Modular Forms: Problem Sheet 8

12 April 2016

1. Let p be a prime and consider the lattice $\Lambda := \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ where $\omega_1, \omega_2 \in \mathbb{C}^*$ and $\omega_1/\omega_2 \notin \mathbb{R}$.

   (a) Show that the lattices $\Lambda' \subset \mathbb{C}$ satisfying $\Lambda' \supset \Lambda$ and $[\Lambda' : \Lambda] = p$ are:

   - $\mathbb{Z}\frac{\omega_1 + b\omega_2}{p} + \mathbb{Z}\omega_2$ with $b = 0, 1 \ldots, p-1$
   - $\mathbb{Z}\omega_1 + \mathbb{Z}\frac{\omega_2}{p}$,

   and that these constitute $p+1$ distinct lattices.

   (b) Provide all details to the claim made in the first sentence of the proof of Proposition 4.4 of the notes.

2. Let p be a prime and consider the lattice $\Lambda := \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ where $\omega_1, \omega_2 \in \mathbb{C}^*$ and $\omega_1/\omega_2 \notin \mathbb{R}$.

   (a) Show that there are exactly $p^2 + p + 1$ lattices $\Lambda' \subset \mathbb{C}$ satisfying $\Lambda' \supset \Lambda$ and $[\Lambda' : \Lambda] = p^2$, and give a list of these.

   (b) Try to generalize part (a) (e.g. replace $[\Lambda' : \Lambda] = p^2$ by $[\Lambda' : \Lambda] = p^k$ with $k \in \mathbb{Z}_{>0}$).

3. Calculate the matrix of the Hecke operator T_2 on the space $\mathrm{S}_{24}(\mathrm{SL}_2(\mathbb{Z}))$ with respect to a basis of your choice. Show that the characteristic polynomial of T_2 is $x^2 - 1080x - 20468736$. (You may use a computer, but not a package in which this exercise can be solved with a one-line command.)

4. Consider the formal (so we do not worry about convergence) generating function of the Hecke operators T_n on $M_k(\Gamma_1(N))$

   $$g(s) := \sum_{n=1}^{\infty} T_n n^{-s}.$$

   Deduce the following formal product expansion (over all primes p):

   $$g(s) = \prod_p \left(\mathrm{id} - T_p p^{-s} + \langle p \rangle p^{k-1-2s}\right)^{-1}.$$

5. Visit the SageMathCloud on `https://cloud.sagemath.com/` and create an account.