# *Possible Answers to*
# **Exam Advanced Logic**

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**1.** **(a)** *Define what it means that a modal formula is valid in a class of frames. (Give your answer in terms of the notion of truth of a formula in a point of a model.)* *(4 pt)*

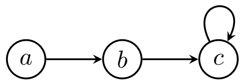
A modal formula $\varphi$ is valid in a frame-class $\mathcal{C}$ if it is valid in all frames of $\mathcal{C}$: $\mathfrak{F} \vDash \varphi$ for all $\mathfrak{F} \in \mathcal{C}$. In turn, $\varphi$ is valid in a frame $\mathfrak{F} = (W, R)$ if for all valuations $V : \{p, q, \ldots\} \to 2^W$ (mapping proposition variables to subsets of $W$) on the frame, and for all points $x \in W$ of the frame, we have $\mathfrak{F}, V, x \vDash \varphi$, where $\mathfrak{F}, V, x \vDash \varphi$ denotes the truth of $\varphi$ in point $x$ of the model $\mathfrak{F}, V$.

**(b)** *Prove that the formula $\Box\Diamond p \to \Box\Diamond\Box\Diamond p$ is valid in all transitive frames.* *(8 pt)*

Let $\mathfrak{F} = (W, R)$ be an arbitrary transitive frame. Let $V$ be an arbitrary valuation on $\mathfrak{F}$, and $x$ an arbitrary point in the model $(\mathfrak{F}, V)$. Assume $x \vDash \Box\Diamond p$. In order to show $x \vDash \Box\Diamond\Box\Diamond p$, we let $y$ be an arbitrary $R$-successor of $x$, and prove that $y \vDash \Diamond\Box\Diamond p$. From the assumption $x \vDash \Box\Diamond p$ it follows that $y \vDash \Diamond p$, and so there exists a point $z$ with $Ryz$ (and $z \vDash p$). It suffices to prove $z \vDash \Box\Diamond p$ (as then $y \vDash \Diamond\Box\Diamond p$), so we let $u$ with $Rzu$ be arbitrary, and show $u \vDash \Diamond p$. By transitivity of $R$ we obtain $Rxz$ (from $Rxy$ and $Ryz$), and also $Rxu$ (from $Rxz$ and $Rzu$). Hence $u \vDash \Diamond p$ follows from the assumption $x \vDash \Box\Diamond p$.

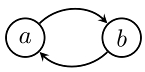
**(c)** *Can the formula of the previous item also be valid in a non-transitive frame? Prove your answer.* *(5 pt)*

Yes that is possible. For example, the formula is valid in the following non-transitive frame:



To see that $\Box\Diamond p \to \Box\Diamond\Box\Diamond p$ is indeed valid in this frame, note that its transition relation $R$ is functional and satisfies $R^2 = R^4$, hence $x \models \Box\Diamond p$ if and only if $x \models \Box\Diamond\Box\Diamond p$, for every $x \in \{a, b, c\}$.

Likewise so, in the following frame (non-transitive since $Rab$ and $Rba$ but $\neg Raa$):



which is again a functional frame whose transition relation $R$ is such that $R^2 = R^4$.

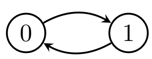
**(d)** *Show that $\mathfrak{F} \models \Diamond r \to \Box r$ implies $\mathfrak{F} \models (\Box p \to \Box q) \to \Box(p \to q)$, for all frames $\mathfrak{F}$. (Here $p, q, r$ are proposition variables.)* *(8 pt)*

Let $\mathfrak{F} = (W, R)$ be an arbitrary frame where $\Diamond r \to \Box r$ is valid (note that this formula forces that $R$ is partially functional, i.e., every point has at most one $R$-successor). Let $V$ be an arbitrary valuation on $\mathfrak{F}$, let $x$ be an arbitrary point of the model $(\mathfrak{F}, V)$, and assume $x \models \Box p \to \Box q$. In order to show that $x \models \Box(p \to q)$, we consider an arbitrary $R$-successor $y$ of $x$, and show $y \models p \to q$. So we assume $y \models p$ (and show $y \models q$). Then we have that $x \models \Diamond p$.[1] Moreover, as we know that $\Diamond r \to \Box r$ is valid in $\mathfrak{F}$ and validity is closed under substitution, we also know that $x \models \Diamond p \to \Box p$. Hence we get $x \models \Box p$, and using the assumption $x \models \Box p \to \Box q$, also $x \models \Box q$. This in turn means that $y \models q$.

