Modular Forms: Problem Sheet 11

3 May 2016

Throughout this sheet, N and k are positive integers.

1. Let g_1 and g_2 be the eigenforms for the operator T_2 on $\mathrm{S}_2(\Gamma_1(16))$ found in problem 2 from problem sheet 10.

   (a) Prove that g_1 and g_2 are in fact eigenforms for the full Hecke algebra $\mathbb{T}(\mathrm{S}_2(\Gamma_1(16)))$. (*Hint:* first show that $\mathrm{S}_2(\Gamma_1(16))$ admits a basis of eigenforms for the full Hecke algebra.)

   (b) Compute the eigenvalues of the diamond operator $\langle 3\rangle$ on g_1 and g_2. (*Hint:* use T_3 and T_9.)

   (c) Prove that the characters of g_1 and g_2 are given by

   $$\langle d\rangle g_j = \chi_j(d) g_j \quad \text{for all } d \in (\mathbb{Z}/16\mathbb{Z})^\times \qquad (j = 1, 2),$$

   where χ_1, χ_2 are the two group homomorphisms $(\mathbb{Z}/16\mathbb{Z})^\times \to \mathbb{C}^\times$ with kernel $\{\pm 1\}$.

   (Do the computations by hand; you may use a computer to check your results.)

2. (a) Use the SageMath command `Newforms` to show that there is exactly one primitive form f of weight 6 for the group $\Gamma_1(4)$. Determine the q-expansion coefficients $a_n(f)$ for $n \le 20$.

   (b) Prove that $a_n(f) = 0$ for all even integers n.

   (c) Give a formula expressing the modular form θ^{12} (see §3.8 of the notes) as a linear combination of $E_6(z)$, $E_6(2z)$, $E_6(4z)$ and f.

   (d) Deduce that for all *even* integers $n \ge 2$, the number of representations of n as a sum of 12 squares is given by the formula

   $$r_{12}(n) = 8 \sum_{d \mid n} d^5 - 512 \sum_{d \,\mid\, n/4} d^5.$$

   (Cf. Theorem 3.17 of the notes; the sums are taken over all positive divisors of n and $n/4$, respectively, and the last sum is omitted if $4 \nmid n$.)

   (As in the lecture, a *primitive form* is an eigenform f in the new subspace, normalised such that $a_1(f) = 1$. These are often also called *newforms*, which explains the name of the SageMath command `Newforms`.)

3. For $f \in \mathrm{S}_k(\Gamma_1(N))$, let $f^* \in \mathrm{S}_k(\Gamma_1(N))$ be the form defined by $f^*(z) = \overline{f(-\bar{z})}$ (see problem 5 from problem sheet 10).

   (a) Show that the map $\mathrm{S}_k(\Gamma_1(N)) \to \mathrm{S}_k(\Gamma_1(N))$ sending f to f^* preserves the subspaces $\mathrm{S}_k(\Gamma_1(N))_{\mathrm{old}}$ and $\mathrm{S}_k(\Gamma_1(N))_{\mathrm{new}}$.

(b) Let $f \in \mathrm{S}_k(\Gamma_1(N))_{\mathrm{new}}$ be a primitive form. Show that the form f^*, which by part (a) is in $\mathrm{S}_k(\Gamma_1(N))_{\mathrm{new}}$, is also a primitive form, and determine the eigenvalues of the operators $\langle d \rangle$ (for $d \in (\mathbb{Z}/N\mathbb{Z})^\times$) and T_m (for $m \geq 1$) on f^*.

4. Recall that the Fricke (or Atkin–Lehner) operator w_N on $\mathrm{S}_k(\Gamma_1(N))$ is the operator T_{α_N} with $\alpha_N = \left(\begin{smallmatrix} 0 & -1 \\ N & 0 \end{smallmatrix}\right)$.

   (a) Show that $w_N^2 = (-N)^k \cdot \mathrm{id}$ and that the adjoint of w_N equals $(-1)^k w_N$.

   (b) Show that for every $d \in (\mathbb{Z}/N\mathbb{Z})^\times$, the diamond operator $\langle d \rangle$ on $\mathrm{S}_k(\Gamma_1(N))$ satisfies $w_N^{-1} \langle d \rangle w_N = \langle d \rangle^{-1}$.

   (c) Show that for every positive integer m such that $\gcd(m, N) = 1$, the Hecke operator T_m satisfies $w_N^{-1} T_m w_N = \langle m \rangle^{-1} T_m$.

5. Let w_N be the Fricke operator on $\mathrm{S}_k(\Gamma_1(N))$; recall that this preserves the new subspace $\mathrm{S}_k(\Gamma_1(N))_{\mathrm{new}}$. Let $f \in \mathrm{S}_k(\Gamma_1(N))_{\mathrm{new}}$ be a primitive form.

   (a) Show that the form $w_N f$ is an eigenform for the operators $\langle d \rangle$ for $d \in (\mathbb{Z}/N\mathbb{Z})^\times$ and T_m for $m \geq 1$ with $\gcd(m, N) = 1$, and determine the eigenvalues of these operators on $w_N f$.

   (b) Deduce that $w_N f = \eta_f f^*$ for some $\eta_f \in \mathbb{C}$, with f^* as in problem 3. (*Hint:* use problem 1 from problem sheet 10 as one ingredient.)

   (c) Prove the identities $\eta_f \eta_{f^*} = (-N)^k$, $\eta_{f^*} = (-1)^k \bar{\eta}_f$ and $|\eta_f| = N^{k/2}$. (*Hint:* consider $\langle w_N f, f^* \rangle_{\Gamma_1(N)}$.)

   You may use results from earlier exercises.

   (The complex number η_f is called the *Atkin–Lehner pseudo-eigenvalue* of f.)