Modular Forms: Problem Sheet 6

15 March 2016

In the exercises below, \( N \) denotes a positive integer.

1. (a) Let \( \chi \) be a Dirichlet character modulo \( N \). Prove that

\[
\sum_{j=0}^{N-1} \chi(j) = \begin{cases} 
\phi(N) & \text{if } \chi = 1_N, \\
0 & \text{otherwise.}
\end{cases}
\]

(b) Let \( j \) be an integer. Prove that

\[
\sum_{\chi \in D_N} \chi(j) = \begin{cases} 
\phi(N) & \text{if } j \in NZ, \\
0 & \text{otherwise.}
\end{cases}
\]

where \( D_N \) is the group of all Dirichlet characters modulo \( N \).

2. For integers \( k > 0 \) and \( n \geq 0 \), write

\[
r_k(n) = \# \{(x_1, \ldots, x_k) \in \mathbb{Z}^k \mid x_1^2 + \cdots + x_k^2 = n\}.
\]

Furthermore, let \( \chi \) be the unique non-trivial Dirichlet character modulo 4. In this exercise you may assume without proof that there exist modular forms \( E_{1,\chi} \in M_1(\Gamma_1(4)) \) and \( E_{3,\chi}, E_{\chi,1} \in M_3(\Gamma_1(4)) \) with \( q \)-expansions

\[
E_{1,\chi} = \frac{1}{4} + \sum_{n=1}^{\infty} \left( \sum_{d \mid n} \chi(d) \right) q^n,
\]

\[
E_{3,\chi} = -\frac{1}{4} + \sum_{n=1}^{\infty} \left( \sum_{d \mid n} \chi(d)d^2 \right) q^n,
\]

\[
E_{\chi,1} = \sum_{n=1}^{\infty} \left( \sum_{d \mid n} \chi(n/d)d^2 \right) q^n.
\]

(These are examples of Eisenstein series for \( \Gamma_1(4) \). For a construction of the last two forms, see exercise 7 below. Eisenstein series of weight 1 will not be constructed in this course.)

(a) Prove the formula

\[
r_2(n) = 4 \sum_{d \mid n} \chi(d) \quad \text{for all } n \geq 1.
\]

(Note: If you know about arithmetic in the ring \( \mathbb{Z}[i] \) of Gaussian integers, you can also prove this formula by counting ideals of norm \( n \) in \( \mathbb{Z}[i] \).)
(b) Prove the formula
\[ r_a(n) = \sum_{d|n} (16\chi(n/d) - 4\chi(d))d^2 \quad \text{for all } n \geq 1. \]

3. Let \( \chi : \mathbb{Z} \to \mathbb{C} \) be a Dirichlet character modulo \( N \). The \textit{L-function} of \( \chi \) is the holomorphic function \( L(\chi, s) \) (of the variable \( s \)) defined by
\[ L(\chi, s) = \sum_{n=1}^{\infty} \chi(n)n^{-s}. \]

(a) Prove that the sum converges absolutely and uniformly on every right half-plane of the form \( \{ s \in \mathbb{C} \mid \Re s \geq \sigma \} \) with \( \sigma > 1 \).

(b) Prove the identity
\[ L(\chi, s) = \prod_{p \text{ prime}} \frac{1}{1 - \chi(p)p^{-s}} \quad \text{for } \Re s > 1. \]

(Hint: expand \( \frac{1}{1 - \chi(p)t} \) in a power series in \( t \).)

Note: The functions \( L(\chi, s) \) were introduced by P. G. Lejeune-Dirichlet in the proof of his famous theorem on primes in arithmetic progressions:

\textbf{Theorem} (Dirichlet, 1837). Let \( N \) and \( a \) be coprime positive integers. Then there exist infinitely many prime numbers \( p \) with \( p \equiv a \pmod{N} \).

4. Let \( \chi \) be a Dirichlet character modulo \( N \). We consider the function \( \mathbb{Z} \to \mathbb{C} \) sending an integer \( m \) to the complex number
\[ \tau(\chi, m) = \sum_{n=0}^{N-1} \chi(n) \exp(2\pi inm/N). \]
(This can be viewed as a discrete Fourier transform of \( \chi \).) The case \( m = 1 \) deserves special mention: the complex number
\[ \tau(\chi) = \tau(\chi, 1) = \sum_{n=0}^{N-1} \chi(n) \exp(2\pi in/N) \]
is called the \textit{Gauss sum} attached to \( \chi \).

(a) Compute \( \tau(\chi) \) for all non-trivial Dirichlet characters \( \chi \) modulo 4 and modulo 5, respectively.

(b) Suppose that \( \chi \) is primitive. Prove that for all \( m \in \mathbb{Z} \) we have
\[ \tau(\chi, m) = \chi(m)\tau(\chi). \]
(Hint: writing \( d = \gcd(m, N) \), distinguish the cases \( d = 1 \) and \( d > 1 \).)

