Non collision periodic solutions for the planar N-center problem with mild topological assumptions

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The 10th AIMS Conference on Dynamical Systems Differential Equations and Applications

July 07 - July 11, 2014 Madrid, Spain

Outline

1. Introduction and motivations
2. Topological setting: definition of admissible classes
3. Statement of the theorem and examples
4. Sketch of the proof

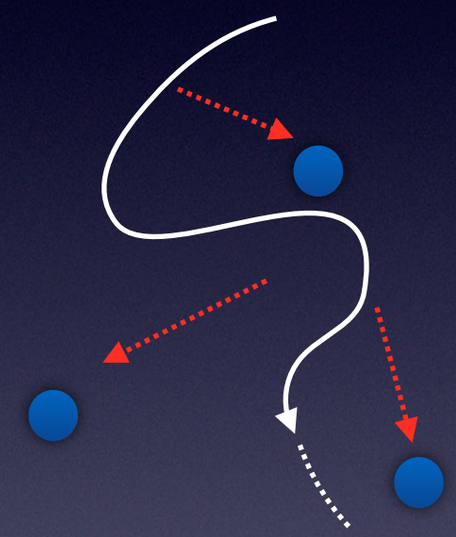
Introduction and motivations

N-center problem: study of the motion of a test particle in the force field generated by N Newton-like center potentials

$$V(x) = \sum_{j=1}^{N} \frac{m_j}{\alpha |x - c_j|^{\alpha}}, \quad x \in \mathbb{R}^2 \setminus \{c_1, \ldots, c_N\}, m_j > 0, \alpha \in [1, 2)$$

Equation of motion: $\ddot{x} = -\sum_j m_j \frac{x(t) - c_j}{|x(t) - c_j|^{\alpha+2}}$

Energy first integral: $h = \frac{1}{2}|\dot{x}|^2 - V(x)$



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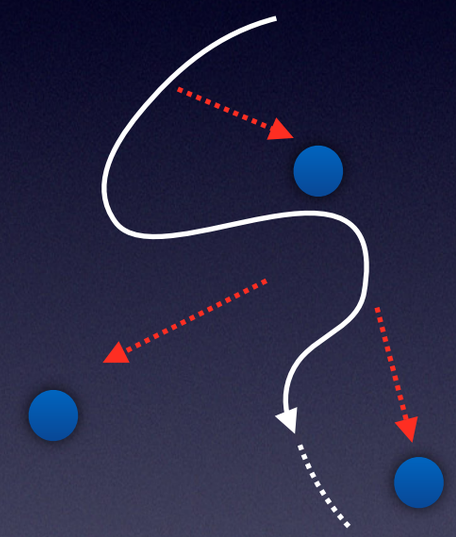

We look for periodic planar solutions with a prescribed energy

Introduction and motivations

Literature review (Not exhaustive…)

N=1,2 The system is integrable (Kepler, Euler)

N>2 The system is not integrable
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More recently

Dimare, L., Chaotic quasi-collision trajectories In the 3-centre problem (2010)

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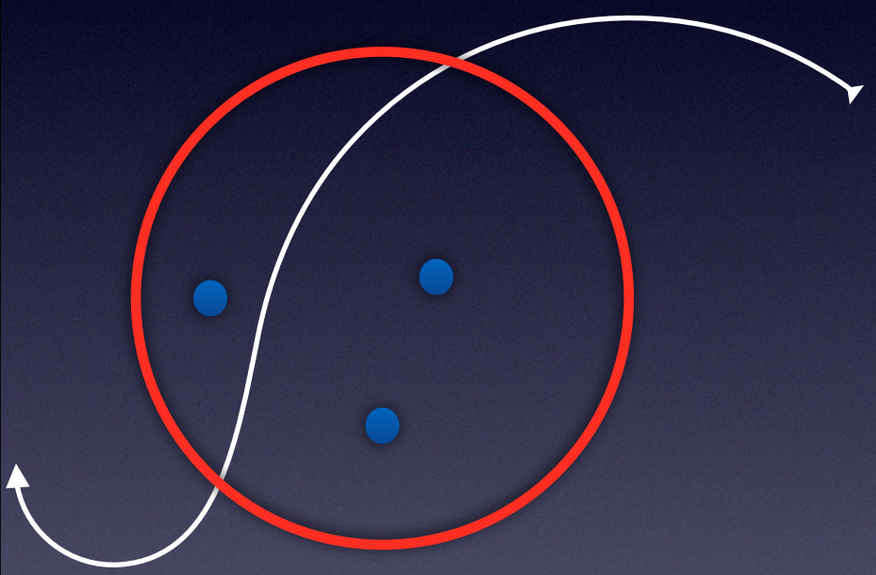

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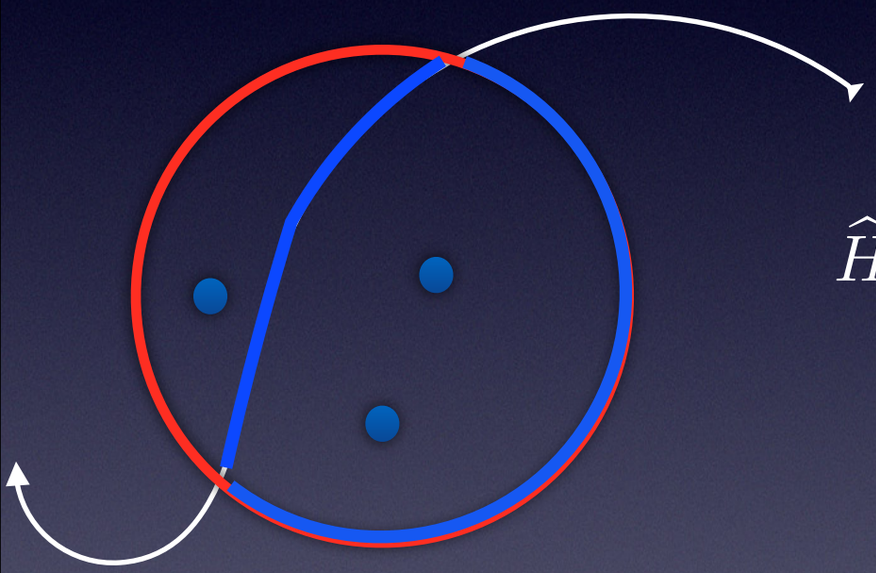

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$$\widehat{H}_l := \{u \in \widehat{H} : Ind(u([a,b]), c_j) \equiv l_j \ mod\ 2,\ \forall j = 1, \ldots, N\}$$

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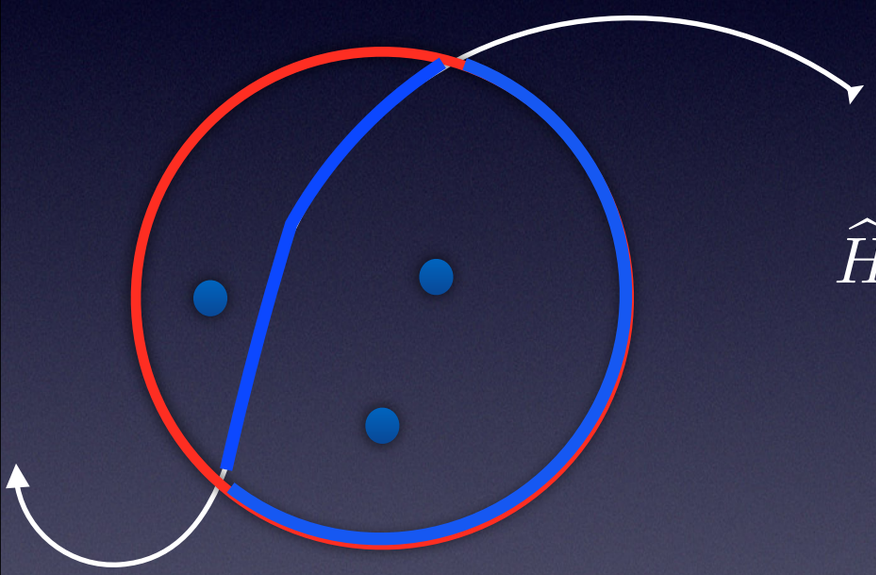

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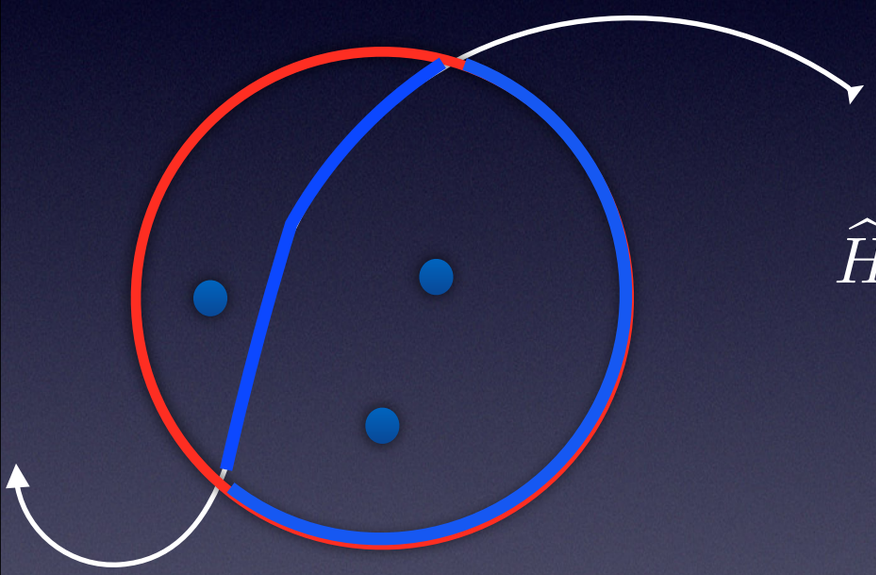

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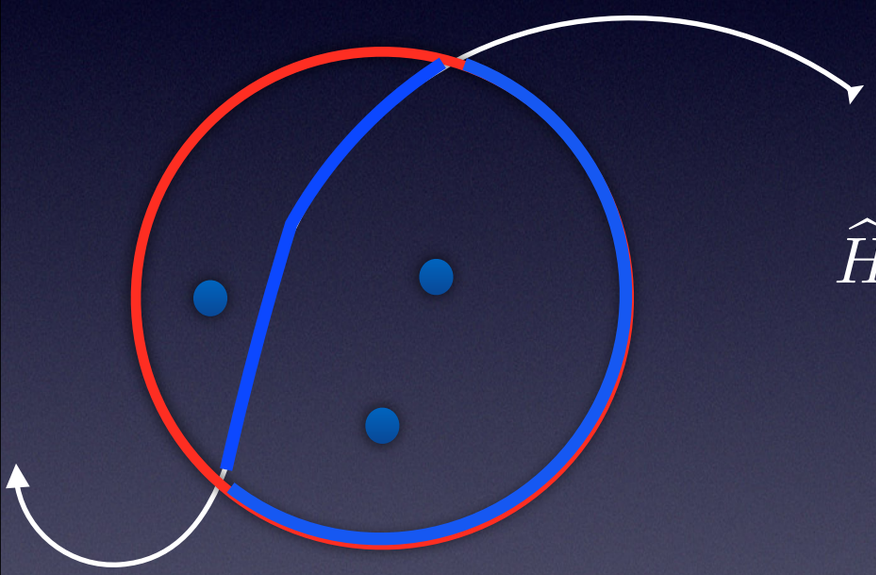

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The minimiser results to be free of self intersections

Goal: weaken the topological assumptions and allow minimisers with intersections

Topological setting:
definition of admissible classes

Topological setting: definition Admissible classes

1. Puntcured plane $\mathbb{R}^2_C := \mathbb{R}^2 \setminus \{c_1, \ldots, c_N\}$

2. Loops space $\mathcal{C}^0(S, \mathbb{R}^2_C) \quad S := [0,1]/\{0,1\}$

3. Homotopy relation $\sim$

$$\gamma_0 \sim \gamma_1 \text{ if } \exists\ \Phi(s,t) : S \times [0,1] \to \mathbb{R}^2_C \text{ continuous s.t. } \Phi(s,0) = \gamma_0(s) \text{ and } \Phi(s,1) = \gamma_1(s)$$

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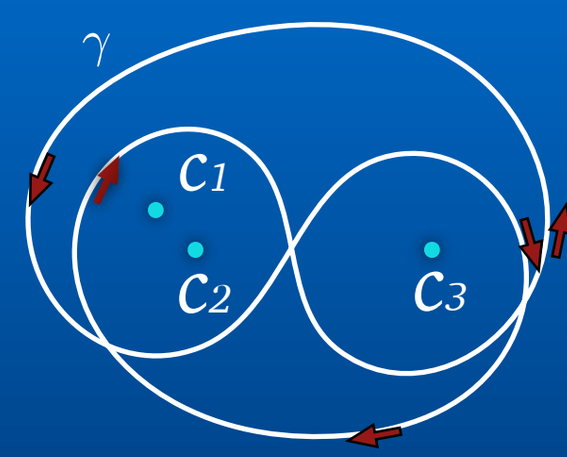

$Ind(\gamma) = (0,0,0)$

There is no exact correspondence between Λ and $\mathbb{Z}^N$

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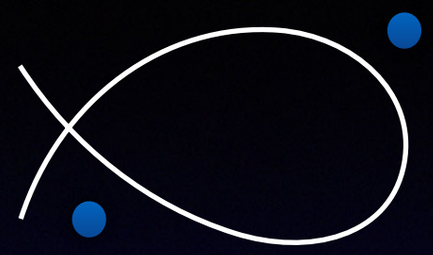

i) *singular 1-gon* if there is a subinterval I of S such that γ identifies the end-points of I and $\gamma|_I$ defines a null-homotopic loop in $\mathbb{R}^2_C$ (i.e. γ has a null-homotopic sub-loop)

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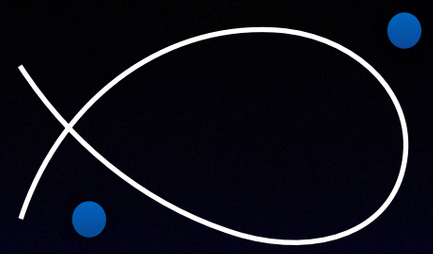

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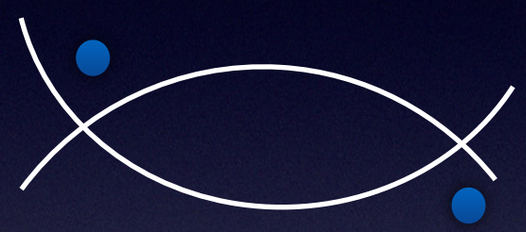

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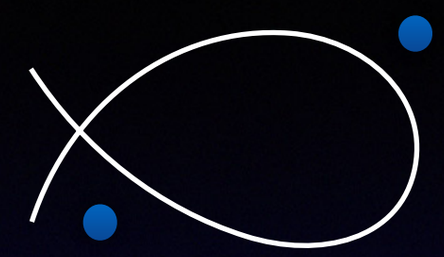

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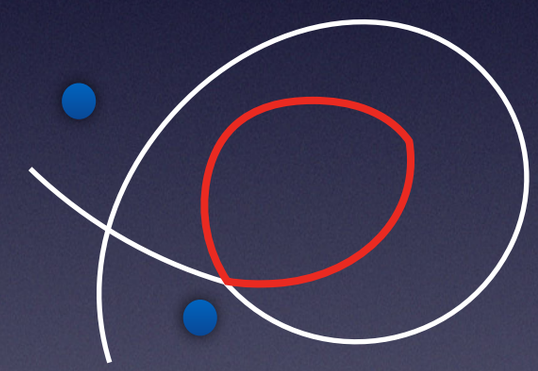
A loop ℓ is innermost if ℓ does not admit any sub loop.



