Modular Forms: Problem Sheet 1

9 February 2016

1. (a) Show that the standard action of $\text{SL}_2(\mathbb{R})$ on $\mathbb{H}$ is transitive.
   
   (b) Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be an element of $\text{SL}_2(\mathbb{R})$ with $\gamma \neq \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Prove that $\gamma$ has exactly one fixed point in $\mathbb{H}$ if $|a + d| < 2$, and no fixed points in $\mathbb{H}$ otherwise.

2. (a) Let $K$ be the stabiliser of $i \in \mathbb{H}$ under the standard action of $\text{SL}_2(\mathbb{R})$ on $\mathbb{H}$. Show that $K = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \bigg| a, b \in \mathbb{R}, a^2 + b^2 = 1 \right\}$ ($= \text{SO}_2(\mathbb{R})$).

   (b) Prove that there is a bijection $\text{SL}_2(\mathbb{R})/K \overset{\sim}{\longrightarrow} \mathbb{H}$
   
   $\gamma K \mapsto \gamma i$.

3. We recall the notation $\sigma_t(n) = \sum_{d | n} d^t$ for all integers $t \geq 0$ and $n \geq 1$, where $d$ runs over the set of positive divisors of $n$.

   (a) Let $m$, $n$ and $t$ be positive integers such that $m$ and $n$ are coprime. Show that $\sigma_t(mn) = \sigma_t(m)\sigma_t(n)$.

   (b) Let $n$ and $t$ be positive integers, and let $n = \prod_{p \text{ prime}} p^{e_p}$ ($e_p \geq 0; e_p = 0$ for all but finitely many $p$)

   be the prime factorisation of $n$. Show that $\sigma_t(n) = \prod_{p \text{ prime}} \frac{p^{(e_p+1)t} - 1}{p^t - 1}$. 

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