Modular Forms: Problem Sheet 10

26 April 2016

Throughout this sheet, N and k are positive integers.

1. Let $f \in \mathrm{S}_k(\Gamma_1(N))$ be a normalised Hecke eigenform with q-expansion $\sum_{n=1}^{\infty} a_n q^n$ (at the cusp ∞) and character $\chi\colon (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$.

   (a) Prove the identity
   $$\overline{a_m} = \chi(m)^{-1} a_m \quad \text{for all } m \geq 1 \text{ with } \gcd(m, N) = 1.$$
   Deduce that the quantity $a_m^2/\chi(m)$ is real for all $m \geq 1$ such that $\gcd(m, N) = 1$.

   (b) Prove the following statement, which you could use without proof in problem 2 of problem sheet 9: Let $f \in \mathrm{M}_k(\mathrm{SL}_2(\mathbb{Z}))$ be a normalised eigenform, and let p be a prime number. Then $a_p(f)$ is real. (*Hint:* treat Eisenstein series and cusp forms separately.)

2. Let V be be the space $\mathrm{S}_2(\Gamma_1(16))$ of cusp forms of weight 2 for $\Gamma_1(16)$. You may use the following fact without proof: a basis for V, expressed in q-expansions at the cusp ∞, is
   $$\begin{aligned} f_1 &= q - 2q^3 - 2q^4 + 2q^6 + 2q^7 + 4q^8 - q^9 + O(q^{10}), \\ f_2 &= q^2 - q^3 - 2q^4 + q^5 + 2q^7 + 2q^8 - q^9 + O(q^{10}). \end{aligned}$$

   (a) Show that $\mathrm{S}_2(\Gamma_1(8)) = \{0\}$ and $V = \mathrm{S}_2(\Gamma_1(16))_{\mathrm{new}}$. (*Hint:* consider the map $i_2^{8,16}$ on q-expansions.)

   (b) Compute the matrix of the Hecke operator T_2 on V with respect to the basis (f_1, f_2).

   (c) Compute a basis (g_1, g_2) of V consisting of eigenforms for T_2.

   (Do the computations by hand; you may use a computer to check your results.)

3. Let M and e be positive integers, let l be a prime number not dividing M, and let $N = l^e M$. Let f be a Hecke eigenform in $\mathrm{S}_k(\Gamma_1(M))$ with character χ. Let V_f be the $\mathbb{C}$-linear subspace of $\mathrm{S}_k(\Gamma_1(N))$ spanned by the forms $f_j = i_{l^j}^{M,N}(f)$ for $0 \leq j \leq e$.

   (a) Prove that the forms $f_0, \ldots, f_e$ are $\mathbb{C}$-linearly independent.

   (b) Show that the Hecke operator T_l on $\mathrm{S}_k(\Gamma_1(N))$ preserves the subspace V_f, and compute the matrix of T_l on V_f with respect to the basis $(f_0, \ldots, f_e)$.

   Answer: $$\begin{pmatrix} a_l & 1 & 0 & 0 & \cdots & 0 \\ -\chi(l)l^{k-1} & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 1 \\ 0 & 0 & \cdots & 0 & 0 & 0 \end{pmatrix}.$$

4. Suppose that $\mathrm{S}_k(\Gamma_0(N))$ contains some normalised eigenform f. Write $g = f^2 \in \mathrm{S}_{2k}(\Gamma_0(N))$. Calculate the first two terms of the q-expansions of g and $T_2 g$, and deduce that the dimension of $\mathrm{S}_{2k}(\Gamma_0(N))$ is at least 2.

5. Let Γ be a congruence subgroup, and let f be a modular form of weight k for Γ. Define a function $f^*\colon \mathbb{H} \to \mathbb{C}$ by

$$f^*(z) = \overline{f(-\bar{z})}.$$

   (a) Prove that f^* is a modular form of weight k for the group $\sigma^{-1}\Gamma\sigma$, where $\sigma = \left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$.

   (b) Suppose (for simplicity) that both Γ and $\sigma^{-1}\Gamma\sigma$ contain the subgroup $\left\{\left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right) \mid b \in \mathbb{Z}\right\}$. Show that the standard q-expansions of f and f^* in the variable $q = \exp(2\pi i z)$ are related by

$$a_n(f^*) = \overline{a_n(f)} \quad \text{for all } n \geq 0.$$

   (c) Show that if $\Gamma = \Gamma_0(N)$ or $\Gamma = \Gamma_1(N)$ for some $N \geq 1$, then $\sigma^{-1}\Gamma\sigma = \Gamma$.

   Bonus problem: Give an example of a congruence subgroup Γ such that $\sigma^{-1}\Gamma\sigma \neq \Gamma$.