Theorem. A loop ℓ is innermost if and only if it it simple.

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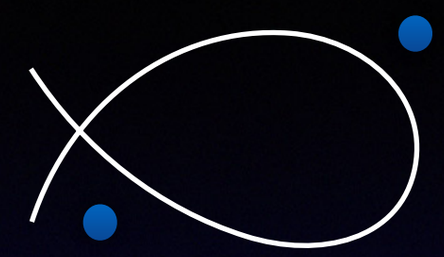

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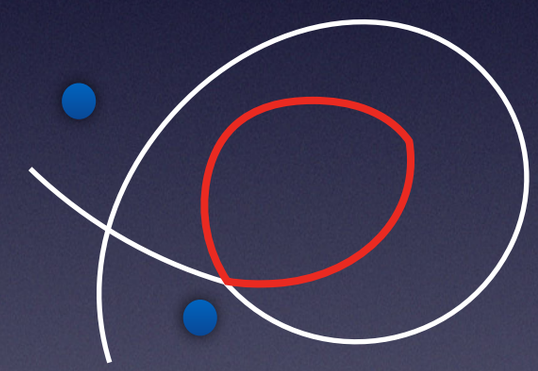

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iii) $Int(\gamma)$ the number of self intersections of γ

iv) $Int([\gamma])$ the minimal number of self intersections of $[\gamma] = min\{Int(\gamma) \ : \ \gamma \in [\gamma]\}$

A loop γ is taut if minimises the number of self intersections.

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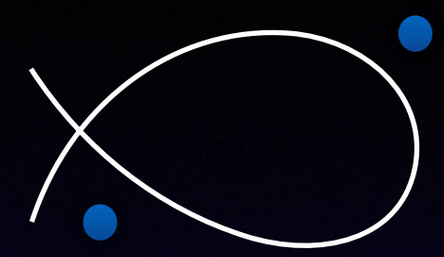

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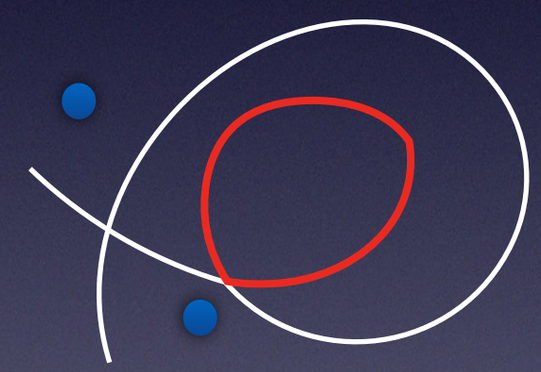

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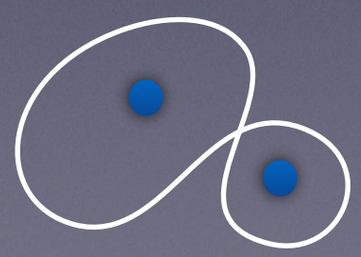
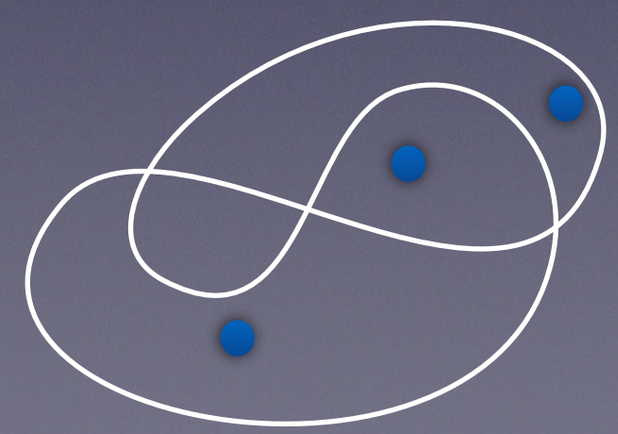


Taut

Non taut



Topological setting: definition Admissible classes

Theorem[HaSc85]
If γ excesses the number of self intersections then γ has a singular $1-gon$ or $2-gon$.

Theorem[HcSc94]
Let γ_1 and γ_2 be two taut representatives of $[\gamma]$. Then there exists an homotopy Φ from γ_1 to γ_2 such that $\Phi_s = \Phi(\cdot, s)$ is self-transverse for all s and the number of self intersections of Φ_s is constant.

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for any innermost loop ℓ of $\hat{\gamma}$ there exist at least two centres c_j, c_k *inside* ℓ. (That is ℓ is not null-homotopic in $\mathbb{R}^2 \setminus c_j$ and in $\mathbb{R}^2 \setminus c_k$).

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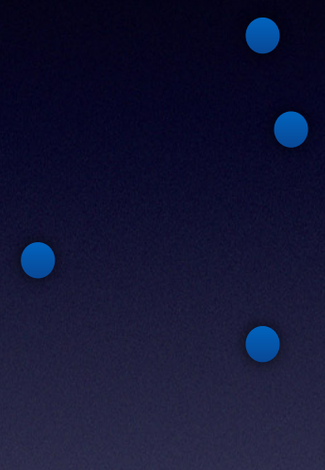
Examples of Admissible Classes



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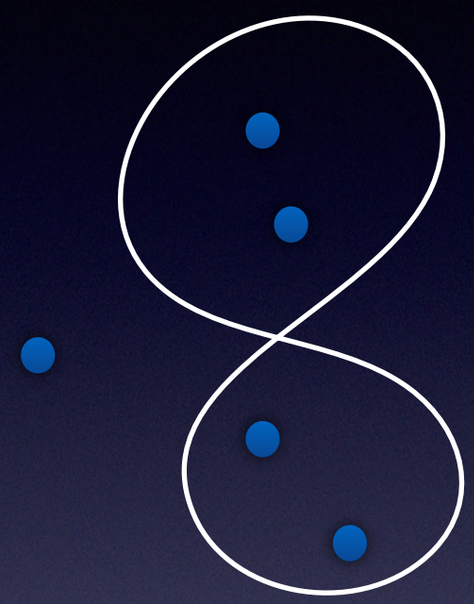

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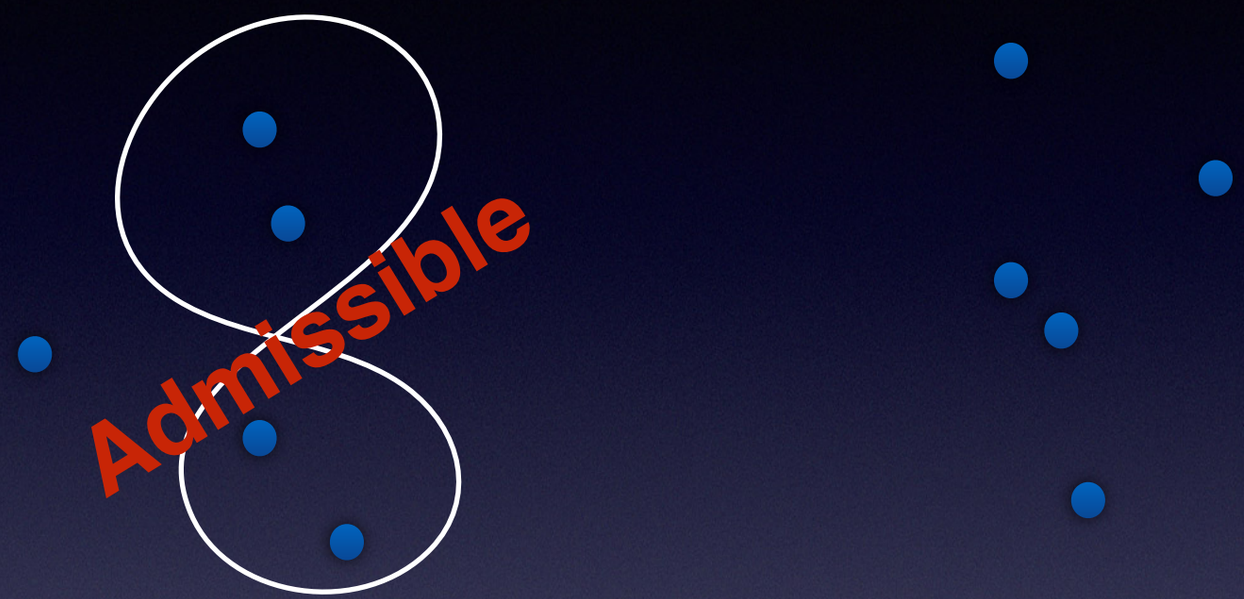

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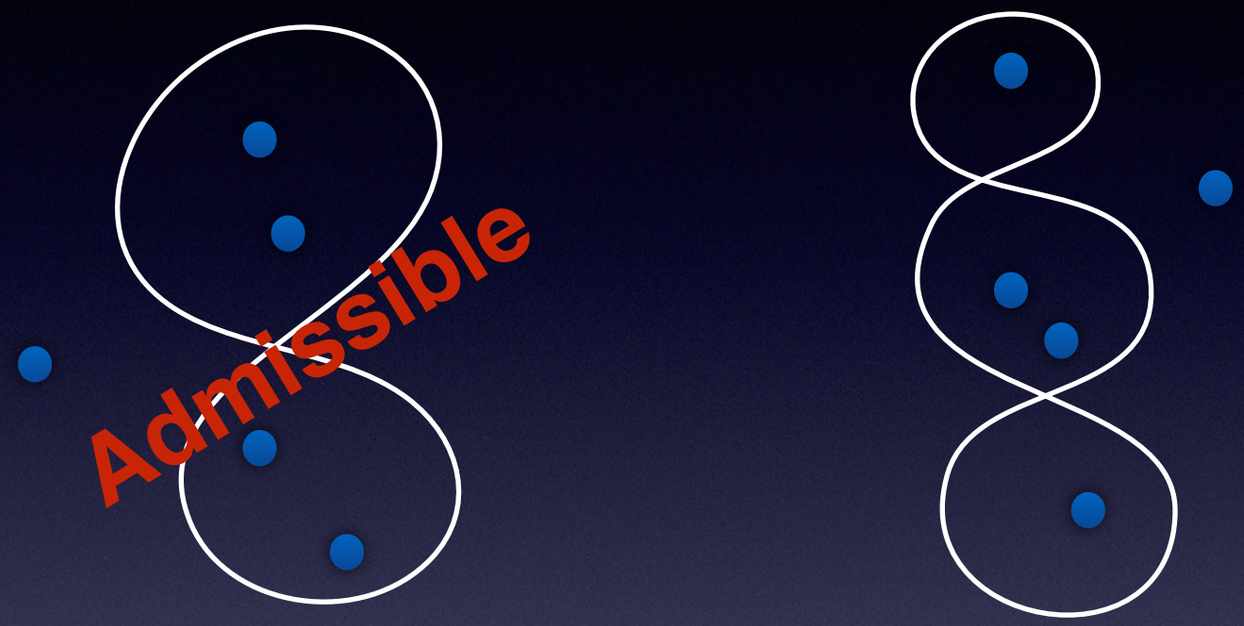

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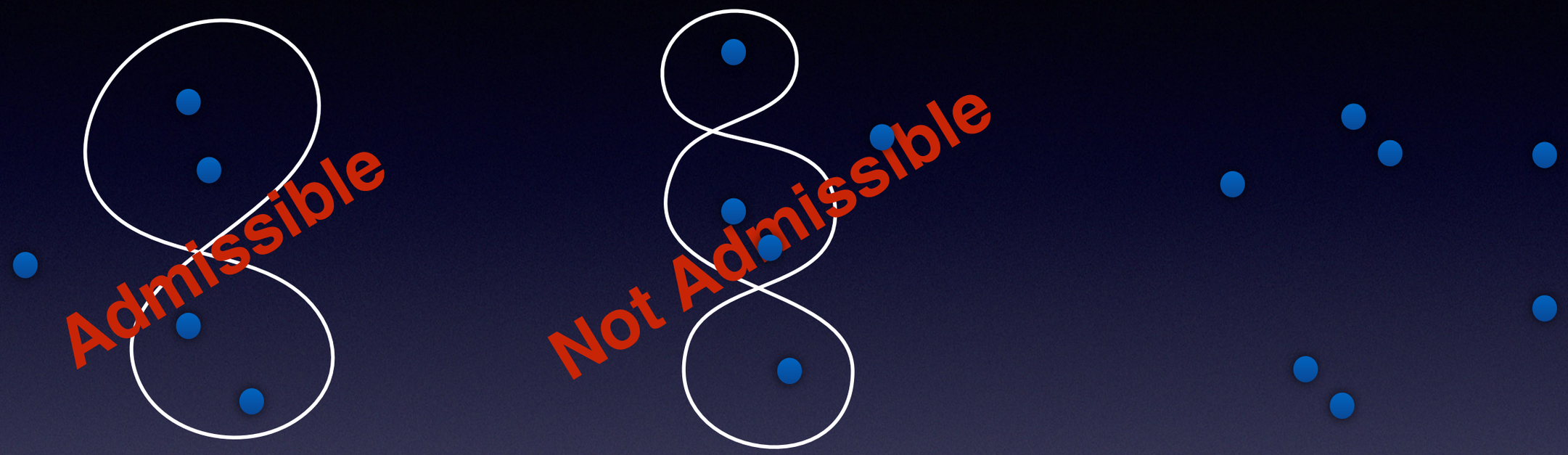

