Modular Forms: Problem Sheet 4

1 March 2016

1. Let $\mathcal{L}_1(N)$ be the set of pairs (Λ, P) where Λ is a lattice in $\mathbb{C}$ and P is a point of order N in the group $\mathbb{C}/\Lambda$.

   (a) Show that on $\mathcal{L}_1(N)$ there is an equivalence relation $\sim$ with the property that $(\Lambda, P) \sim (\Lambda', P')$ if and only if there exists $\alpha \in \mathbb{C}^\times$ such that for any $\omega \in \mathbb{C}$ with $\omega + \Lambda = P$ in $\mathbb{C}/\Lambda$ we have $\alpha\Lambda = \Lambda'$ and $\alpha\omega + \Lambda' = P'$ in $\mathbb{C}/\Lambda'$.

   (b) Recall that $\Gamma_1(N)$ is the subgroup of $\mathrm{SL}_2(\mathbb{Z})$ consisting of matrices of the form $\left(\begin{smallmatrix} a & b \\ Nc & d \end{smallmatrix}\right)$ with $a, b, c, d \in \mathbb{Z}$, $a \equiv d \equiv 1 \pmod{N}$ and $ad - Nbc = 1$. Prove that there is a bijection

   $$\mathcal{L}_1(N)/{\sim} \cong \Gamma_1(N)\backslash\mathbb{H}.$$

   (*Hint:* consider lattices together with a suitable $\mathbb{Z}$-basis (ω_1, ω_2), and use a similar argument as for the bijection $\mathcal{L}_0(N)/{\sim} \cong \Gamma_0(N)\backslash\mathbb{H}$) constructed in the lecture.)

2. Show that the cusps of $\Gamma_1(4)$, viewed as $\Gamma_1(4)$-orbits in $\mathbb{P}^1(\mathbb{Q})$, are represented by the elements 0, 1/2 and ∞ of $\mathbb{P}^1(\mathbb{Q})$. For each of these cusps $\mathfrak{c}$, determine whether $\mathfrak{c}$ is regular or irregular, and compute its width $h_\Gamma(\mathfrak{c})$.

3. Let p be an odd prime number. Determine a set of representatives for the $\Gamma_1(p)$-orbits in $\mathbb{P}^1(\mathbb{Q})$. For each of the corresponding cusps $\mathfrak{c}$ of $\Gamma_1(p)$, compute its width $h_\Gamma(\mathfrak{c})$.

4. Let N be a positive integer, and let H be a subgroup of $(\mathbb{Z}/N\mathbb{Z})^\times$. Show that the set

   $$\Gamma_H = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \;\middle|\; a, d \bmod N \text{ are in } H \text{ and } c \equiv 0 \pmod{N} \right\}$$

   is a congruence subgroup, and determine its level.