Modular Forms: Problem Sheet 5

8 March 2016

1. Let N be a positive integer. We consider the set

$$C_N = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in (\mathbb{Z}/N\mathbb{Z})^2 \mid \langle x, y \rangle = \mathbb{Z}/N\mathbb{Z} \right\} \Big/ \{\pm 1\},$$

where $\langle x, y \rangle$ denotes the (additive) subgroup of $\mathbb{Z}/N\mathbb{Z}$ generated by x and y, and where the group $\{\pm 1\}$ acts from the right on C_N by $\binom{x}{y}\epsilon = \binom{\epsilon x}{\epsilon y}$. Note that the set C_N has a natural left $\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$-action.

   (a) Prove that there is a natural bijection

$$\mathrm{Cusps}(\Gamma(N)) \cong C_N.$$

   (b) Let $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$ be a congruence subgroup of level N. Let H be the image of Γ under the map $\mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$. Show that there is a natural bijection

$$\mathrm{Cusps}(\Gamma) \cong H \backslash C_N.$$

   (c) Describe how the widths of the cusps of a given congruence subgroup of level N can be determined using computations "in characteristic N", i.e. involving $\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$ and C_N instead of $\mathrm{SL}_2(\mathbb{Z})$ and $\mathbb{P}^1(\mathbb{Q})$.

   (d) Use parts (b) and (c) to solve problem 3 of the previous exercise sheet: given an odd prime number p, describe the set $\mathrm{Cusps}(\Gamma_1(p))$, and for each $\mathfrak{c} \in \mathrm{Cusps}(\Gamma_1(p))$, compute $h_\Gamma(\mathfrak{c})$.

2. The goal of this exercise is to prove the implication (ii) $\Rightarrow$ (i) of Theorem 3.5 in the notes. Let Γ be a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$, and let k be an integer. Let $f\colon \mathbb{H} \to \mathbb{C}$ be a holomorphic function that is weakly modular of weight k for Γ and holomorphic at the cusp ∞. Suppose that there exist positive real numbers C, d such that the coefficients a_n in the Fourier expansion

$$f(z) = \sum_{n=0}^{\infty} a_n q_\infty^n$$

satisfy

$$|a_n| \le C n^d \quad \text{for all } n \in \mathbb{Z}_{>0}.$$

   (a) Prove that there exist positive real numbers C_1 and C_2 such that for all $z \in \mathbb{H}$ we have

$$|f(z)| \le C_1 + C_2 (\Im z)^{-d-1}.$$

   (*Hint:* bound $|f(z)|$ by comparing $\sum_{n=1}^{\infty} |a_n q_\infty^n|$ to an integral of the form $\int_0^\infty t^d \exp(-at)dt$.)

(b) Prove that for any $\alpha \in \mathrm{SL}_2(\mathbb{Z})$, the function $z \mapsto (f|_k\alpha)(z)$ grows at most polynomially when $\Im z \to \infty$, i.e. that there exist positive real numbers C_3 and e such that

$$\left|(f|_k\alpha)(z)\right| \leq C_3(\Im z)^e \quad \text{for all } z \in \mathbb{H} \text{ with } \Im z \geq 1.$$

(c) Deduce that f is a modular form of weight k for Γ.

(*Hint:* A version of this exercise with more intermediate steps is in the book of Diamond and Shurman, Exercise 1.2.6.)