Modular Forms: Problem Sheet 2

16 February 2016

1. (a) Show that $G_4(\exp(2\pi i/3)) = 0$. (Hint: $G_4(-1/z) = z^4 G_4(z)$.)

   (b) Show that $G_6(i) = 0$.

2. Define $f : \mathbb{H} \to \mathbb{C}$ by
$$f(z) := G_2(z) - \frac{\pi}{\Im z}.$$

   (a) Show that
$$f(\gamma z) = j(\gamma, z)^2 f(z) \quad \text{for all } \gamma \in \mathrm{SL}_2(\mathbb{Z}) \text{ and } z \in \mathbb{H}.$$

   (b) Is $f(z)$ a modular form?

3. Let $f : \mathbb{H} \to \mathbb{C}$ be a modular form of weight 0.

   (a) Show that there exists some $C \in \mathbb{R}_{>0}$ such that: any element in $\mathbb{H}$ is $\mathrm{SL}_2(\mathbb{Z})$-equivalent to some $z \in \mathbb{H}$ with $\Im(z) \geq C$. (Take e.g. $C = \sqrt{3}/2$.)

   (b) Deduce that $|f|$ attains a maximum.

   (c) Conclude that the space of modular forms of weight zero consists exactly of the ($\mathbb{C}$-)constant functions. (Hint: maximum modulus principle.)