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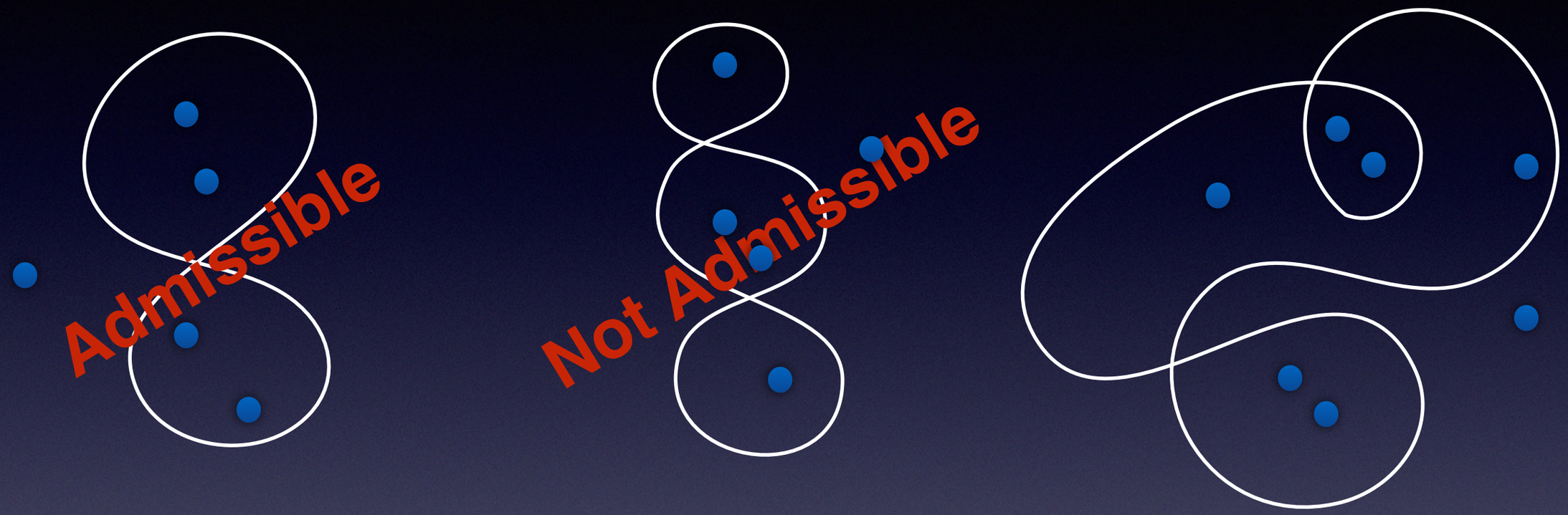

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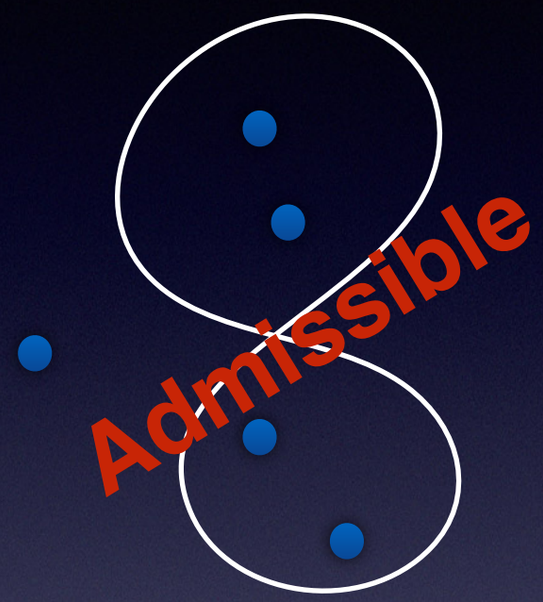

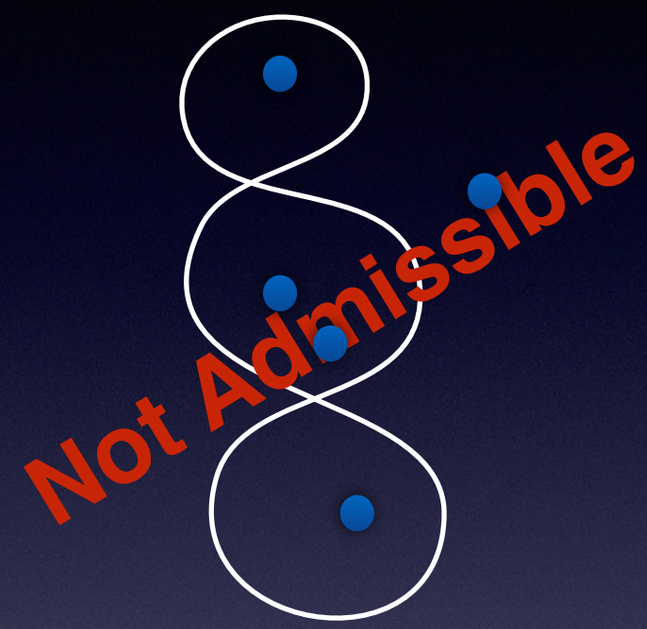

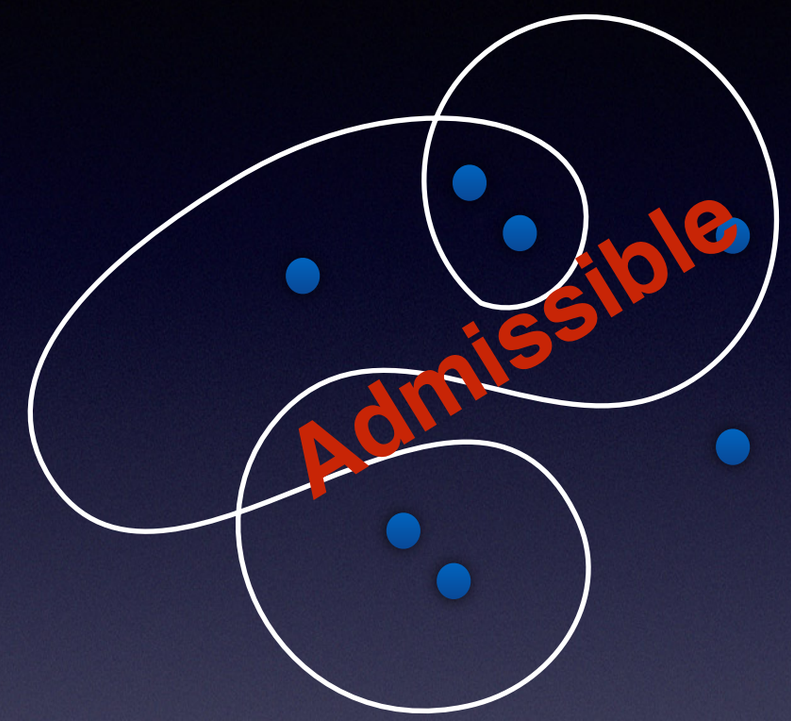

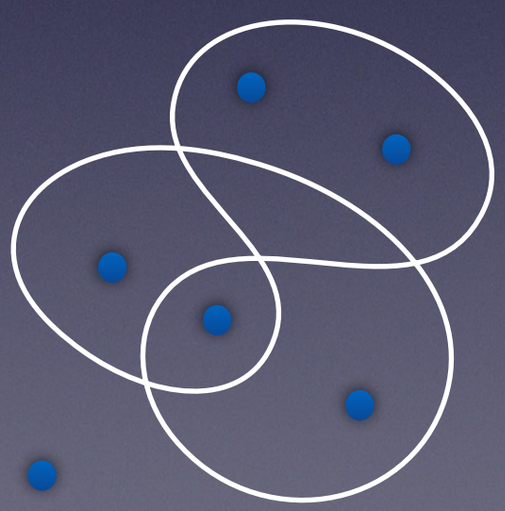

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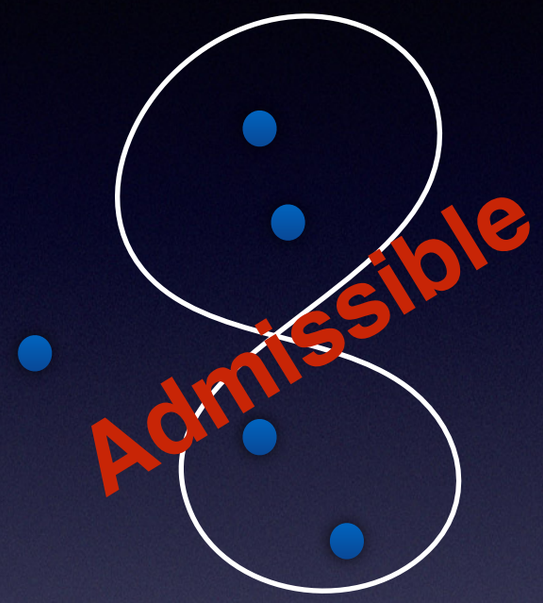

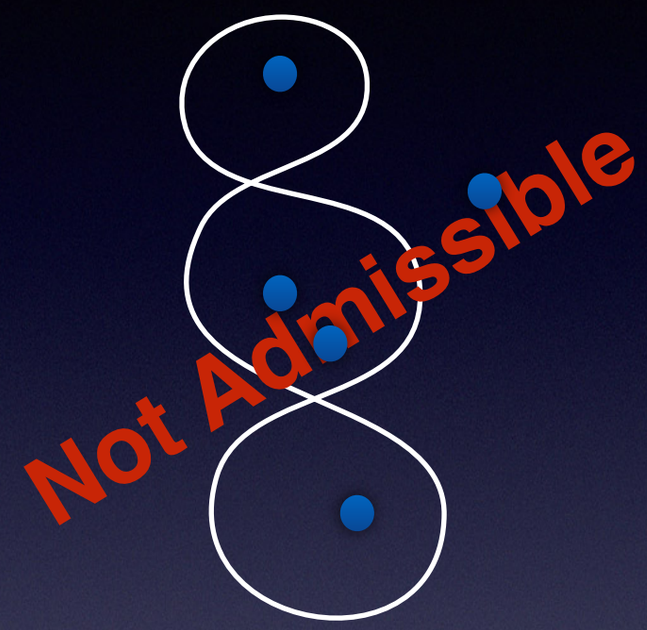

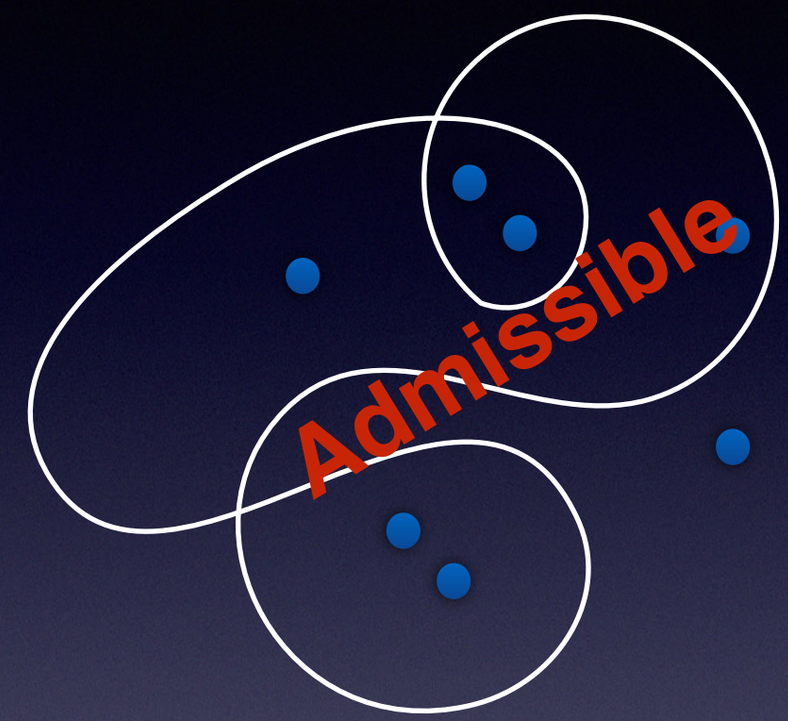

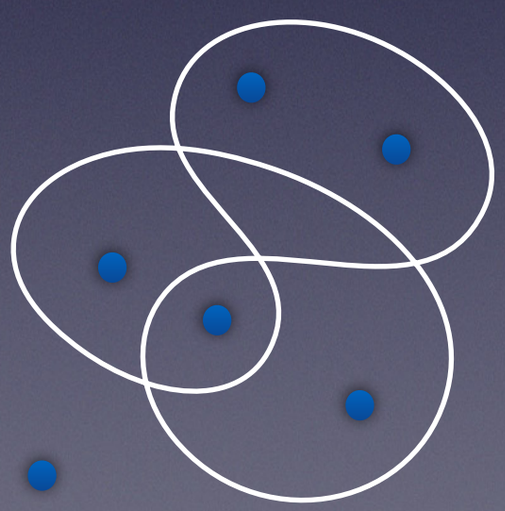

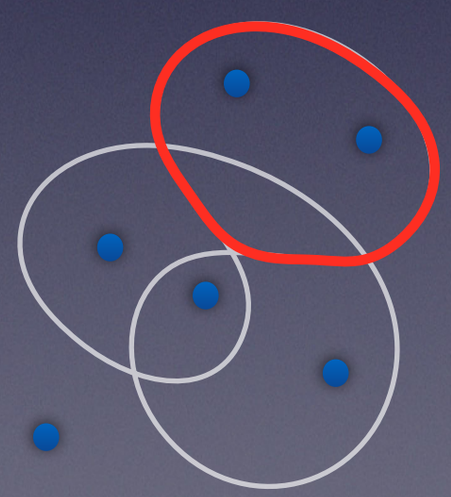

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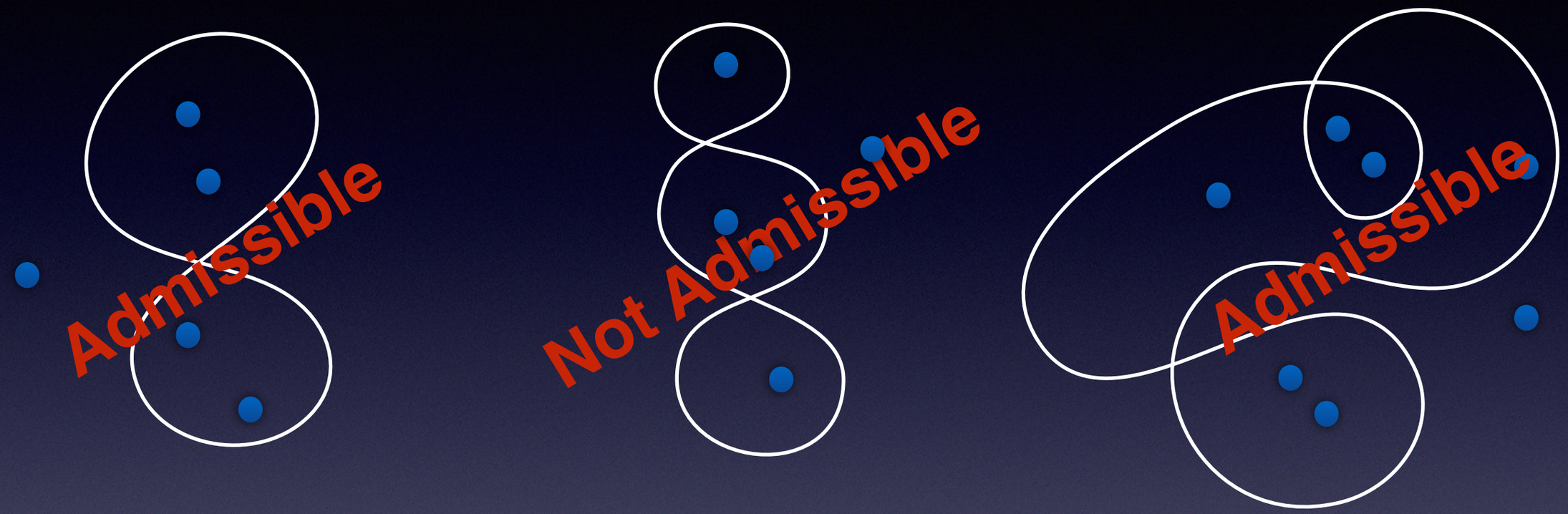

