Modular Forms: Problem Sheet 3

23 February 2016

1. (a) Prove the formula

$$\sigma_9(n) = \frac{21}{11}\sigma_5(n) - \frac{10}{11}\sigma_3(n) + \frac{5040}{11}\sum_{j=1}^{n-1}\sigma_3(j)\sigma_5(n-j) \quad \text{for all } n \in \mathbb{Z}_{>0}.$$

   (b) Find similar expressions for σ_{13} in terms of σ_3 and σ_9, and in terms of σ_5 and σ_7.

2. (a) Find rational numbers λ and μ such that

$$\Delta = \lambda E_4^3 + \mu E_{12}.$$

   (b) Let $\tau(n)$ be the n-th coefficient in the q-expansion of Δ, so that

$$\Delta = \sum_{n=1}^{\infty} \tau(n) q^n.$$

   Prove *Ramanujan's congruence*:

$$\tau(n) \equiv \sigma_{11}(n) \pmod{691}.$$

3. Show that the ring $\mathbb{C}[E_2, E_4, E_6]$ is closed under differentiation.

4. (a) Show that the modular functions (for $\mathrm{SL}_2(\mathbb{Z})$) form a field F (with addition and multiplication defined pointwise).

   (b) Prove that $F = \mathbb{C}(j)$ and that j is transcendental over $\mathbb{C}$.

5. Consider the modular function $j\colon \mathbb{H} \to \mathbb{C}$.

   (a) Show that $j(i) = 1728$ and $j(\rho) = 0$ (where $\rho = \exp(2\pi i/3)$).

   (b) Let $z \in \mathcal{D}$ (the standard fundamental domain for $\mathrm{SL}_2(\mathbb{Z})$). Prove:

$$(z \text{ lies on the boundary of } \mathcal{D} \text{ or } \Re z = 0) \;\Rightarrow\; j(z) \in \mathbb{R}.$$

   (c) Show that $\overline{j} : \mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H} \to \mathbb{C}$ given by $\overline{j}([z]) := j(z)$ is well-defined and prove that $\overline{j}$ is bijective.
   (Here $[z]$ denotes the orbit of z under the action of $\mathrm{SL}_2(\mathbb{Z})$.)

   (d) Prove the converse to part (b).

6. (a) Show that M_k is spanned by all $E_4^a E_6^b$ with $a, b \in \mathbb{Z}_{\geq 0}$ and $4a + 6b = k$.

   (b) Show that E_4 and E_6 are algebraically independent over $\mathbb{C}$.

   We remark that this exercise shows that the *ring of modular forms* (for $\mathrm{SL}_2(\mathbb{Z})$) $M := \bigoplus_{k\in\mathbb{Z}} M_k$ is isomorphic to the ring of polynomials over $\mathbb{C}$ in two variables $\mathbb{C}[x, y]$ with isomorphism $\mathbb{C}[x, y] \xrightarrow{\sim} M$ given by $(x, y) \mapsto (E_4, E_6)$. (If we *grade* the rings by assigning grade k to a modular form of weight k and grades 4 and 6 to x and y respectively, we get an isomorphism of graded rings.)