Modular Forms: Problem Sheet 1

9 February 2016

1. (a) Show that the standard action of $\mathrm{SL}_2(\mathbb{R})$ on $\mathbb{H}$ is transitive.

   (b) Let $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ be an element of $\mathrm{SL}_2(\mathbb{R})$ with $\gamma \neq \pm\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Prove that γ has exactly one fixed point in $\mathbb{H}$ if $|a+d| < 2$, and no fixed points in $\mathbb{H}$ otherwise.

2. (a) Let K be the stabiliser of $i \in \mathbb{H}$ under the standard action of $\mathrm{SL}_2(\mathbb{R})$ on $\mathbb{H}$. Show that

$$K = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \;\middle|\; a, b \in \mathbb{R}, a^2 + b^2 = 1 \right\} \quad (= \mathrm{SO}_2(\mathbb{R})).$$

   (b) Prove that there is a bijection

$$\begin{aligned} \mathrm{SL}_2(\mathbb{R})/K &\xrightarrow{\sim} \mathbb{H} \\ \gamma K &\longmapsto \gamma i. \end{aligned}$$

3. We recall the notation

$$\sigma_t(n) = \sum_{d \mid n} d^t \quad \text{for all integers } t \geq 0 \text{ and } n \geq 1,$$

   where d runs over the set of positive divisors of n.

   (a) Let m, n and t be positive integers such that m and n are coprime. Show that

$$\sigma_t(mn) = \sigma_t(m)\sigma_t(n).$$

   (b) Let n and t be positive integers, and let

$$n = \prod_{p \text{ prime}} p^{e_p} \quad (e_p \geq 0;\ e_p = 0 \text{ for all but finitely many } p)$$

   be the prime factorisation of n. Show that

$$\sigma_t(n) = \prod_{p \text{ prime}} \frac{p^{(e_p+1)t} - 1}{p^t - 1}.$$