Modular Forms: Problem Sheet 12

10 May 2016

Throughout this sheet, N and k are positive integers.

1. Consider functions $f\colon \mathbb{R} \to \mathbb{C}$ that are infinitely continuously differentiable and such that for all $m, n \ge 0$ the function $x^m f^{(n)}(x)$ (where $f^{(n)}$ is the n-th derivative of x) tends to zero as $|x| \to \infty$. Recall that the Fourier transform of such a function f is defined as

$$\hat{f}(t) = \int_{-\infty}^{\infty} f(x)\exp(-2\pi itx)dx,$$

and that f can be recovered from $\hat{f}$ using the Fourier inversion formula,

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(t)\exp(2\pi ixt)dt.$$

Use this to give a proof of the Mellin inversion formula (see §6.1 of the notes) for 'sufficiently nice' functions $g(t)$.

2. (a) Suppose k is even and $k \ge 4$. Prove that the L-function of the Eisenstein series E_k admits the factorisation

$$L(E_k, s) = \zeta(s)\zeta(s-k+1),$$

where $\zeta(s)$ is the Riemann ζ-function.

(b) (This part is optional and depends on the optional exercises from problem sheet 6.) Let α, β be primitive Dirichlet characters modulo M and N, respectively, satisfying $\alpha(-1)\beta(-1) = (-1)^k$. Prove that the L-function of the Eisenstein series $E_k^{\alpha,\beta} \in \mathrm{M}_k(\Gamma_1(MN))$ associated to the pair (α, β) admits the factorisation

$$L(E_k^{\alpha,\beta}, s) = L(\alpha, s)L(\beta, s-k+1).$$

Note: The fact that the L-function of an Eisenstein series has such a factorisation is one of the manifestations of the rule of thumb that Eisenstein series are 'easier' than cusp forms.

3. Let $f \in \mathrm{S}_k(\Gamma_1(N))$ be an eigenform such that all coefficients $a_n(f)$ for $n \ge 1$ are real.

(a) Show that the complex number ϵ_f defined in Theorem 6.4 of the notes is either $+1$ or -1.

(b) Let r be the order of vanishing of the holomorphic function $L(f, s)$ in $s = k/2$. Prove that r is even if $\epsilon_f = +1$ and that r is odd if $\epsilon_f = -1$. (*Hint:* expand the completed L-function $\Lambda(f, s)$ in a power series around $s = k/2$.)