1. Let $p$ be a prime and consider the lattice $\Lambda := \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ where $\omega_1, \omega_2 \in \mathbb{C}^*$ and $\omega_1/\omega_2 \notin \mathbb{R}$.
   
   (a) Show that the lattices $\Lambda' \subset \mathbb{C}$ satisfying $\Lambda' \supset \Lambda$ and $[\Lambda' : \Lambda] = p$ are:
   
   - $\mathbb{Z}\omega_1 + b\omega_2$ with $b = 0, 1, \ldots, p - 1$
   - $\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2/p$,
   
   and that these constitute $p + 1$ distinct lattices.
   
   (b) Provide all details to the claim made in the first sentence of the proof of Proposition 4.4 of the notes.

2. Let $p$ be a prime and consider the lattice $\Lambda := \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ where $\omega_1, \omega_2 \in \mathbb{C}^*$ and $\omega_1/\omega_2 \notin \mathbb{R}$.
   
   (a) Show that there are exactly $p^2 + p + 1$ lattices $\Lambda' \subset \mathbb{C}$ satisfying $\Lambda' \supset \Lambda$ and $[\Lambda' : \Lambda] = p^2$, and give a list of these.
   
   (b) Try to generalize part (a) (e.g. replace $[\Lambda' : \Lambda] = p^2$ by $[\Lambda' : \Lambda] = pk$ with $k \in \mathbb{Z}_{>0}$).

3. Calculate the matrix of the Hecke operator $T_2$ on the space $S_{24}(\text{SL}_2(\mathbb{Z}))$ with respect to a basis of your choice. Show that the characteristic polynomial of $T_2$ is $x^2 - 1080x - 20468736$. (You may use a computer, but not a package in which this exercise can be solved with a one-line command.)

4. Consider the formal (so we do not worry about convergence) generating function of the Hecke operators $T_n$ on $M_k(\Gamma_1(N))$ 
   
   $$g(s) := \sum_{n=1}^{\infty} T_n n^{-s}.$$ 

   Deduce the following formal product expansion (over all primes $p$):
   
   $$g(s) = \prod_p (\text{id} - T_p p^{-s} + (p)p^{k-1-2s})^{-1}.$$ 