Modular Forms: Problem Sheet 9

19 April 2016

1. Let $k, N \in \mathbb{Z}_{>0}$, and let χ be a Dirichlet character modulo N.

   (a) For $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, denote by d_γ the lower-right entry of γ. Show that

   $$M_k(N, \chi) = \{f \in M_k(\Gamma_1(N)) : f|_k\gamma = \chi(d_\gamma)f \text{ for all } \gamma \in \Gamma_0(N)\}$$

   and

   $$S_k(N, \chi) = \{f \in S_k(\Gamma_1(N)) : f|_k\gamma = \chi(d_\gamma)f \text{ for all } \gamma \in \Gamma_0(N)\}.$$

   (b) Let 1_N denote the trivial character modulo N. Show that

   $$M_k(N, 1_N) = M_k(\Gamma_0(N)) \quad \text{and} \quad S_k(N, 1_N) = S_k(\Gamma_0(N)).$$

2. Let $k \in \mathbb{Z}_{>0}$, let $f \in \mathrm{M}_k(\mathrm{SL}_2(\mathbb{Z}))$ be an eigenform, normalised such that $a_1(f) = 1$, and let p be a prime number. Let $\alpha, \beta \in \mathbb{C}$ be the roots of the polynomial $t^2 - a_p(f)t + p^{k-1}$.
   You may use without proof that $a_p(f)$ is real.

   (a) Prove the formula

   $$a_{p^r}(f) = \sum_{j=0}^{r} \alpha^j \beta^{r-j} \quad \text{for all } r \geq 0.$$

   (b) Show that the following conditions are equivalent: (1) $|a_p(f)| \leq 2p^{(k-1)/2}$; (2) α and β are complex conjugates of absolute value $p^{(k-1)/2}$.

   (c) Show that if the equivalent conditions of part (b) hold for all prime numbers p, then the q-expansion coefficients of f satisfy the bound

   $$|a_n(f)| \leq \sigma_0(n)n^{(k-1)/2} \quad \text{for all } n \geq 1,$$

   where $\sigma_0(n)$ is the number of (positive) divisors of n.

   Note: If f is a cusp form, then the conditions of part (b) do in fact hold. This follows from two very deep theorems proved by P. Deligne in 1968 and 1974.

3. Let $k, N \in \mathbb{Z}_{>0}$, and let $f \in \mathrm{S}_k(\Gamma_1(N))$ be a normalised Hecke eigenform with q-expansion $\sum_{n=1}^{\infty} a_n q^n$ (at the cusp ∞) and character $\chi\colon (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$. Prove the identity

   $$\overline{a_m} = \chi(m)^{-1} a_m \quad \text{for all } m \geq 1 \text{ with } \gcd(m, N) = 1.$$

   Deduce that the quantity $a_m^2/\chi(m)$ is real for all $m \geq 1$ with $\gcd(m, N) = 1$.

4. Play around with the functions in SageMath for some of your favorite choices of congruence subgroup, modular forms space, etc.

5. In this exercise you are supposed to make (partly) use of SageMath. Please attach your code when handing in the exercise, preferably by using the print button on the top right in the SageMath worksheet with which you can generate a pdf file. (Any comments can also be written down in the SageMath worksheet.)

   (a) Compute a basis for $S_2(\Gamma_0(26))$.

   (b) Find a basis B for $S_2(\Gamma_0(26))$ such that all the basis elements are eigenvectors for the Hecke operator T_2.

   (c) Check that all the basis elements in B are eigenvectors for the Hecke operators T_n with $1 \leq n \leq 101$.
   Remark: in fact, these basis elements are Hecke eigenforms.

   (d) The following equation defines a curve in the plane:

   $$E_1 : y^2 + xy + y = x^3. \tag{1}$$

   (It is a so-called affine equation for an elliptic curve.) Tell SageMath about this object by typing `E1=EllipticCurve([1,0,1,0,0])` .
   For a prime number p, the number of solutions to (1) with $x, y \in \mathbb{F}_p := \mathbb{Z}/p\mathbb{Z}$ *plus one* is denoted by $N_p(E_1)$, i.e.

   $$N_p(E_1) = \#\{(x,y) \in \mathbb{F}_p^2 : y^2 + xy + y = x^3\} + 1.$$

   (The $+1$ is there, because it is more natural to count solutions in the so-called *projective closure*, which boils down to one more solution 'at infinity'.) SageMath can compute these numbers by typing `E1.Np(prime)` where `prime` is some prime number (e.g. `prime=79`).
   Find an explicit relation between $N_p(E_1)$ and $a_p(f)$ for one of the basis elements f in B and all primes $p < 1000$.

   (e) Now consider

   $$E_2 : y^2 + xy + y = x^3 - x^2 - 3x + 3,$$

   which is give in SageMath by `E2=EllipticCurve([1,-1,1,-3,3])` .
   Similarly as in (d), we set

   $$N_p(E_2) = \#\{(x,y) \in \mathbb{F}_p^2 : y^2 + xy + y = x^3 - x^2 - 3x + 3\} + 1$$

   an this can be computed in SageMath by typing `E2.Np(prime)` .
   Find an explicit relation between $N_p(E_2)$ and $a_p(g)$ for one of the basis elements g in B and all primes $p < 1000$.

   Note: This illustrates the *modularity* of the elliptic curves E_1 and E_2.