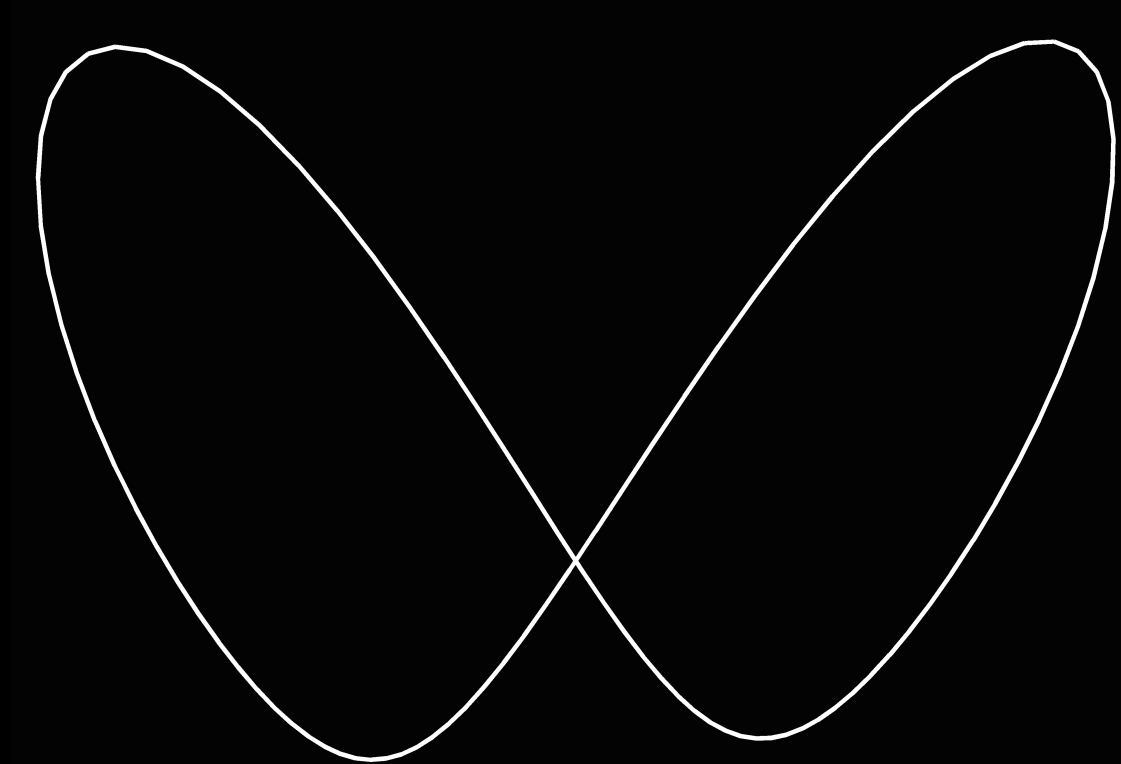
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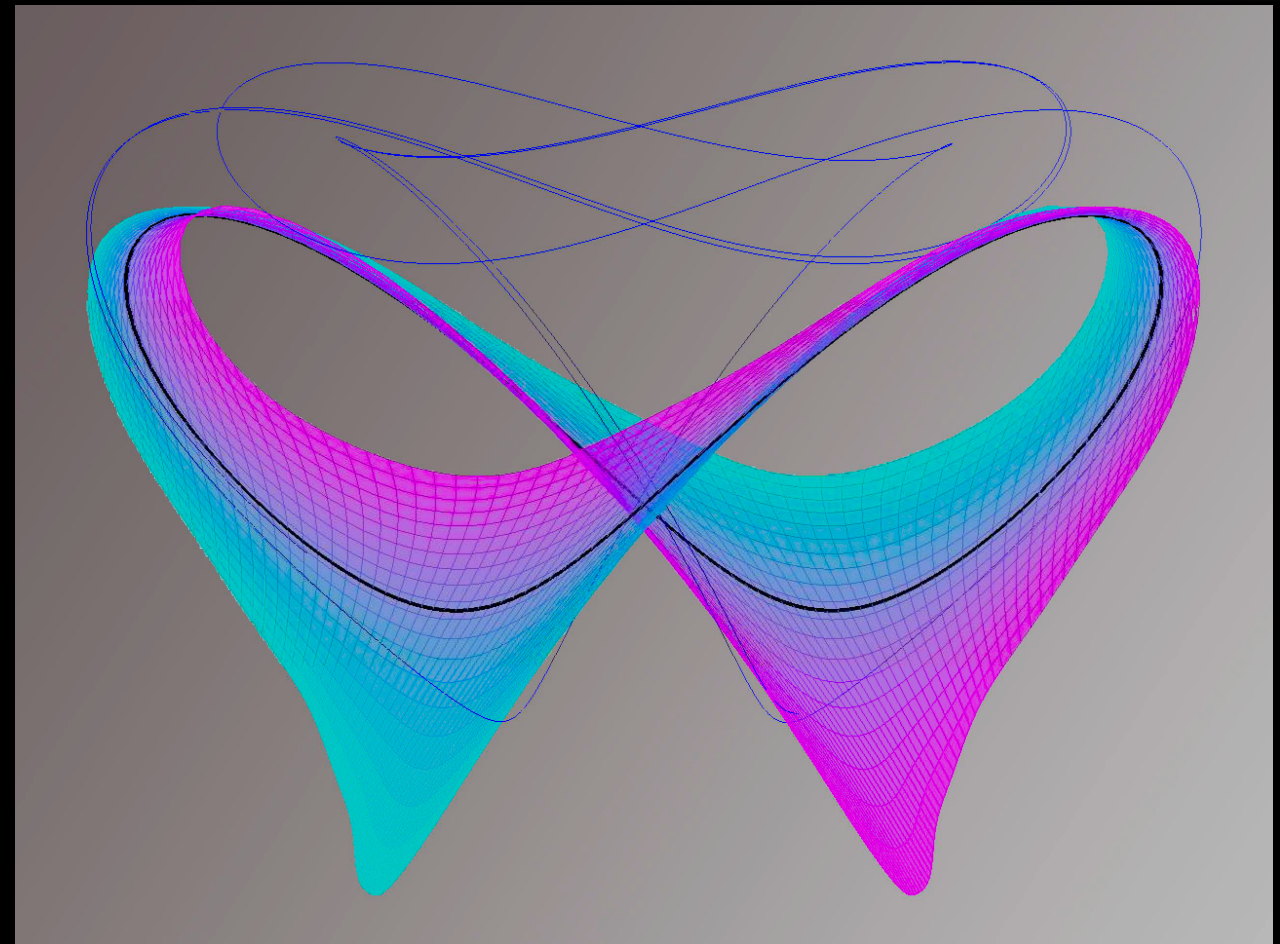
