Practice Exam	**Modular Forms**
Mastermath	**May 17, 2016 (three hours)**

• The use of books, lecture notes, calculators, etc. is *not* allowed.
• If you are unable to answer a subitem, you are still allowed to use the result in the remainder of the exercise.

Note: Throughout this exam, N and k denote positive integers.

1. (a) Define the notion of a *congruence subgroup* of $\mathrm{SL}_2(\mathbb{Z})$, and of the *level* of a congruence subgroup. Define the congruence subgroups $\Gamma(N)$, $\Gamma_0(N)$ and $\Gamma_1(N)$.

   (b) Let p be a prime number not dividing N, and let $\alpha = \left(\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix}\right)$. Show that the group

   $$\Gamma' = \Gamma_1(N) \cap (\alpha^{-1}\Gamma_1(N)\alpha)$$

   is a congruence subgroup, and determine the level of Γ'.

   (c) Describe the action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{P}^1(\mathbb{Q}) = \mathbb{Q} \cup \{\infty\}$, and show that for every $x \in \mathbb{P}^1(\mathbb{Q})$ there exists $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ such that $\gamma\infty = x$.

   (d) Let Γ be a congruence subgroup. Define the set $\mathrm{Cusps}(\Gamma)$ of *cusps* of Γ. Show that $\mathrm{SL}_2(\mathbb{Z})$ has exactly one cusp.

   (e) Let E be an elliptic curve over $\mathbb{Q}$, let $E(\mathbb{Q})$ be its Mordell–Weil group, and let $L(E, s)$ be the L-function of E. Recall that $E(\mathbb{Q})$ is finitely generated by the Mordell–Weil theorem, and that $L(E, s)$ can be continued to an analytic function on the whole complex plane. What does the conjecture of Birch and Swinnerton-Dyer predict about the relation between $E(\mathbb{Q})$ and $L(E, s)$?

2. (a) Show that the space $\mathrm{S}_{12}(\mathrm{SL}_2(\mathbb{Z}))$ contains a unique normalised eigenform Δ.

   In the lectures, certain maps on (and between) the spaces $\mathrm{S}_k(\Gamma_1(N))$ were introduced. You may use without proof that these are also defined for $\mathrm{S}_k(\Gamma_0(N))$, and that the relevant properties for this question are the same for $\Gamma_0(N)$ as for $\Gamma_1(N)$.

   (b) Show that the space $\mathrm{S}_{12}(\Gamma_0(6))$ has dimension at least 4.

   (c) Define the *Fricke operator* w_N on $\mathrm{S}_k(\Gamma_0(N))$ (also known as the Atkin–Lehner operator).

   In parts (d) and (e) of this question, the function $F\colon \mathbb{H} \to \mathbb{C}$ is defined as

   $$F(z) = \Delta(z)\Delta(5z).$$

   (d) Prove that F is a cusp form of weight 24 for the group $\Gamma_0(5)$, and compute the order of vanishing of F at the cusps 0 and ∞.

   (e) Prove that F is an eigenform for the Fricke operator w_5 on $\mathrm{S}_{24}(\Gamma_0(5))$, with eigenvalue 5^{12}.

3. (a) Define the *new subspace* $\mathrm{S}_k(\Gamma_1(N))_{\mathrm{new}}$ (also known as the space of *newforms*) of $\mathrm{S}_k(\Gamma_1(N))$.

   (b) Sketch a proof of the fact that the $\mathbb{C}$-vector space $\mathrm{S}_k(\Gamma_1(N))_{\mathrm{new}}$ admits a basis of simultaneous eigenforms for all Hecke operators T_m for $m \geq 1$ and all diamond operators $\langle d \rangle$ for $d \in (\mathbb{Z}/N\mathbb{Z})^\times$.
   (You do not have to reproduce long computations, but you should explain the main steps and ingredients of the proof.)

4. Let $f \in \mathrm{S}_k(\Gamma_1(N))$ be a primitive form, let f^* be the newform defined by $f^*(z) = \overline{f(-\bar{z})}$, and let η_f be the complex number such that $w_N f = \eta_f f^*$.

(a) Define the completed L-function $\Lambda(f, s)$.

(b) The *incomplete* Γ-*function* is defined by

$$\Gamma(s, x) = \int_x^\infty \exp(-t) t^s \frac{dt}{t} \quad \text{for } s \in \mathbb{C} \text{ and } x > 0.$$

Prove the formula

$$\begin{aligned}\Lambda(f, s) = N^{s/2} &\sum_{n=1}^\infty a_n (2\pi n)^{-s} \Gamma\left(s, \frac{2\pi n}{\sqrt{N}}\right) \\ &+ i^k \eta_f N^{-s/2} \sum_{n=1}^\infty \bar{a}_n (2\pi n)^{s-k} \Gamma\left(k - s, \frac{2\pi n}{\sqrt{N}}\right).\end{aligned}$$

(You may use without proof the identity

$$\Lambda(f, s) = N^{s/2} \int_{1/\sqrt{N}}^\infty f(it) t^s \frac{dt}{t} + i^k \eta_f N^{-s/2} \int_{1/\sqrt{N}}^\infty f^*(it) t^{k-s} \frac{dt}{t},$$

which follows from a formula given in the lecture by the assumption $a_0(f) = 0$.)

(c) Assume that k equals 2, and take $s = 1$. Prove the formula

$$\Lambda(f, 1) = \sum_{n=1}^\infty \left(a_n - \frac{\eta_f}{N} \bar{a}_n\right) \frac{\sqrt{N}}{2\pi n} \exp\left(-\frac{2\pi n}{\sqrt{N}}\right).$$

(This gives an efficient way to approximate $\Lambda(f, s)$ in the *critical point* $s = 1$, and hence the quantity $L(f, 1) = \frac{2\pi}{\sqrt{N}} \Lambda(f, 1)$, which is important for the conjecture of Birch and Swinnerton-Dyer.)

Maximum scores per subitem			
1a: 5	2a: 6	3a: 6	4a: 6
1b: 6	2b: 4	3b: 10	4b: 6
1c: 6	2c: 4		4c: 8
1d: 6	2d: 6		
1e: 5	2e: 6		
Maximum total = 90			
Mark = 1 + Total/10			