(c) Deduce that if \( \chi \) is primitive, we have
\[ \tau(\chi)\tau(\bar{\chi}) = \chi(-1)N \]
and
\[ \tau(\chi)\overline{\tau(\chi)} = N. \]
The following exercises are optional. The goal is to construct Eisenstein series with character. In each exercise you may use the results of all preceding exercises.

5. Let $\chi$ be a primitive Dirichlet character modulo $N$. The generalised Bernoulli numbers attached to $\chi$ are the complex numbers $B_k(\chi)$ for $k \geq 0$ defined by the identity

$$\sum_{k=0}^{\infty} \frac{B_k(\chi)}{k!} t^k = \frac{t}{\exp(Nt) - 1} \sum_{j=1}^{N} \chi(j) \exp(jt)$$

in the ring $\mathbb{C}[[t]]$ of formal power series in $t$.

(a) Let $\zeta$ be a primitive $N$-th root of unity in $\mathbb{C}$. Prove that if $\chi$ is non-trivial (i.e. $N > 1$), then we have

$$\sum_{j=0}^{N-1} \chi(j) \frac{x + \zeta^j}{x - \zeta^j} = \frac{2N}{\tau(\chi)(x^N - 1)} \sum_{m=0}^{N-1} \bar{\chi}(m)x^n$$

in the field $\mathbb{C}(x)$ of rational functions in the variable $x$. (Hint: compute residues.)

(b) Prove that for every integer $k \geq 2$ such that $(-1)^k = \chi(-1)$, the special value of the Dirichlet $L$-function of $\chi$ at $k$ is

$$L(\chi, k) = -\frac{(2\pi i)^k B_k(\bar{\chi})}{2\tau(\chi)N^{k-1}k!}.$$ 

6. Let $k \geq 3$, and let $\alpha$ and $\beta$ be Dirichlet characters modulo $M$ and $N$, respectively. For all $k \geq 3$, we define a function $G_{\alpha,\beta}^k: \mathbb{H} \to \mathbb{C}$ by

$$G_{\alpha,\beta}^k(z) = \sum_{m,n \in \mathbb{Z}} \frac{\alpha(m)\beta(n)}{(mz + n)^k}.$$ 

(a) Prove that the function $G_{\alpha,\beta}^k$ is weakly modular of weight $k$ for the congruence subgroup

$$\Gamma_1(M, N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z}) \right| \text{\begin{tabular}{c} $a \equiv d \equiv 1 \pmod{\text{lcm}(M, N)}$, \\
$c \equiv 0 \pmod{M}$, \\
$b \equiv 0 \pmod{N}$ \end{tabular}} \right\}.$$ 

(b) Show that $G_{\alpha,\beta}^k$ is the zero function unless $\alpha(-1)\beta(-1) = (-1)^k$.

(c) Prove the identity

$$G_{\alpha,\beta}^k(z) = 2\alpha(0) \sum_{n>0} \frac{\beta(n)}{n^k} + 2 \sum_{m>0} \alpha(m) \sum_{n \in \mathbb{Z}} \frac{\beta(n)}{(mz + n)^k}.$$ 

7. Keeping the notation of the previous exercise, assume in addition that $\alpha(-1)\beta(-1) = (-1)^k$ and that the character $\beta$ is primitive.

(a) Prove that for all $w \in \mathbb{H}$ we have

$$\sum_{n \in \mathbb{Z}} \frac{\beta(n)}{(w + n)^k} = \frac{(-2\pi i)^k \tau(\bar{\beta})}{N^k(k-1)!} \sum_{d=1}^{\infty} \beta(d)d^{k-1} \exp(2\pi idw/N).$$
(b) Deduce the formula
\[ G_{k}^{\alpha,\beta}(z) = -\alpha(0) \left( \frac{(2\pi i)^k B_k(\beta)}{\tau(\beta) N^{k-1} k!} \right) \]
\[ + \frac{2(-2\pi i)^k \tau(\beta)}{N^k (k-1)!} \sum_{d=1}^{\infty} \left( \sum_{d|\alpha} \alpha(n/d) \beta(d) d^{k-1} \right) \exp(2\pi i n z/N). \]

(c) Let \( E_{k}^{\alpha,\beta}(z) \) be the unique scalar multiple of \( G_{k}^{\alpha,\beta}(Nz) \) such that the coefficient of \( q \) in the \( q \)-expansion of \( E_{k}^{\alpha,\beta} \) equals 1. Prove the identity
\[ E_{k}^{\alpha,\beta}(z) = -\alpha(0) \frac{B_k(\beta)}{2k} + \sum_{n=1}^{\infty} \left( \sum_{d|\alpha} \alpha(n/d) \beta(d) d^{k-1} \right) q^n. \]

(d) Prove that \( E_{k}^{\alpha,\beta}(z) \) is a modular form of weight \( k \) for \( \Gamma_1(MN) \).