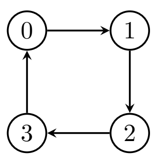
**2.** For $n = 1, 2, 3, \ldots$, let the 'looping frame' $\mathcal{L}_n = (W_n, R_n)$ be defined by

$$W_n = \{0, \ldots, n-1\}$$
$$R_n = \{\, (k, k') \mid k' = k+1 \text{ if } k+1 < n \text{ and } k' = 0 \text{ otherwise} \,\}$$

**(a)** *Draw the frames $\mathcal{L}_2$ and $\mathcal{L}_4$.* *(2 pt)*



$\mathcal{L}_2$



$\mathcal{L}_4$

**(b)** *Give a modal formula that distinguishes frame $\mathcal{L}_2$ from $\mathcal{L}_4$, that is, a formula $\varphi$ such that $\mathcal{L}_2 \models \varphi$ and $\mathcal{L}_4 \not\models \varphi$. Prove your answer.* *(8 pt)*

An example of a formula that is valid in $\mathcal{L}_2$ but not in $\mathcal{L}_4$ is $p \leftrightarrow \Diamond\Diamond p$.[1] To see that $\mathcal{L}_2 \models p \leftrightarrow \Diamond\Diamond p$ (more generally $\mathcal{L}_n \models p \leftrightarrow \Diamond^n p$ for all $n > 0$), observe that $R_2^2 = (R_2; R_2) = \Delta$ (in general $R_n^n = \Delta$), where $\Delta$ denotes the identity relation on $W_2$, $\Delta = \{(x, x) \mid x \in W_2\}$. Let $x \in W_2 = \{0, 1\}$, and $V$ an arbitrary valuation on $\mathcal{L}_2$. Then in the model $\mathcal{M} = (\mathcal{L}_2, V)$, it clearly holds that $x \models p$ if and only if $x \models \Diamond\Diamond p$.

To see that $\mathcal{L}_4 \not\models p \leftrightarrow \Diamond\Diamond p$ (more generally $\mathcal{L}_m \not\models p \leftrightarrow \Diamond^n p$ for all $n, m > 0$ with $n < m$), consider for example the valuation $V$ on $\mathcal{L}_4$ defined by $V(p) = \{0\}$. In the model $(\mathcal{L}_4, V)$ we then have $0 \models p$, but $0 \not\models \Diamond\Diamond p$ since $2 \not\models p$ (for $n < m$, in the model $(\mathcal{L}_m, V)$, with $V(p) = \{0\}$, we similarly get $0 \not\models \Diamond^n p$ since $n \not\models p$).

*For questions **(c)** and **(d)** you have to define a bisimulation, but reporting on the verification of the bisimulation conditions is not required.*

**(c)** *Let $\mathcal{M}_2$ be some model based on $\mathcal{L}_2$. Define a model $\mathcal{M}_4$ based on $\mathcal{L}_4$ such that $\mathcal{M}_2, 0 \leftrightarroweq \mathcal{M}_4, 0$.* *(6 pt)*