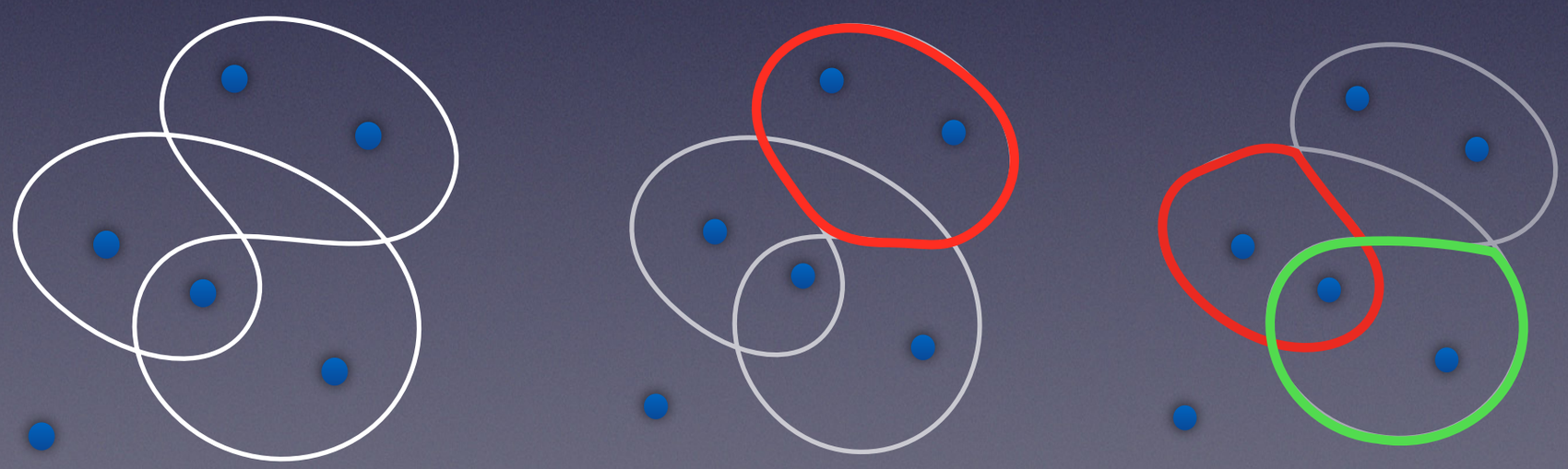

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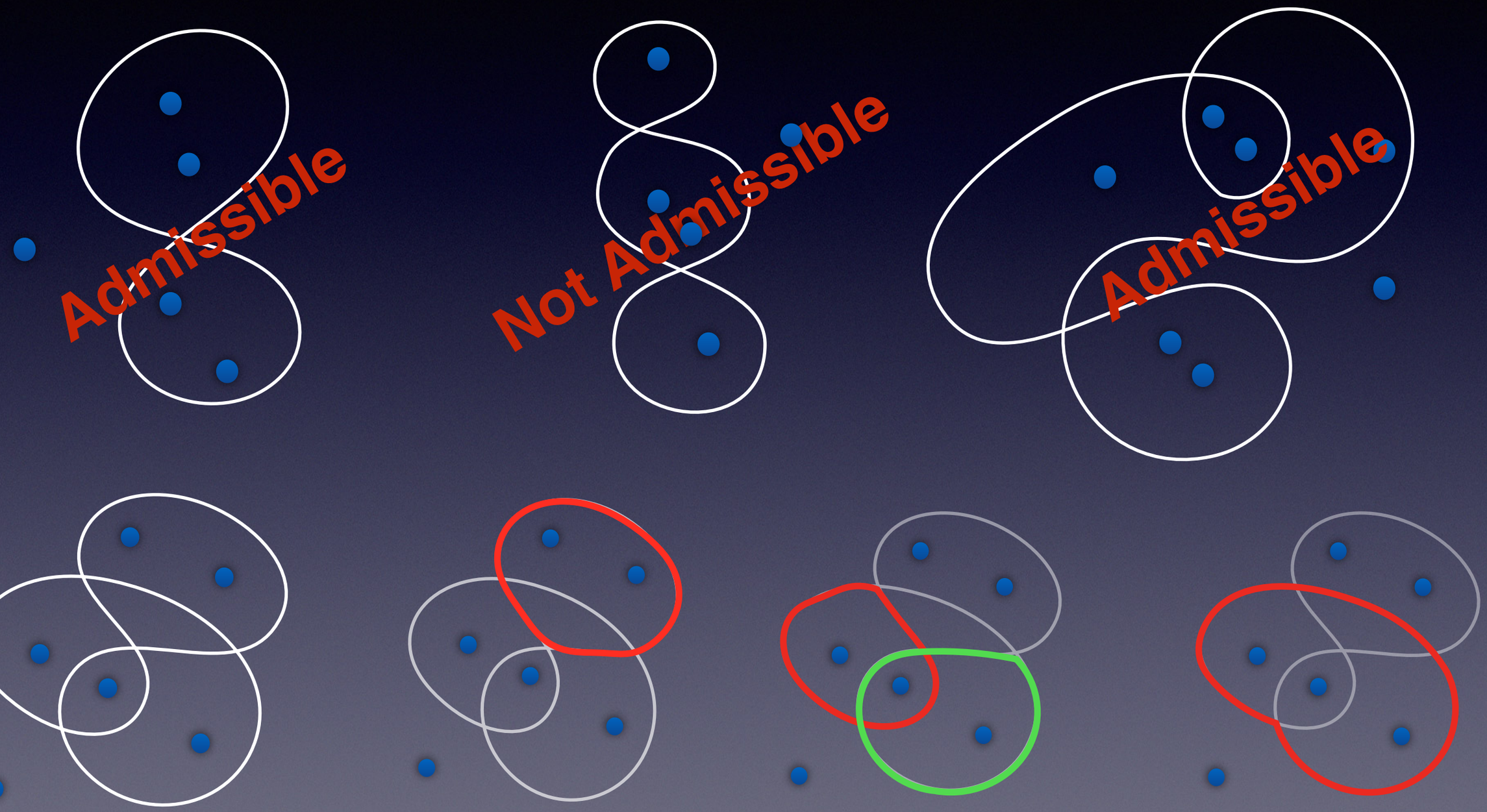

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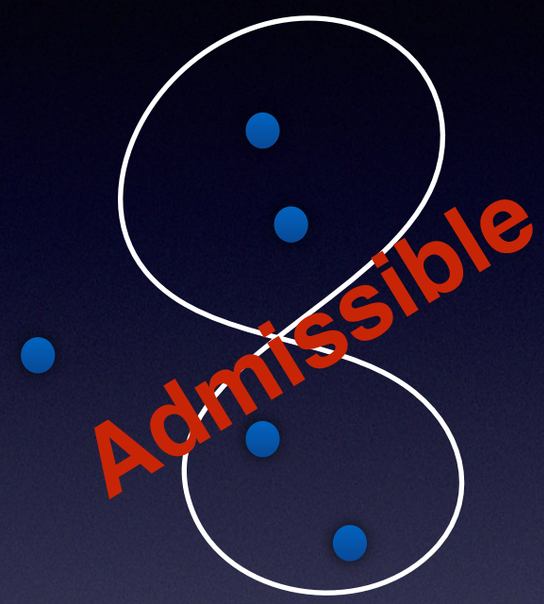

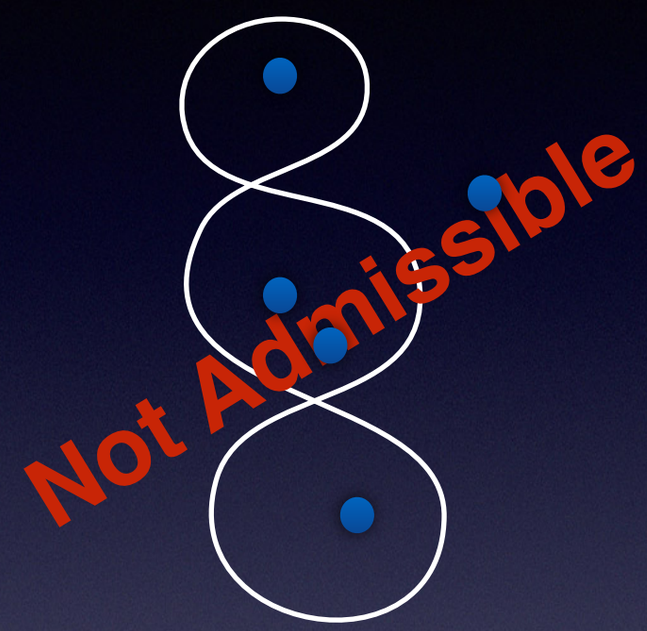

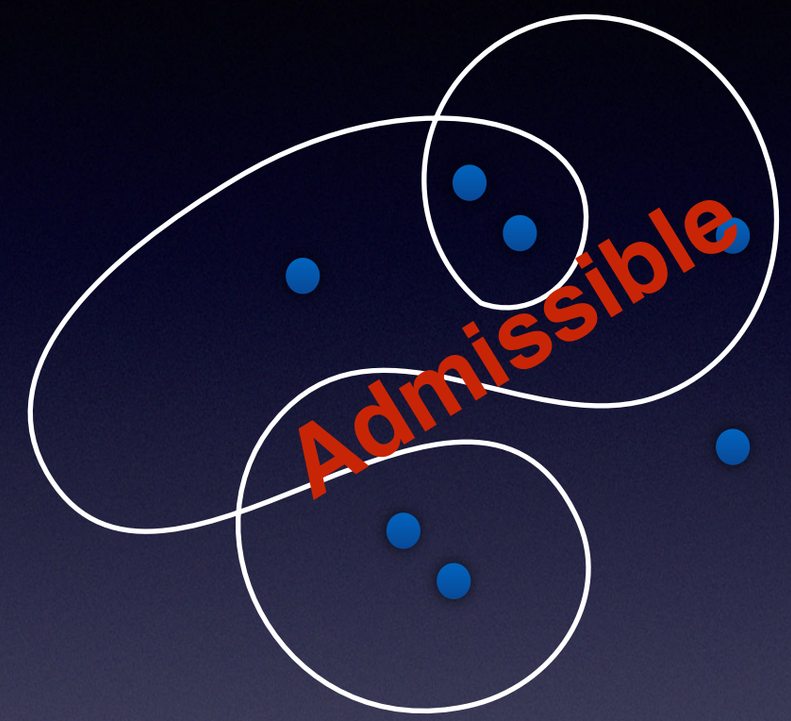

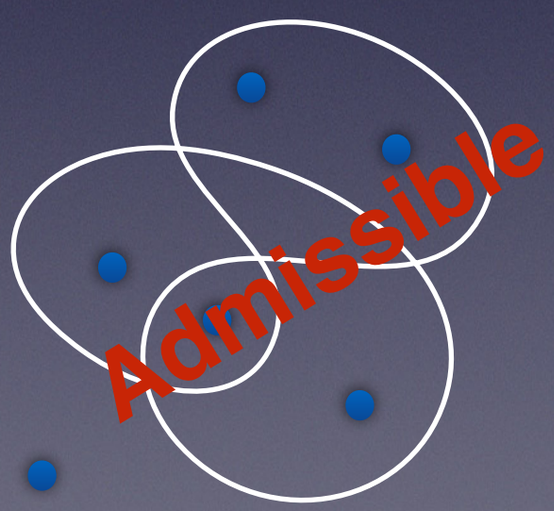

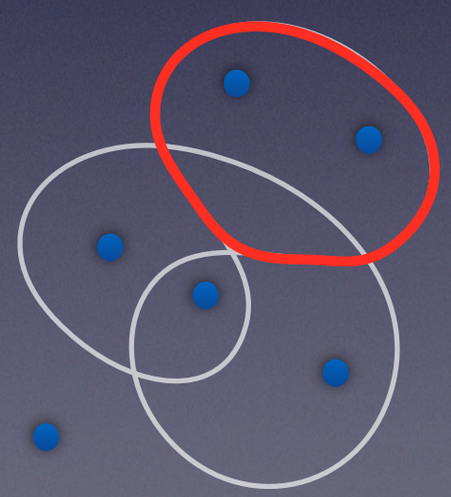

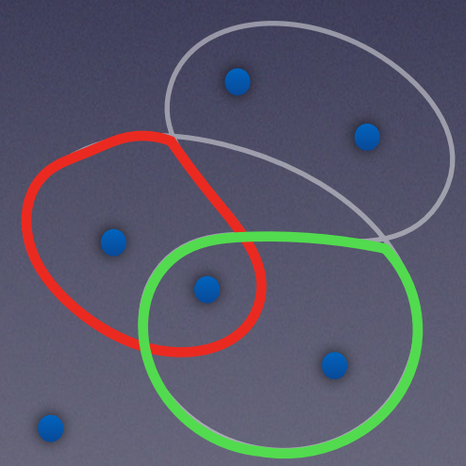

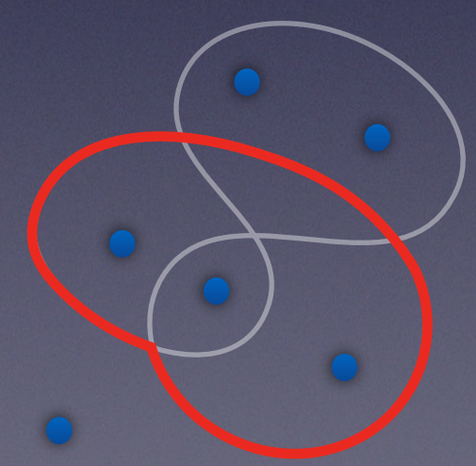

Statement of the Theorem and examples

Statement of the Theorem

Theorem [Kepler case $\alpha = 1$]

Let $\alpha = 1$ and let $C = \{c_1, \ldots, c_N\}$, $c_i \in \mathbb{R}^2$ be any set of centres. Then, for any admissible class $[\gamma]$ and any $h > 0$, one of the following holds:

1. it exists a non-collision periodic solution $x(s)$ for the N-center problem with energy h so that $x(s)$ parameterizes a curve in $[\gamma]$,
2. it exists a collision-reflection solution.

Theorem $[\alpha \in (1, 2)]$

Let $\alpha \in (1, 2)$ and let $C = \{c_1, \ldots, c_N\}$, $c_i \in \mathbb{R}^2$ be any set of centres. Then, for any admissible class $[\gamma]$ and any $h > 0$ it exists a non-collision periodic solution $x(s)$ for the N-center problem with energy h so that $x(s)$ parameterizes a curve in $[\gamma]$.

