Supporting Information

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SI Text

SI Materials and Methods

Derivation of a General Equation Describing Density-Dependent Movement. We now derive a general equation describing the rate of change of local density M of a population of organisms, in our case mussels. In principle, we consider a general *n*-dimensional case; we will later restrict ourselves to a 2D space, i.e., where n = 2.

We study a population of organisms that perform a random walk with an individual movement speed that is a function of the local density M, denoted by V(M). The organisms change direction with a density-dependent turning rate $\tau(M)$. Because the speed of the particles depends on M, the flux J_{ν} is given by the following:

$$J_{\nu} = -\frac{V(M)}{n\tau(M)} \nabla(V(M)M), \qquad [S1]$$

as derived by Schnitzer (1) (equation 4.14 of ref. 1). This relation is valid under the assumptions that $V|\nabla \tau| \ll \tau^2$, which states that the fractional change of the turning rate over the typical distance traveled between turning must be small (1). More complicated forms of the population flux depends not only on close-neighborhood densities but also on spatial third-order derivatives of population densities (arising from the nonlocal interactions). We incorporate the effect of long-range movement in the model which results in a second contribution to the flux (2, 3) (see ref. 2, pp 408–416 for details),

$$J_{nl} = \kappa \nabla(\Delta M), \qquad [S2]$$

for some constant $\kappa > 0$ (here $\Delta = \nabla^2$). See refs. 2 and 4 for a similar approach.

We study the population on relatively short timescales of maximally 1 d, at which birth and mortality processes play a relatively minor role. For this reason, we do not consider demographic processes in our model analysis. Combining the above assumptions, changes in the local density of organisms can be described by the following:

$$\frac{\partial M}{\partial t} = -\nabla (J_v + J_{nl}),$$

in which M is—by construction—a conserved quantity. Combining Eqs. **S1** and **S2** leads to the following:

$$\frac{\partial M}{\partial t} = \nabla [f(M)\nabla M - \kappa \nabla (\Delta M)], \qquad [S3]$$

where $f(M) = \frac{1}{2\tau}V\left(V + M\frac{\partial V}{\partial M}\right)$, and V and τ are, in general, functions of M.

For simplicity, we consider the turning rate τ to be independent of M. Moreover, we restrict the problem to two dimensions, and hence n = 2. Note that because V is the speed of the organisms in the population, V(M) > 0, for all M, and because $f(M) = \frac{1}{2\tau}V\left(V + M\frac{\partial V}{\partial M}\right)$, the occurrence of zeros in f(M) is controlled by $V + M\frac{\partial V}{\partial M}$, and thus by the parameters in V.

The Derivation of the Mussel Movement Model (Eq. 4 in Main Text). Based on the data obtained from the experiments and the analysis provided in the main text, we assume a parabolic relation between speed V and density M as follows:

$$V(M) = aM^2 - bM + c,$$

where the values of the constants *a*, *b*, and *c* can be obtained from the experimental data. The quadratic term model is highly preferable over the linear one from all statistical results in Table S1.

With this definition of V, we can derive function f(M) as follows:

$$f(M) = \frac{1}{2\tau} \left(aM^2 - bM + c \right) \left(3aM^2 - 2bM + c \right),$$

so that by introducing $m = \sqrt{\frac{a}{c}}M$ and $\beta = \frac{b}{\sqrt{ac}}$, Eq. S3 can be written as follows:

$$\frac{\partial m}{\partial t} = \mathcal{D}_0 \nabla [g(m) \nabla m - \kappa_1 \nabla (\Delta m)], \qquad [S4]$$

where $g(m) = (m^2 - \beta m + 1)(3m^2 - 2\beta m + 1)$, with $\mathcal{D}_0 = \frac{c^2}{2\tau}$, $\kappa_1 = \frac{2\tau \kappa}{2\tau}$, and $\beta^2 < 4$ [because V(m) > 0].

The Standard Cahn–Hilliard Equation. The original Cahn–Hilliard (from here abbreviated as CH) equation describes the separation of a binary fluid into two phases. Assuming *s* is the concentration of this fluid, it follows the general structure:

$$\frac{ds}{\partial t} = \mathcal{D}\nabla^2[\mathcal{P}(s) - \kappa\Delta s],$$
[S5]

where in the most standard setting, proposed by Cahn and Hilliard in 1958 (5), $\mathcal{P}(s) = s^3 - s$, and \mathcal{D} is the diffusion coefficient. This equation generates patterns of the type shown in Fig. S2, consisting of two phases characterized by $s_+ = 1$ and $s_- = -1$. (Notice the striking similarity between Fig. 1D and Fig. S2.) Mathematically, the values s_+ and s_- are given by the minima of the potential function Q(s), where $Q'(s) = \mathcal{P}(s)$. For $\mathcal{P}(s) = s^3 - s$, this potential function is given by $Q(s) = \frac{1}{4}s^4 - \frac{1}{2}s^2 + \frac{1}{4}$, where Q(s) has a symmetric double-well shape.

For the generation of CH-type patterns, it is not strictly necessary that Q(s) has the precise standard form given above. The condition needed is that two minima exists in Q(s) (6, 7), separated by a local maximum that acts as a third, unstable state, s_0 . This implies that Q'(s) must have three—and no more than three zeros, at $s_- < s_0 < s_+$.

In the context of model Eq. **S5**, this means that one expects CH-type dynamics in case $\mathcal{P}(s)$ has three zeros. This implies that $\mathcal{P}'(s)$ must have two zeros, and one negative minimum between these points. Vice versa, if $\mathcal{P}'(s)$ does not have zeros, Eq. **S5** cannot generate patterns because the associated potential well does not have two preferred stable states.

