Modular Forms: Problem Sheet 10

26 April 2016

Throughout this sheet, \( N \) and \( k \) are positive integers.

1. Let \( f \in S_k(\Gamma_1(N)) \) be a normalised Hecke eigenform with \( q \)-expansion \( \sum_{n=1}^{\infty} a_n q^n \) (at the cusp \( \infty \)) and character \( \chi: (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times \).

(a) Prove the identity
\[
a_m = \chi(m)^{-1} a_m \quad \text{for all} \quad m \geq 1 \quad \text{with} \quad \gcd(m, N) = 1.
\]
Deduce that the quantity \( a^2_m/\chi(m) \) is real for all \( m \geq 1 \) such that \( \gcd(m, N) = 1 \).

(b) Prove the following statement, which you could use without proof in problem 2 of problem sheet 9: Let \( f \in M_k(\text{SL}_2(\mathbb{Z})) \) be a normalised eigenform, and let \( p \) be a prime number. Then \( a_p(f) \) is real. (Hint: treat Eisenstein series and cusp forms separately.)

2. Let \( V \) be the space \( S_2(\Gamma_1(16)) \) of cusp forms of weight 2 for \( \Gamma_1(16) \). You may use the following fact without proof: a basis for \( V \), expressed in \( q \)-expansions at the cusp \( \infty \), is
\[
\begin{align*}
f_1 &= q - 2q^3 - 2q^4 + 2q^6 + 4q^7 + 4q^9 + O(q^{10}), \\
f_2 &= q^2 - q^3 - 2q^4 + 2q^6 + 2q^7 + 2q^8 - q^9 + O(q^{10}).
\end{align*}
\]

(a) Show that \( S_2(\Gamma_1(8)) = \{0\} \) and \( V = S_2(\Gamma_1(16))_{\text{new}} \). (Hint: consider the map \( i_{8,16}^2 \) on \( q \)-expansions.)

(b) Compute the matrix of the Hecke operator \( T_2 \) on \( V \) with respect to the basis \( (f_1, f_2) \).

(c) Compute a basis \( (g_1, g_2) \) of \( V \) consisting of eigenforms for \( T_2 \).

(Do the computations by hand; you may use a computer to check your results.)

3. Let \( M \) and \( e \) be positive integers, let \( l \) be a prime number not dividing \( M \), and let \( N = l^e M \). Let \( f \) be a Hecke eigenform in \( S_k(\Gamma_1(M)) \) with character \( \chi \). Let \( V_f \) be the \( \mathbb{C} \)-linear subspace of \( S_k(\Gamma_1(N)) \) spanned by the forms \( f_j = i^{M,N}_{M,N}(f) \) for \( 0 \leq j \leq e \).

(a) Prove that the forms \( f_0, \ldots, f_e \) are \( \mathbb{C} \)-linearly independent.

(b) Show that the Hecke operator \( T_l \) on \( S_k(\Gamma_1(N)) \) preserves the subspace \( V_f \), and compute the matrix of \( T_l \) on \( V_f \) with respect to the basis \( (f_0, \ldots, f_e) \).

Answer:
\[
\begin{pmatrix}
a_l & 1 & 0 & 0 & \cdots & 0 \\
-\chi(l) l^{k-1} & 0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 0 & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & 1 & 0 \\
0 & 0 & \cdots & 0 & 0 & 1 \\
0 & 0 & \cdots & 0 & 0 & 0
\end{pmatrix}
\]
4. Suppose that $S_k(\Gamma_0(N))$ contains some normalised eigenform $f$. Write $g = f^2 \in S_{2k}(\Gamma_0(N))$. Calculate the first two terms of the $q$-expansions of $g$ and $T_2g$, and deduce that the dimension of $S_{2k}(\Gamma_0(N))$ is at least 2.

5. Let $\Gamma$ be a congruence subgroup, and let $f$ be a modular form of weight $k$ for $\Gamma$. Define a function $f^* : \mathbb{H} \rightarrow \mathbb{C}$ by

$$f^*(z) = \overline{f(-z)}.$$  

(a) Prove that $f^*$ is a modular form of weight $k$ for the group $\sigma^{-1}\Gamma\sigma$, where $\sigma = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$.

(b) Suppose (for simplicity) that both $\Gamma$ and $\sigma^{-1}\Gamma\sigma$ contain the subgroup $\{(1 \ a) \mid a \in \mathbb{Z}\}$. Show that the standard $q$-expansions of $f$ and $f^*$ in the variable $q = \exp(2\pi i z)$ are related by

$$a_n(f^*) = \overline{a_n(f)}$$

for all $n \geq 0$.

(c) Show that if $\Gamma = \Gamma_0(N)$ or $\Gamma = \Gamma_1(N)$ for some $N \geq 1$, then $\sigma^{-1}\Gamma\sigma = \Gamma$.

**Bonus problem:** Give an example of a congruence subgroup $\Gamma$ such that $\sigma^{-1}\Gamma\sigma \neq \Gamma$. 