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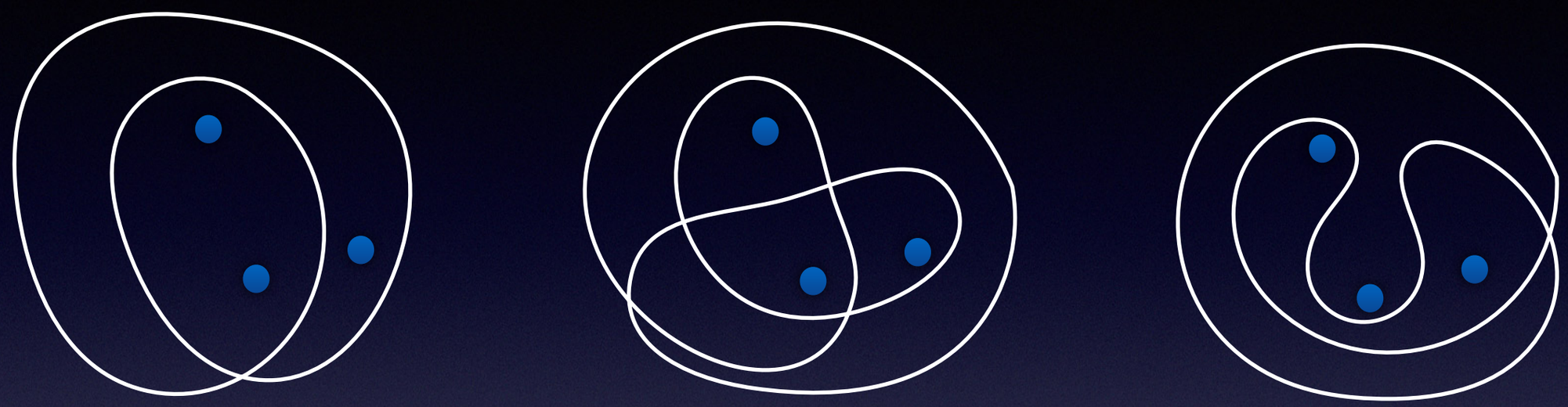
Remark

The alternative 2 in first Thm applies only if the closure of $[\gamma]$ admits the collision reflection path. Otherwise only the first alternative holds.

Statement of the Theorem: examples

$\alpha = 1, N = 3$



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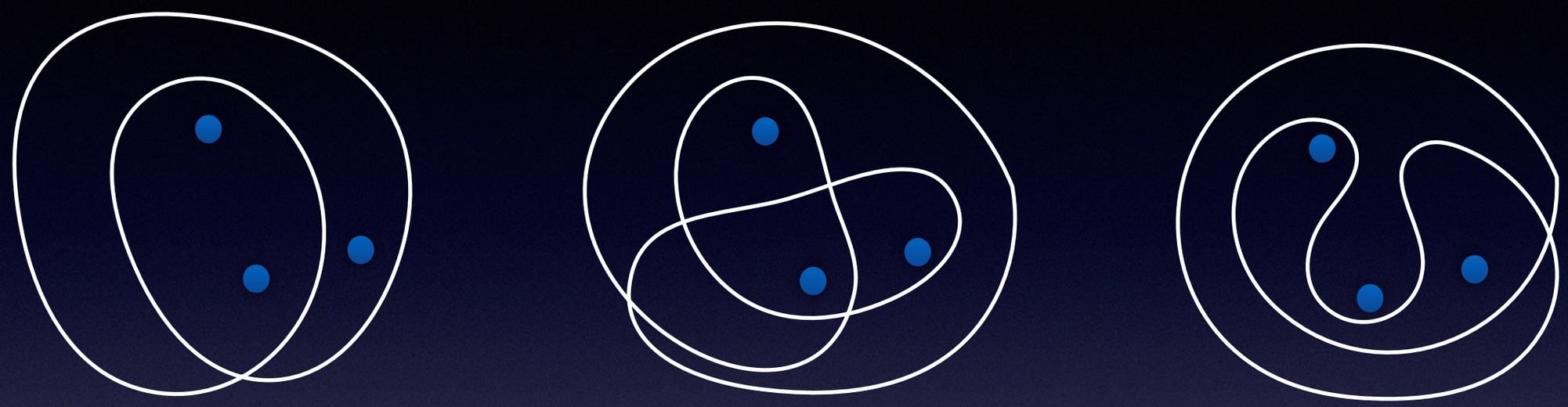
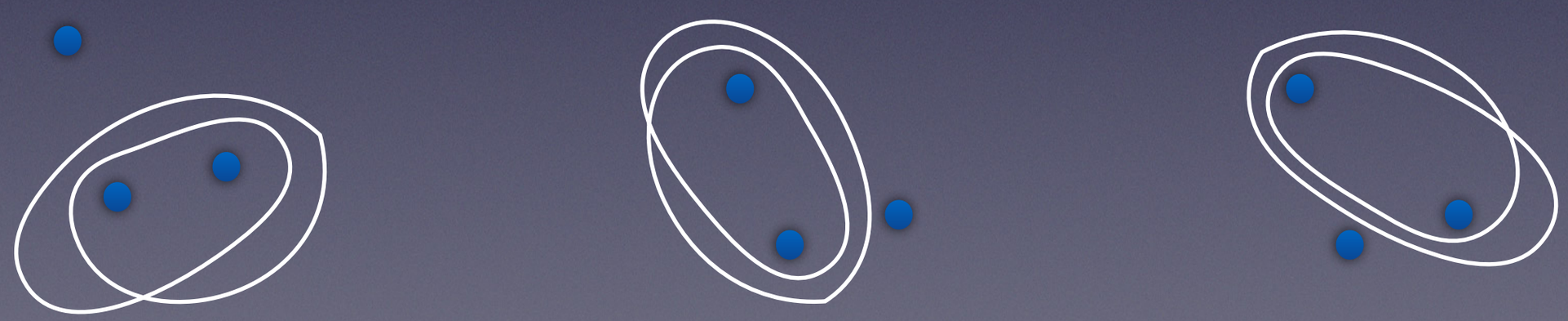


A collision reflection solution may result only if $[\gamma]$ consists of paths that wind around only two centres.



Statement of the Theorem: examples

GOAL: Given N centers, prescribe the shape of a periodic solution

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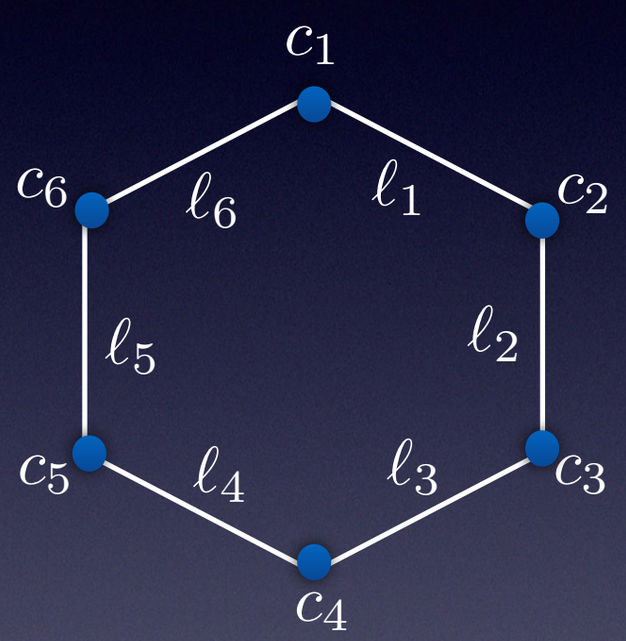
To begin with, suppose the centers are placed on vertices of a regular N-gon



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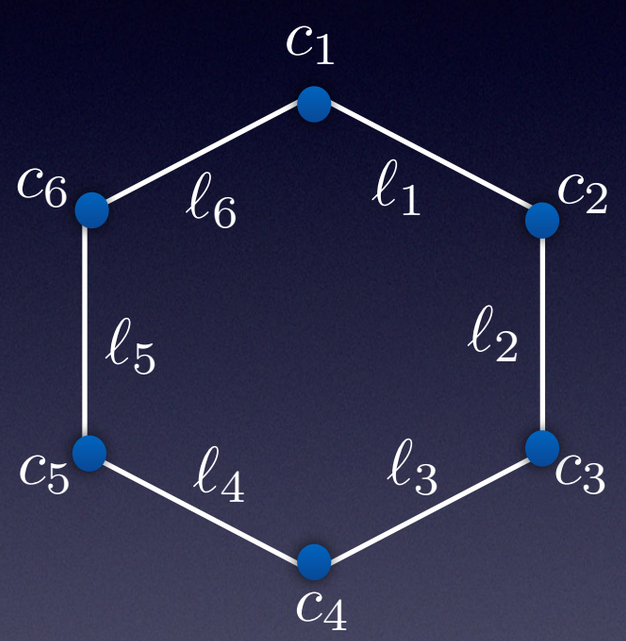

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Define $d(\ell_j, \ell_k) = \min\{(j-k)\ modN, (k-j)\ modN\}$

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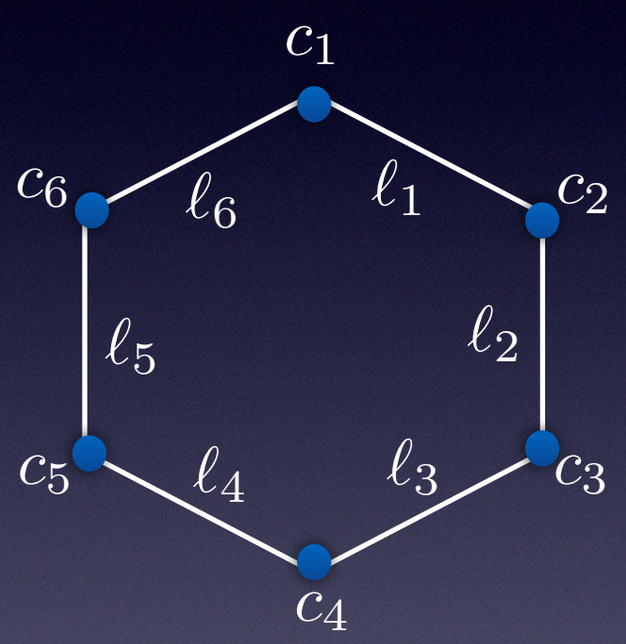

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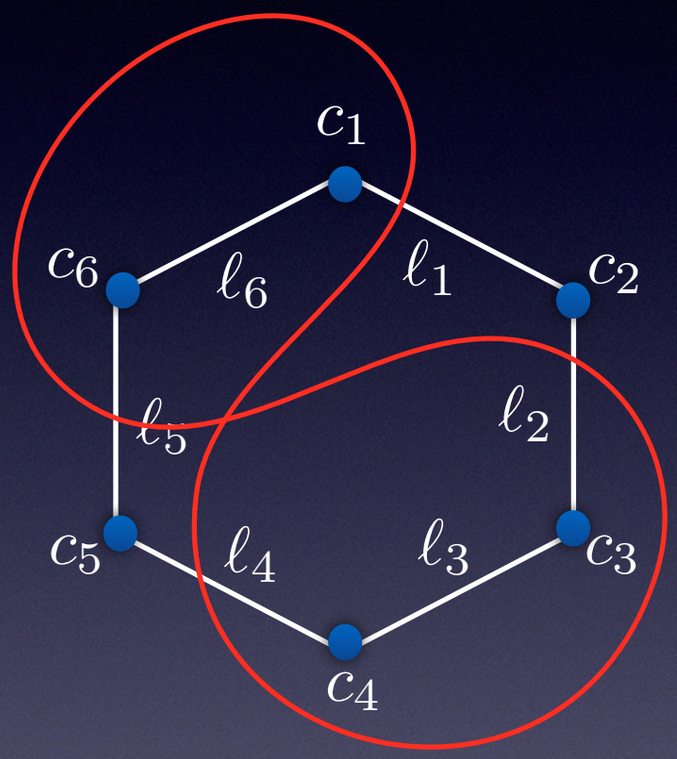

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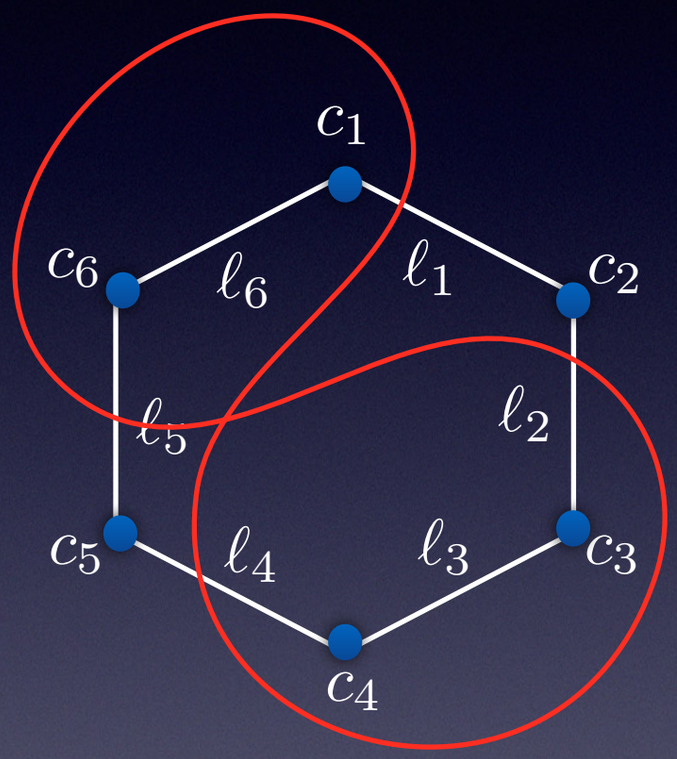

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If $d(w_j, w_{j+1}) \geq 2$ for any j, the class $[\gamma_W]$ is Admissible

Statement of the Theorem: examples

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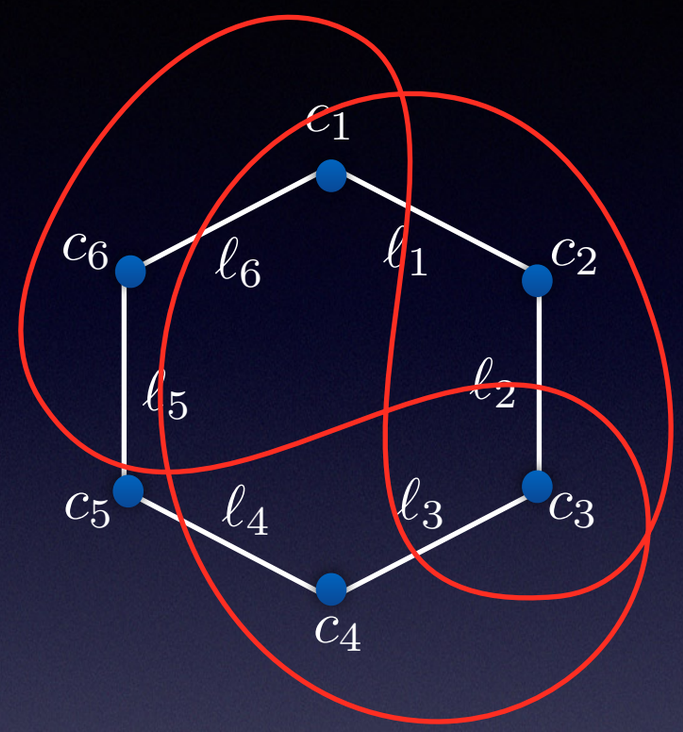