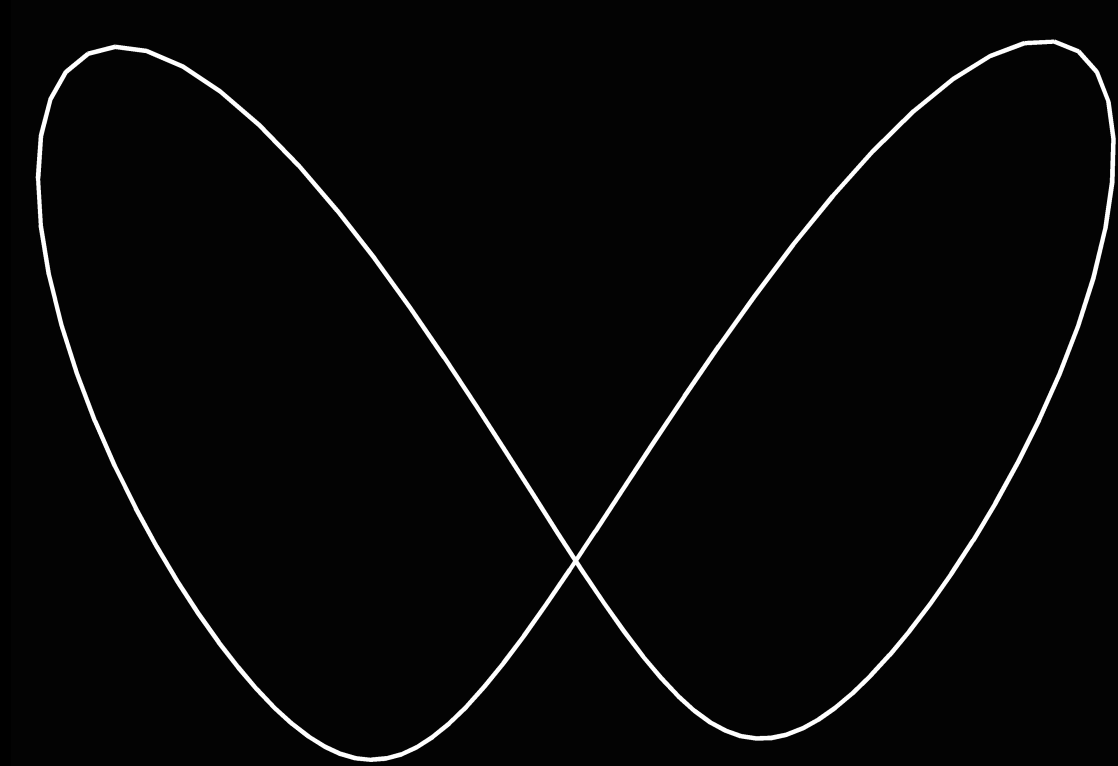
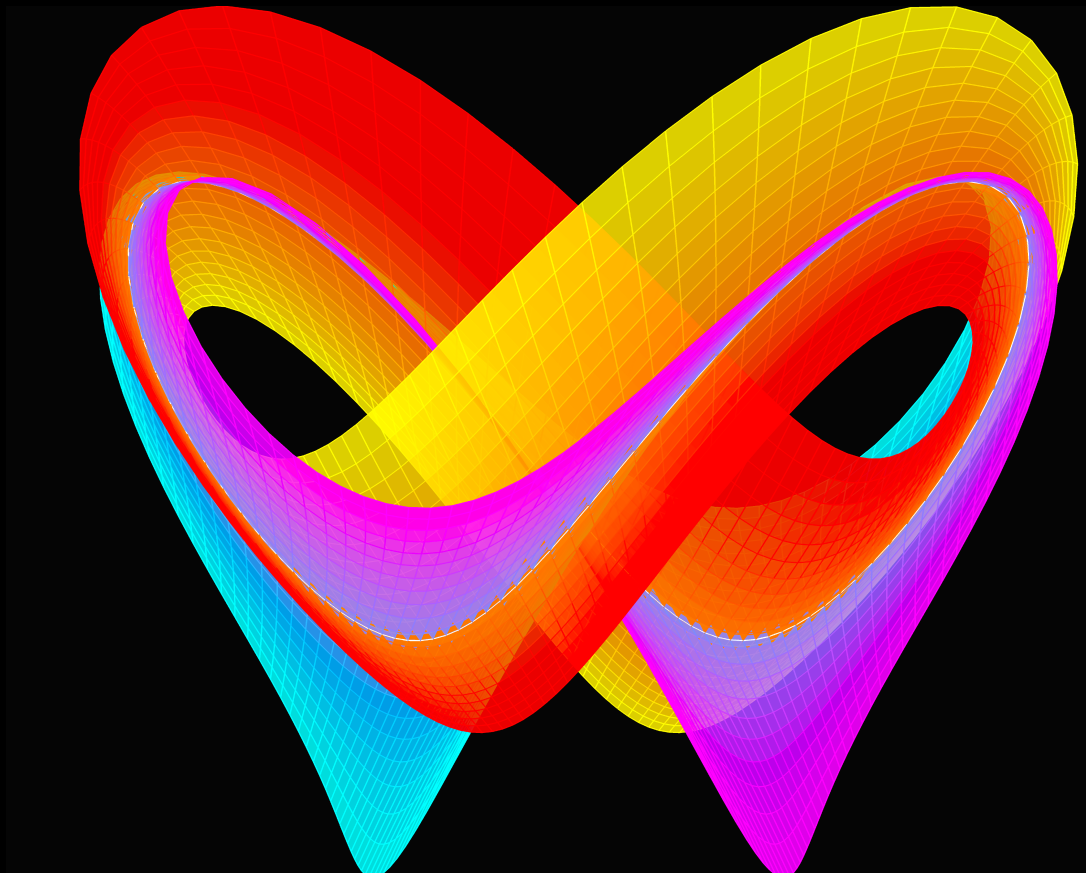
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A homoclinic orbit in the 10-dimensional Galerkin projection of the Kuramoto-Sivashinsky PDE

Thank you