Translating the Mussel Model to an Extended Cahn-Hilliard Model. Based on the data obtained from the experiments given in Fig. 1*C* of the main text, we adopted a general form of the relation between movement speed V(M) and mussel density *M* in which V(M) is minimal at a certain density M^* : *V* increases as *M* moves away from M^* . Hence, we consider V(M) with V'(M) < 0 for $M < M^*$ and V'(M) > 0 for $M > M^*$. In this case, it can be shown that f(M) has one global minimum. The position of this minimum with respect to *M* is controlled by the precise structure of V(M). Thus, the general shape of V(M) found in the experiments is such

that Eq. S4 could generate phase separation and spatial patterns, but patterns arise only when the minimum of f(M) is negative.

Eq. S4 can be written in the CH form Eq. S5 with $\mathcal{P}(m)$ such that $\mathcal{P}'(m) = g(m)$ (see Eq. S6 in *Remark* for details). Because g(m) is a quartic polynomial with respect to m, $\mathcal{P}(m)$ will be a quintic: significantly different from the standard CH case in which $\mathcal{P}(m)$ is a cubic polynomial. However, in the model that we consider, f(M) can have at most one minimum that lies below the axis because of the condition on movement speed that V(M) > 0. This can be deduced by straightforward arguments from the observational fact that V(M) only has one minimum at $M = M^*$ (and decreases, respectively increases, for $M < M^*$, respectively, $M > M^*$). Thus, in the present model only CH-type patterns may develop. This is also typical behavior if we drop the assumption that the turning rate τ is constant, and take it to depend on the mussel concentration M. Because the turning rate must remain positive for all M, it is not possible to create additional zeroes in f(M) by varying τ . Hence, also in this more general case, the dynamics generated by Eq. S3 remain of CH type.

It is straightforward to "control" the appearance of zeros of g(m): g(m) > 0 for all *m* when $\beta < \sqrt{3}$. As β crosses through $\sqrt{3}$, two zeros appear: thus, CH-like patterns will appear as β^2 increases through 3. This is confirmed by our simulations in the bifurcation analysis in Fig. S1 and the simulated pattern in Fig. 1*D*.

Remark: We can easily obtain the exact expression of the CH formula of mussel model Eq. **S4** as follows:

$$\frac{\partial m}{\partial t} = \mathcal{D}_0 \nabla^2 [\mathcal{P}(m) - \kappa_1 \Delta m], \qquad [S6]$$

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with $\mathcal{P}(m) = \frac{3m^5}{5} - \frac{5\beta m^4}{4} + \frac{(4+2\beta^2)m^3}{3} - \frac{3\beta m^2}{2} + m + \mathcal{H}$. Here, $\mathcal{H} \in \mathbf{R}$ is a (essentially irrelevant) new parameter.

Correlations Analysis. The comparison of images obtained from the mussel beds on the tidal flats near Menai Bridge with results of the numerical solution of model Eq. **S4** reveals a remarkable similarity of the real mussel beds with the model prediction (Fig. 2 in main text). Of course, due to the inherent stochastic nature of the real mussel ecological system, the snapshots do not match precisely. To reach a quantitative assessment on the validity of the model Eq. **S4** to describe the spatial properties of the mussel system in the short timescale, we have computed spatial correlation function for the system's spatial patterns.

We consider equal-time spatial correlation functions (in fact, the system displays coarsening at long timescale, then we must choose the appropriate timescale), which yield information about the size of the emerging patterns. Here, we focus on the correlation function, where $\mathcal{G}(\mathbf{r}) = \frac{\langle m(\mathbf{r}+\mathbf{r}')m(\mathbf{r})\rangle - \langle m(\mathbf{r},t)\rangle^2}{\langle m(\mathbf{r},t)^2 \rangle - \langle m(\mathbf{r},t) \rangle^2}$, which expressed how the value at position $m(\mathbf{r}, \mathbf{t})$ is related to data points at some distance \mathbf{r}' (8). The spatial correlation function, $\mathcal{G}(\mathbf{r})$, averaged for specie distance classes over the entire density field, reveals the global behavior of the pattern as a function of spatial scale. The position of the first peak gives the mean wavelength of spatial patterns. In Fig. S4, we show the spatial correlation function obtained for both field patterns and from the predicted patterns of model Eq. S4, after a timescale of about 24 h, revealing an excellent agreement.

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Fig. S1. Bifurcation diagram of Eq. S4 on spatial pattern formation.



Fig. S2. Spatial pattern of CH Eq. **S5** with $\kappa = 0.01$.



Fig. S3. Laboratory images of mussels pattern formation on a surface of 30 cm \times 50 cm, where the number of mussels is equal to 200, 400, 600, and 1,100 individuals in the arena from A to D, respectively.



Fig. S4. Correlation functions of the patterns. The spatial correlation as function of *r* comes from the experimental aggregation patterns and simulation patterns about 24-h timescale. Results were obtained from the experiment in Fig. 1*B*, and numerical solutions of the model Eq. **S4**, and show an excellent agreement.



Fig. S5. The numerical schemes of the kernel ∇^4 was used on Eq. S4 in 2D space.

Table S1. Likelihood-ratio test, AIC weights, adjusted R^2 , and significant level of the fitted functions for mussel movement speed data

Model	LRT	AIC weights	Adjusted R ²	P values
Quadratic	5.4194	2.838788	0.43030	0.00001 (<i>t</i> = 5.717)*
Linear	-7.4215	20.84295	0.00999	0.44900 (<i>t</i> = -0.764)

The observed data are best fitted by a quadratic function. AIC, Akaike information criterion; LRT, likelihood-ratio test.

*The significant difference is referred to second-order term of Eq. 1 in the main text.