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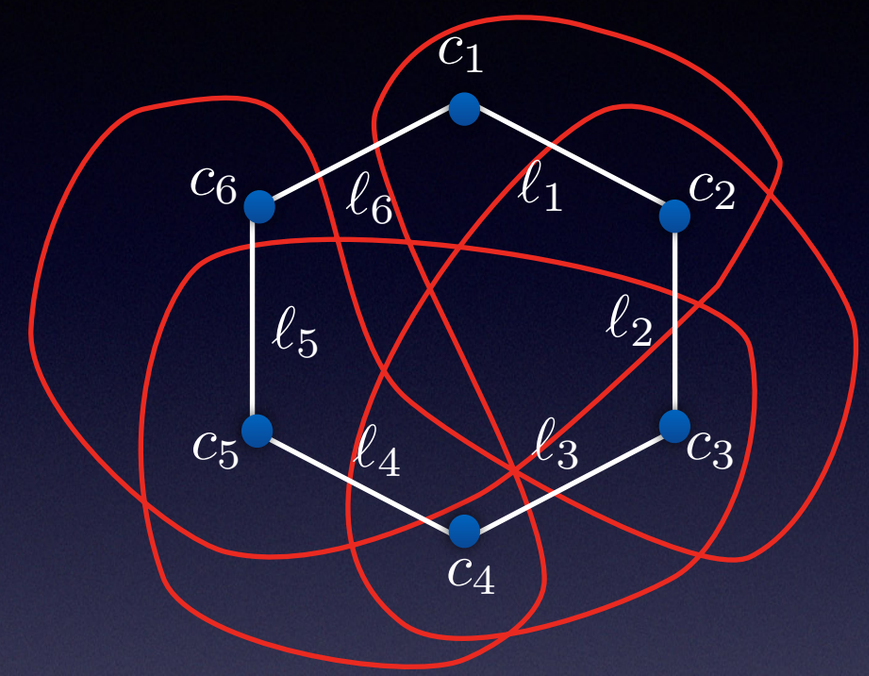

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Statement of the Theorem: examples

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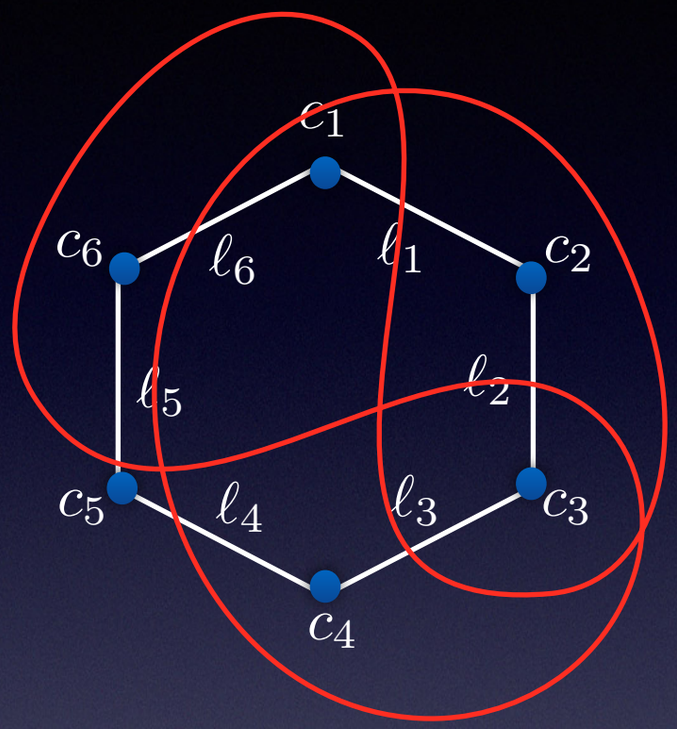

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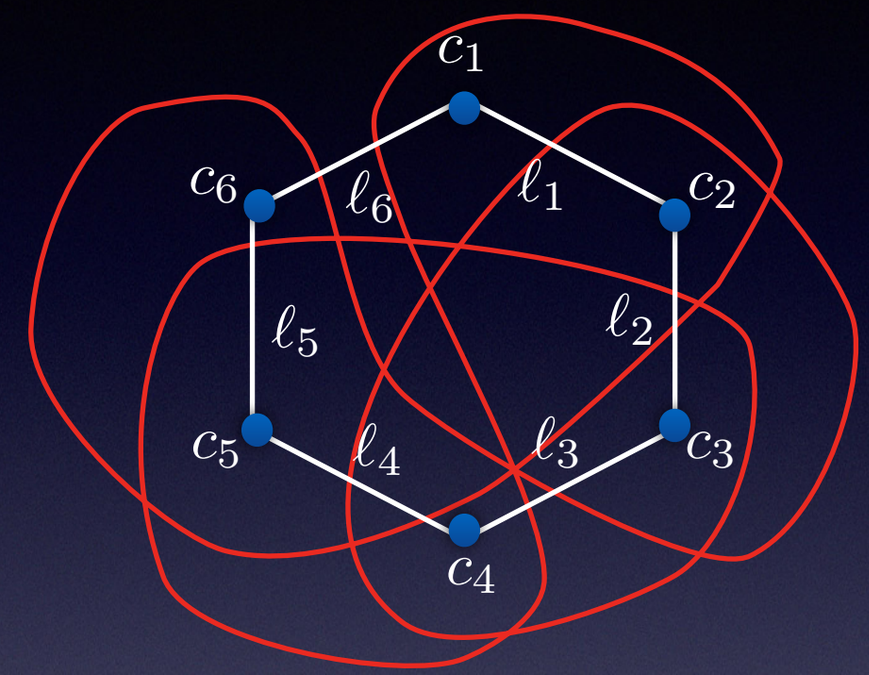

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General position of N centres

Statement of the Theorem: examples

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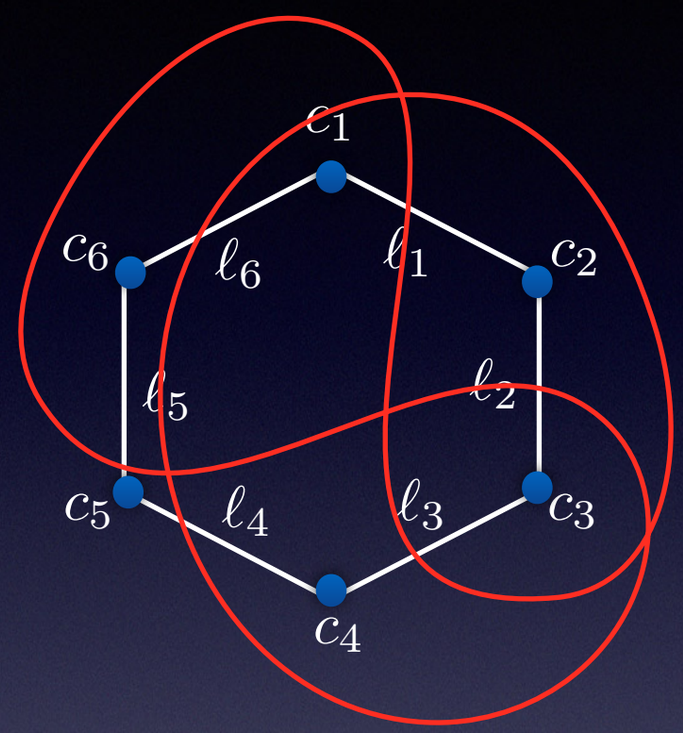

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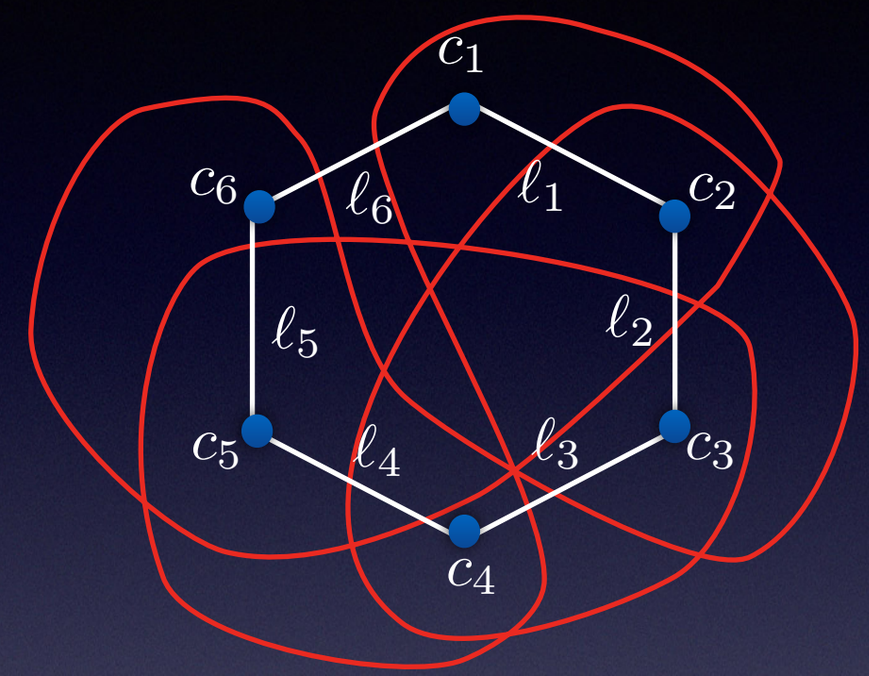

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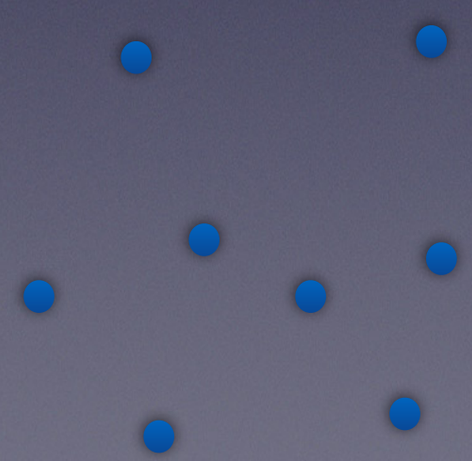

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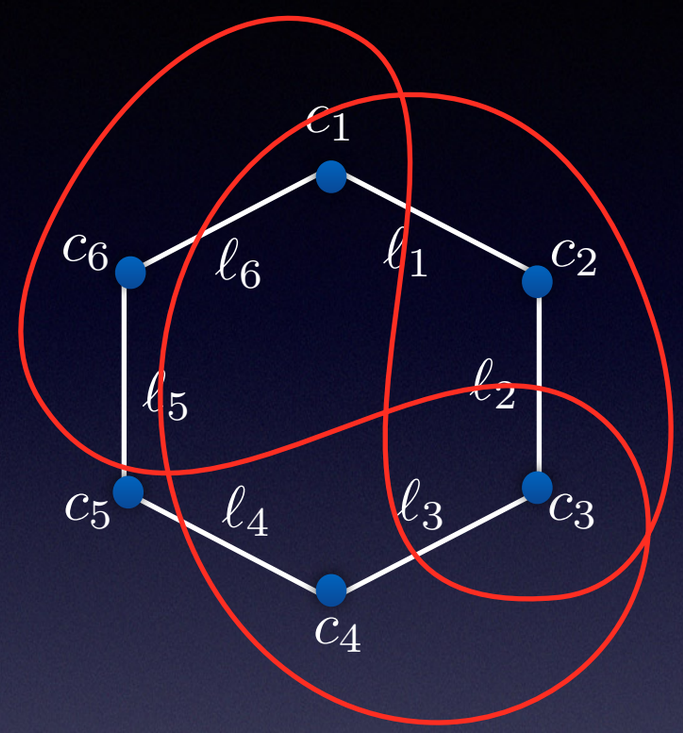

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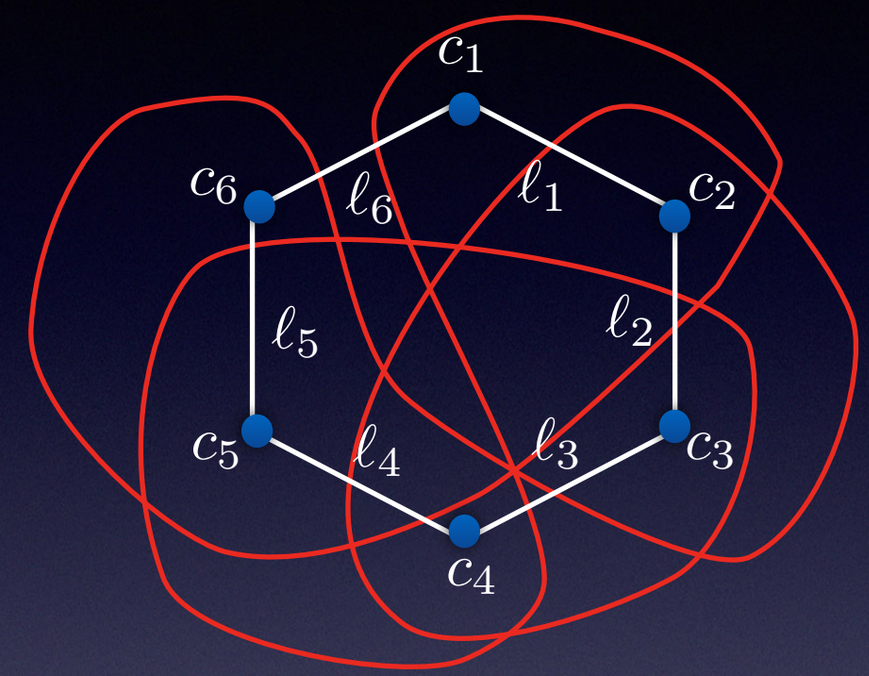

