1 March 2016

1. Let $L_1(N)$ be the set of pairs $(\Lambda, P)$ where $\Lambda$ is a lattice in $\mathbb{C}$ and $P$ is a point of order $N$ in the group $\mathbb{C}/\Lambda$.

   (a) Show that on $L_1(N)$ there is an equivalence relation $\sim$ with the property that $(\Lambda, P) \sim (\Lambda', P')$ if and only if there exists $\alpha \in \mathbb{C}^\times$ such that for any $\omega \in \mathbb{C}$ with $\omega + \Lambda = P$ in $\mathbb{C}/\Lambda$ we have $\alpha \Lambda = \Lambda'$ and $\alpha \omega + \Lambda' = P'$ in $\mathbb{C}/\Lambda'$.

   (b) Recall that $\Gamma_1(N)$ is the subgroup of $SL_2(\mathbb{Z})$ consisting of matrices of the form $\begin{pmatrix} a & b \\ Nc & d \end{pmatrix}$ with $a, b, c, d \in \mathbb{Z}$, $a \equiv d \equiv 1 \mod N$ and $ad - Nbc = 1$. Prove that there is a bijection $L_1(N)/\sim \cong \Gamma_1(N)\backslash \mathbb{H}$.

   (Hint: consider lattices together with a suitable $\mathbb{Z}$-basis $(\omega_1, \omega_2)$, and use a similar argument as for the bijection $L_0(N)/\sim \cong \Gamma_0(N)\backslash \mathbb{H}$ constructed in the lecture.)

2. Show that the cusps of $\Gamma_1(4)$, viewed as $\Gamma_1(4)$-orbits in $\mathbb{P}^1(\mathbb{Q})$, are represented by the elements $0, 1/2$ and $\infty$ of $\mathbb{P}^1(\mathbb{Q})$. For each of these cusps $\epsilon$, determine whether $\epsilon$ is regular or irregular, and compute its width $h_{\Gamma}(\epsilon)$.

3. Let $p$ be an odd prime number. Determine a set of representatives for the $\Gamma_1(p)$-orbits in $\mathbb{P}^1(\mathbb{Q})$. For each of the corresponding cusps $\epsilon$ of $\Gamma_1(p)$, compute its width $h_{\Gamma}(\epsilon)$.

4. Let $N$ be a positive integer, and let $H$ be a subgroup of $(\mathbb{Z}/N\mathbb{Z})^\times$. Show that the set

   $$\Gamma_H = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \mid a, d \mod N \text{ are in } H \text{ and } c \equiv 0 \pmod N \right\}$$

is a congruence subgroup, and determine its level.