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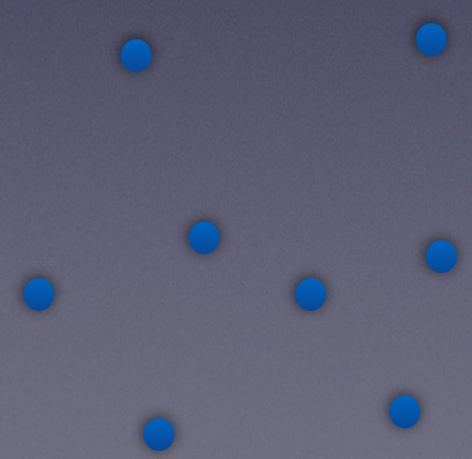
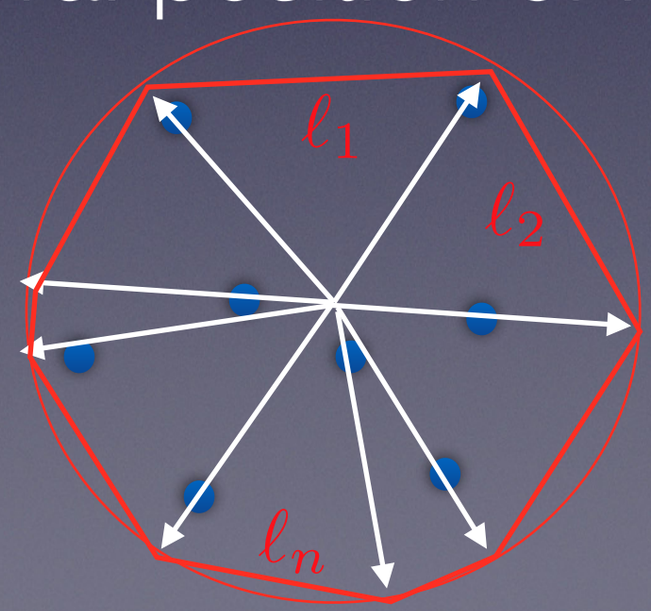
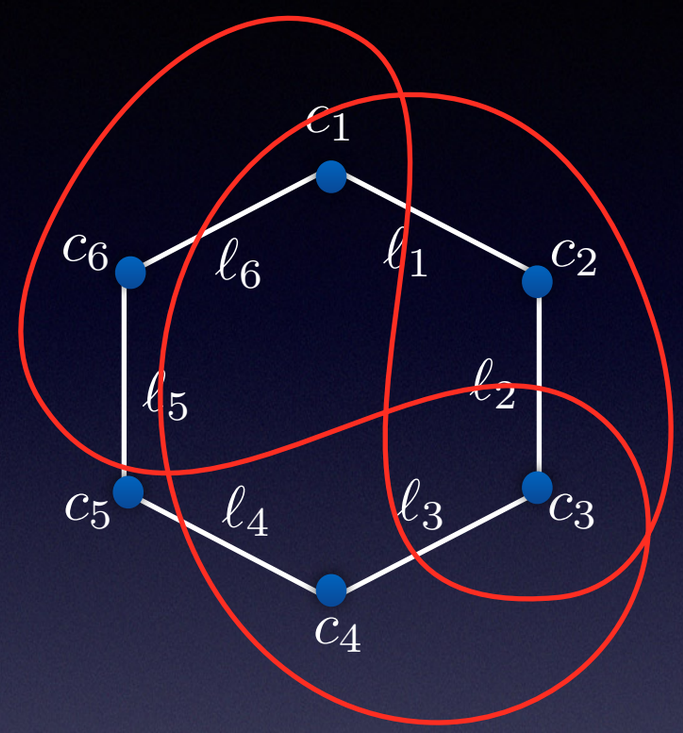
General position of N centres





Statement of the Theorem: examples

GOAL: Given N centers, prescribe the shape of a periodic solution



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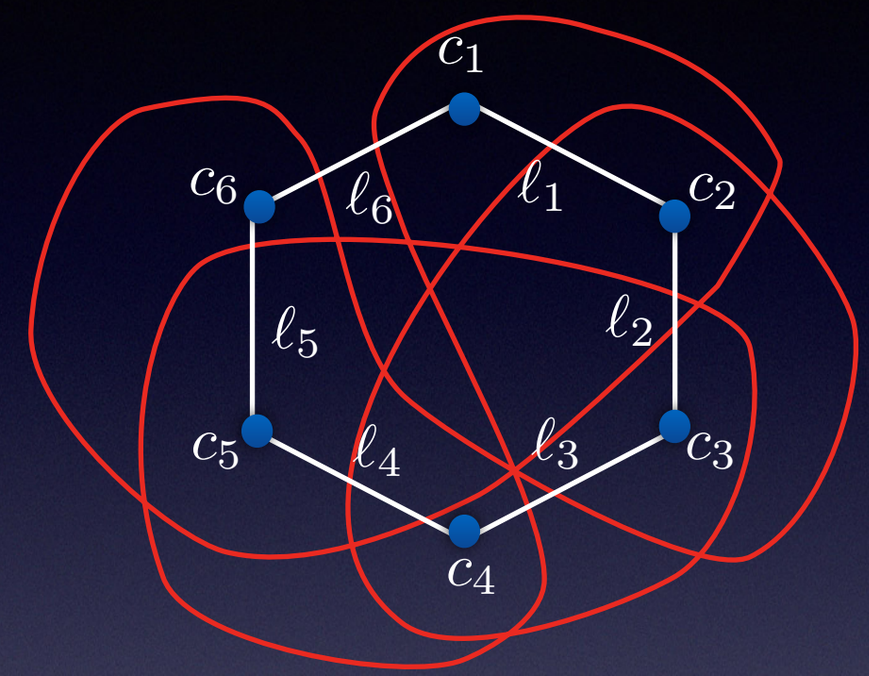

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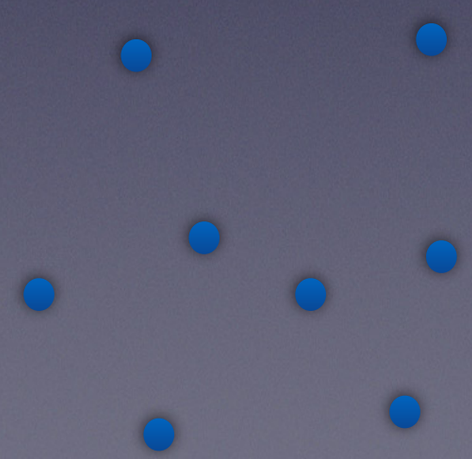

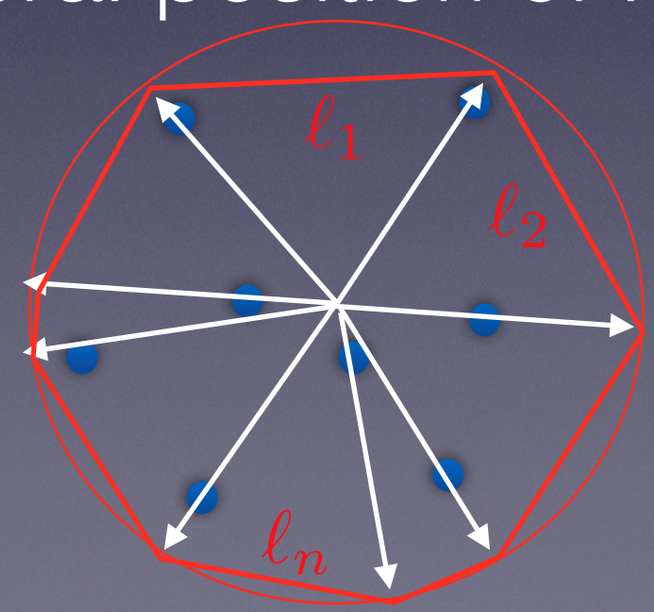

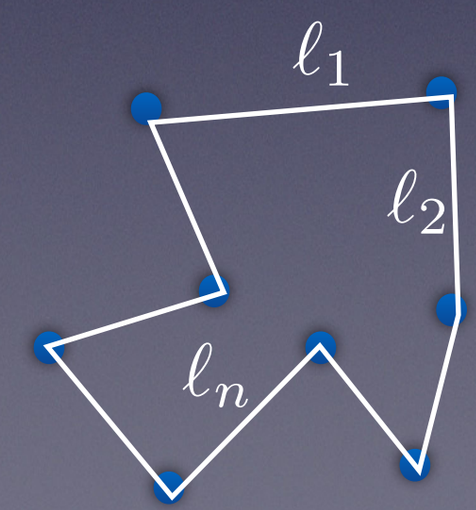

Sketch of the proof

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We look for solution of the N-center problem with Newtonian-like potential

$$V(x) = \sum_j \frac{m_j}{\alpha |x - c_j|^\alpha}, \quad \alpha \in [1, 2), m_j > 0$$

Equation of motion: $\ddot{x} = -\sum_j m_j \frac{x(t) - c_j}{|x(t) - c_j|^{\alpha+2}}$

Energy first integral: $h = \frac{1}{2}|\dot{x}|^2 - V(x)$

Outline of the proof

Variational approach: look for minimiser of Maupertuis functional

Apply the *obstacle problem* technique

Prove that the minimising sequence constructed locally around one center can not have intersections

Sketch of the proof

Variational setting

Functional spaces

$$H := \{u \in H^1(S, \mathbb{R}^2)\}, \quad \hat{H} = \{u \in H : u(t) \in \mathbb{R}^2_C\}$$

$$\hat{H}_\gamma := \hat{H} \cap [\gamma], \quad H_\gamma := \overline{\hat{H}_\gamma} \text{ in the weak } \sigma(H^1, (H^1)^*) \text{ topology}$$

$$\widehat{H}_{p_1,p_2}(a,b) := \{u \in H^1([a,b], \mathbb{R}^2) : u(a) = p_1, u(b) = p_2, u(t) \in \mathbb{R}^2_C \forall t \in (a,b)\}$$

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Maupertuis Functional

$$\mathcal{M} : H \to \mathbb{R} \cup \{\infty\} \qquad \text{given by } \mathcal{M}(u) := \int_S \frac{1}{2}|\dot{u}(t)|^2 dt \int_S \Big(V(u) + h\Big)\, dt$$

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Maupertuis principle

Let $u \in \widehat{H}$ be a critical point of $\mathcal{M}$. Then $x(t) := u(\omega t)$ is a periodic classical solution of $\ddot{x} = \nabla V(x)$ with energy h.

$$\omega^2 = \frac{\int_S (V(u) + h)\, dt}{\int_S \frac{1}{2}|\dot{u}(t)|^2\, dt}.$$

Sketch of the proof

Properties of the Maupertuis functional

- $\mathcal{M}(u)$ and $\mathcal{M}_{p_1,p_2}(u)$ are invariant under time rescale of u.

Sketch of the proof

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  Let $u \in H$ be a non constant minimizer for $\mathcal{M}$. Let $[c,d] \subset S$ and set $p_1 = u(c)$, $p_2 = u(d)$. Then $u|_{[c,d]}$ is a minimizer for $\mathcal{M}_{p_1,p_2}$ in $H_{p_1,p_2}(c,d)$. Moreover suppose v is any minimizer for $\mathcal{M}_{p_1,p_2}$ on $H_{p_1,p_2}(c,d)$. Then the path $w = v\#u|_{S\setminus[c,d]}$ is a minimizer of $\mathcal{M}$ on H.

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implies coercivity

By standard arguments of calculus of variations, $\mathcal{M}$ achieves the minimum. Define

$$\bar{u} : \mathcal{M}(\bar{u}) = \min_{u \in H_{[\gamma]}} \mathcal{M}(u)$$

In principle, $\bar{u}$ may have collisions.

Sketch of the proof

Some properties of the minimiser

1. The energy is conserved through the collisions

Sketch of the proof

Some properties of the minimiser

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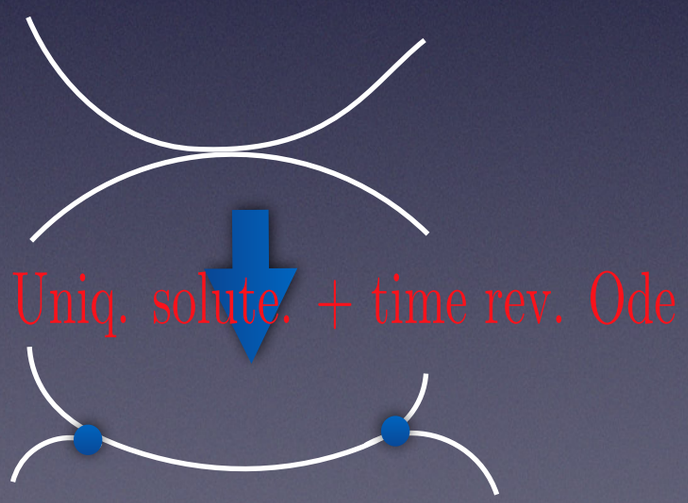
2. If a self tangency occurs, then it extends up to the previous and next collisions



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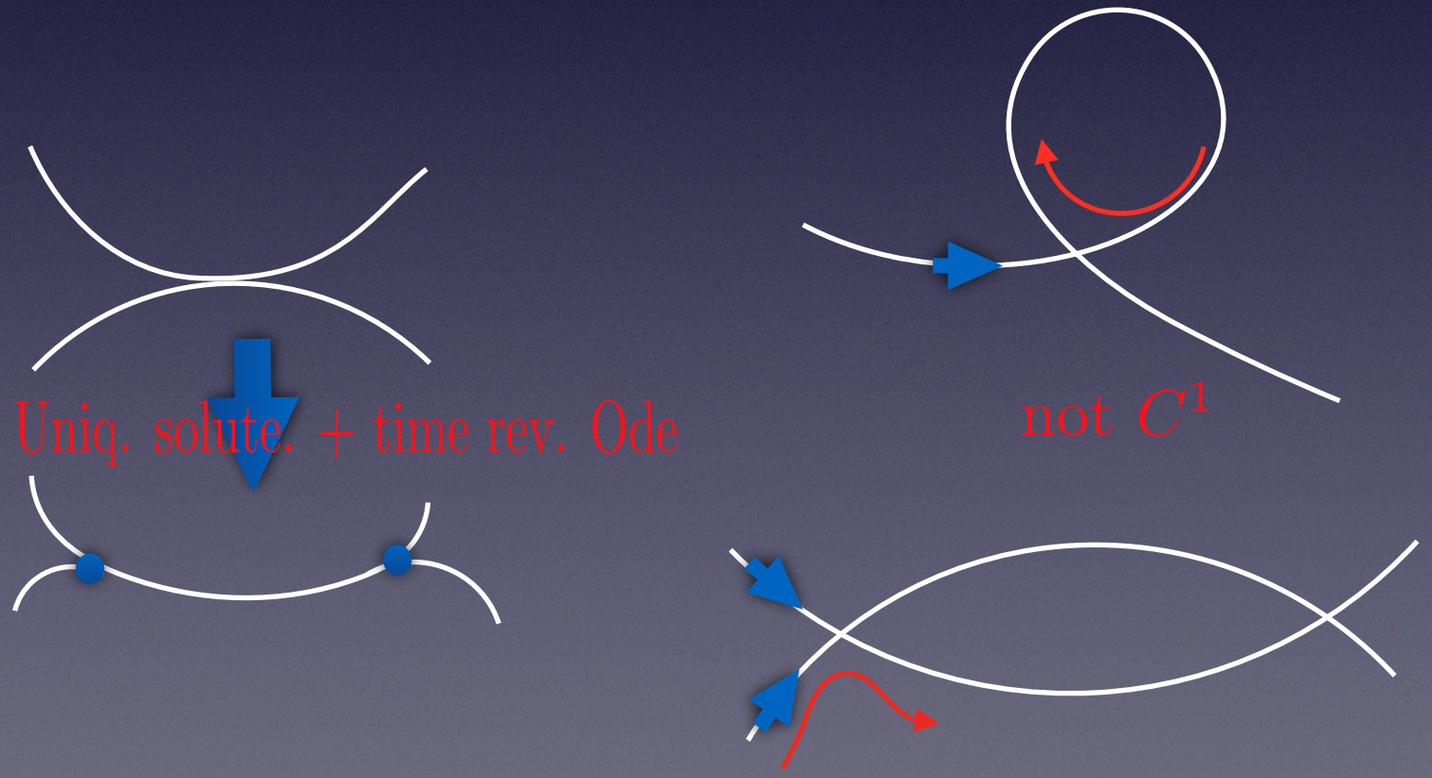

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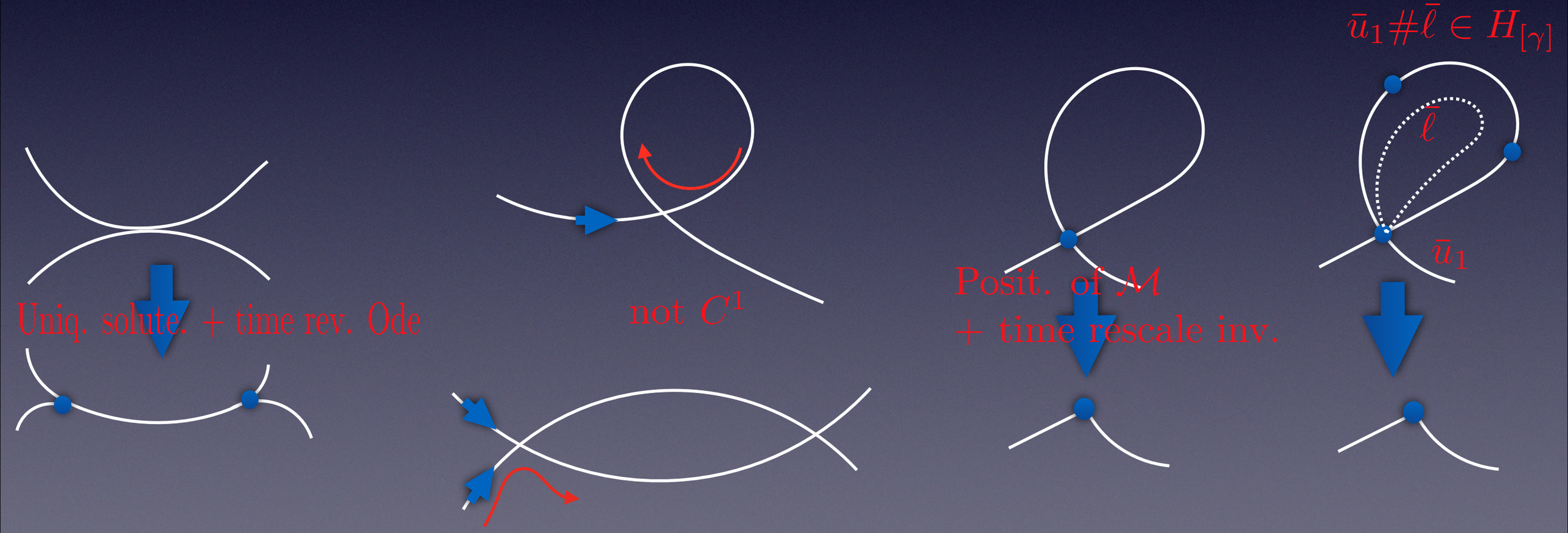
4. The minimiser does not admits any 1-gon as depicted below

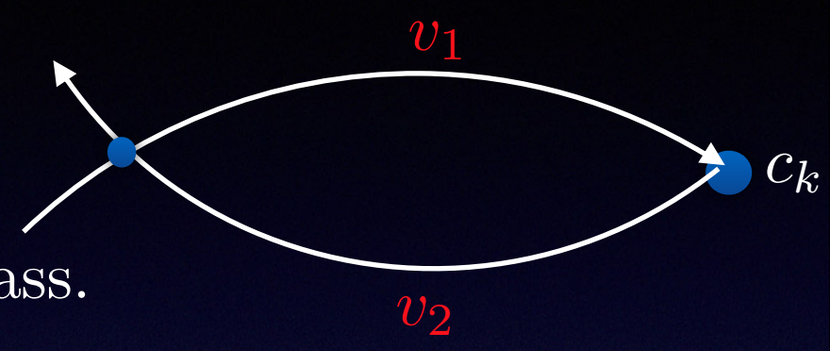


Sketch of the proof

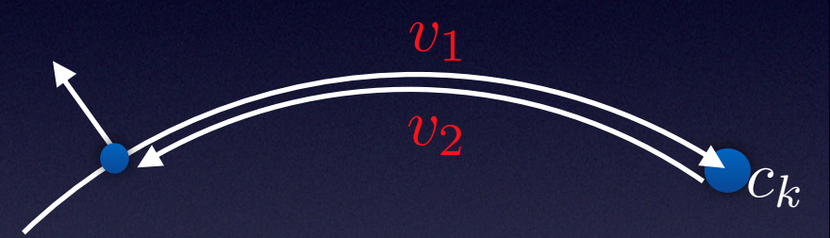
Suitable selection of the minimiser

Suppose the following configuration holds

a)



$v_1 \sim_b -v_2$ and v_1, v_2 minimises $\mathcal{M}_{c_k,c_j}$ in they homotopy class.
Then $\mathcal{M}_{c_k,c_j}(v_1) = \mathcal{M}_{c_k,c_j}(v_2)$.
Then $\bar{u}_2$ with the reflection at c_k is a minimiser for $\mathcal{M}_h$.



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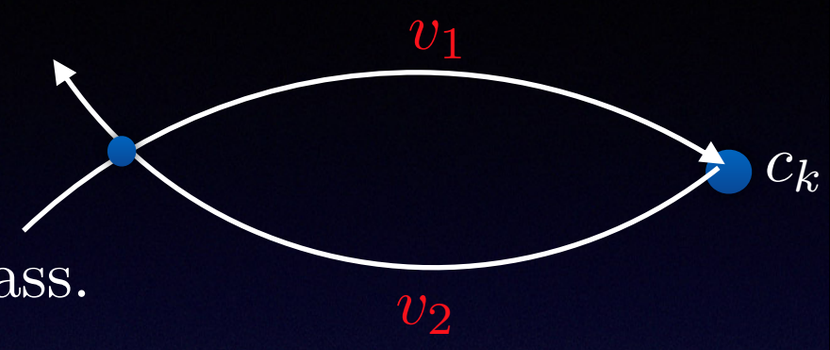

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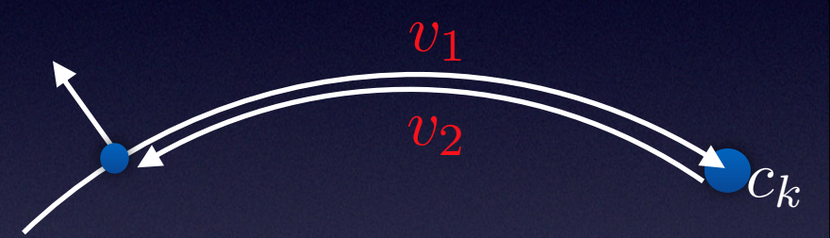

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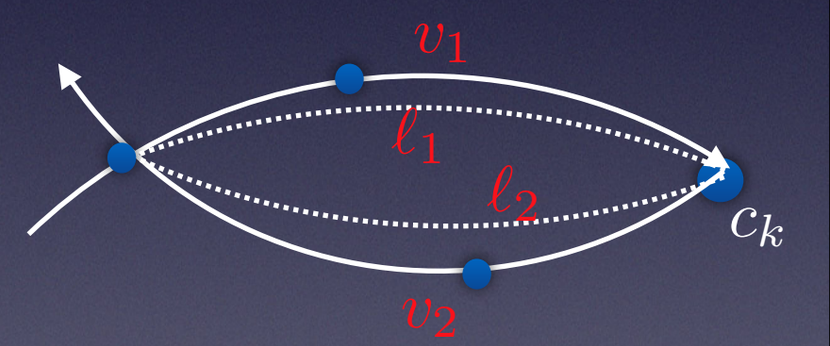

Similarly, in the situation when v_1 and/or v_2 intersect other centres and there exist ℓ_1, ℓ_2 so that $u_1 \# \ell_1 \# \ell_2 \in H_{[\gamma]}$, then v_2 may be replaced by $-v_1$.

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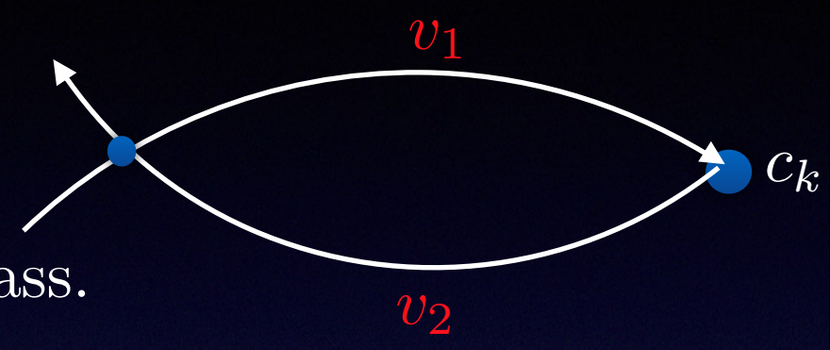

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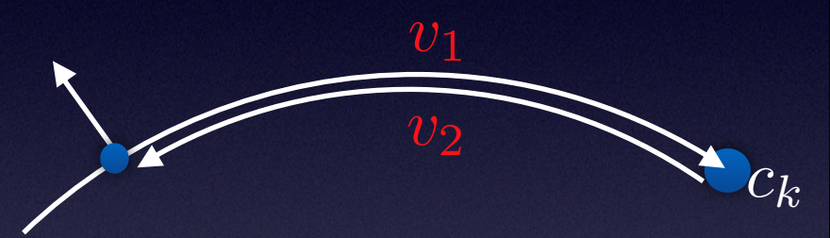

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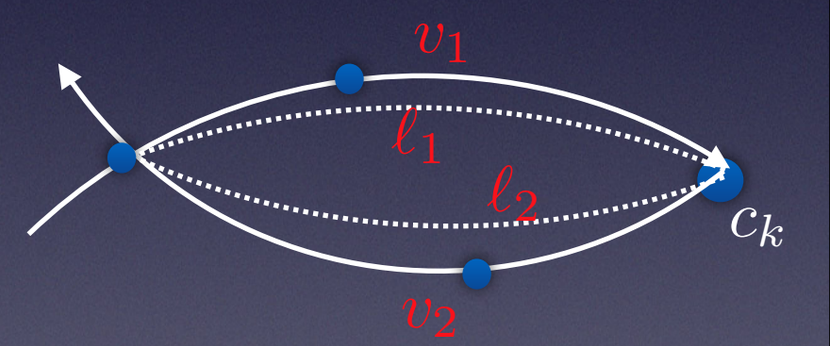

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We select the minimiser $\bar{u}$ si to have a collision-reflection behaviour in situations (a) and (b).

Sketch of the proof

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If $\bar{u}$ does not have collisions with C we are done.

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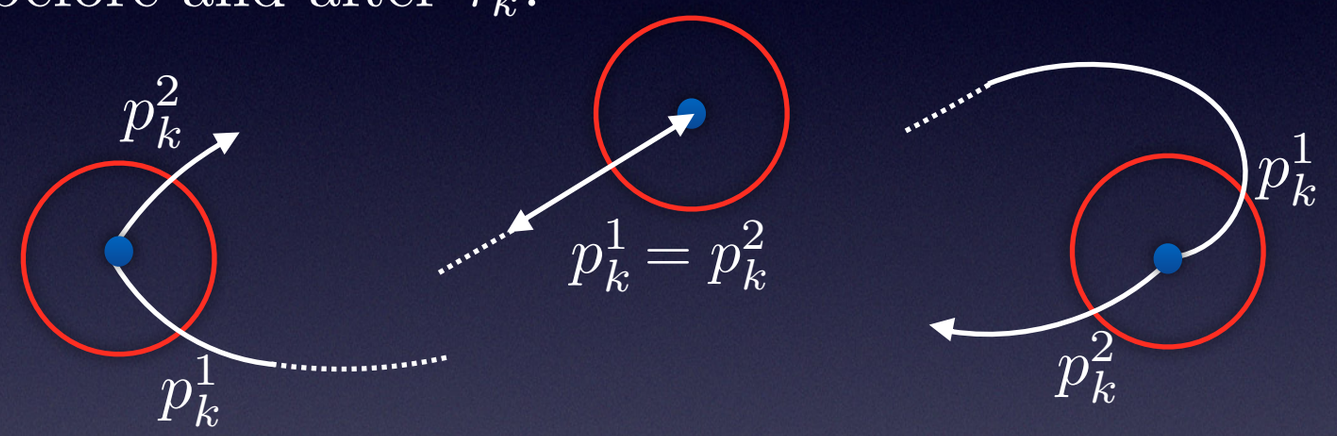

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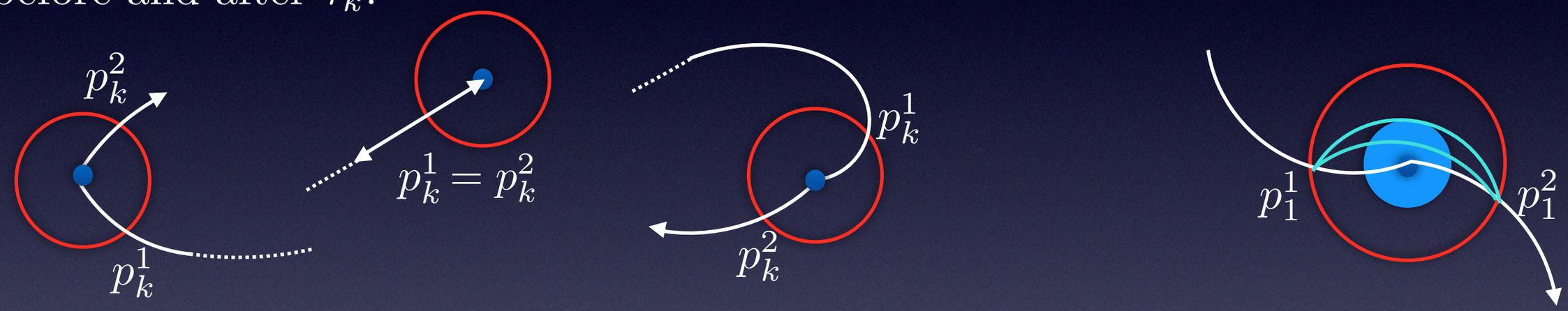

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Develop the obstacle problem technique around τ_1.

Sketch of the proof

CLAIM: The minimising sequence v_n constructed inside B according to the obstacle problem technique is free of intersections.

The proposition results from

The fact that the minimiser is in the closure of an admissible class

The minimiser has been suitable selected

The minimiser does not admits singular 1-gon and 2-gon

Then v_n converges to a collision reflection path.
That implies $p_1^1 = p_1^2$. Contradiction

Thank you for your attention

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