Let $\mathcal{M}_2 = (\mathcal{L}_2, V_2)$ for some valuation $V_2$. We define a valuation $V_4$ on $\mathcal{L}_4$ by

$$V_4(p) = \{x \mid (x \bmod 2) \in V_2\}$$

(So $V_4(p) = \varnothing$ if $V_2(p) = \varnothing$, $V_4(p) = \{0, 2\}$ if $V_2(p) = \{0\}$, $V_4(p) = \{1, 3\}$ if $V_2(p) = \{1\}$, and $V_4(p) = W_4$ if $V_2(p) = W_2$.) Then the relation $E \subseteq W_2 \times W_4$ given by $E = \{(0, 0), (0, 2), (1, 1), (1, 3)\}$ is a bisimulation between the models $\mathcal{M}_2$ and $\mathcal{M}_4 = (\mathcal{L}_4, V_4)$. As $(0, 0) \in E$, we conclude $\mathcal{M}_2, 0 \leftrightarroweq \mathcal{M}_4, 0$.

**(d)** *Let $\mathcal{M}_3$ be some model based on $\mathcal{L}_3$. Define an acyclic model $\mathcal{N}$ bisimilar to $\mathcal{M}_3$.* *(6 pt)*

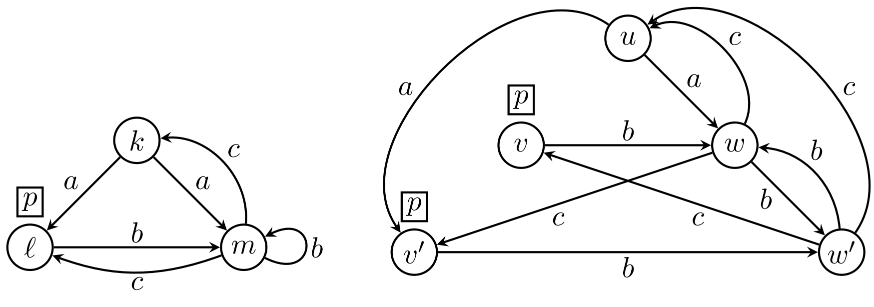
Every model is bisimilar to its tree unfolding (and trees are acyclic). Let $\mathcal{M}_3 = (\mathcal{L}_3, V)$. The tree unfolding of $\mathcal{M}_3$ is the model $\mathcal{N} = (\mathbb{N}, S, U)$ where $S = \{(n, n+1) \mid n \in \mathbb{N}\}$, and with valuation $U$ defined for all variables $p$, and all $n \in \mathbb{N}$ by

$$\begin{aligned} 3n \in U(p) &\iff 0 \in V(p) \\ 3n + 1 \in U(p) &\iff 1 \in V(p) \\ 3n + 2 \in U(p) &\iff 2 \in V(p) \end{aligned}$$

[1] Other examples are $p \to \Diamond\Diamond p$, $\Diamond\Diamond p \to p$, or any variation of them obtained by replacing $\Diamond$'s with $\Box$'s (whose meanings coincide on these functional frames).

Then the relation $\{(0, 3n), (1, 3n+1), (2, 3n+2)\}$ is a bisimulation between the models $\mathcal{M}_3$ and $\mathcal{N}$.

**3.** *Consider the* $\{a, b, c\}$*-models* $\mathcal{A}$ *and* $\mathcal{B}$ *defined by:*




$\mathcal{A}$ $\qquad\qquad$ $\mathcal{B}$

**(a)** *Is there a modal formula that distinguishes state $k$ in model $\mathcal{A}$ from state $u$ in model $\mathcal{B}$ ? Prove your answer.* *(8 pt)*

No such formula exists, because, as we will prove, states $k$ and $u$ are bisimilar, and we know that modal truth is invariant under bisimilarity, that is, if $x$ is bisimilar to $x'$, then $x \vDash \varphi$ if and only if $x' \vDash \varphi$.

We show that the relation

$$Z = \{(k, u), (\ell, v), (\ell, v'), (m, w), (m, w')\}$$

is a bisimulation.

For $i \in \{a, b, c\}$, we denote by $R_i^{\mathcal{A}}$, the transition relation labeled of model $\mathcal{A}$ labeled $i$. Likewise so for model $\mathcal{B}$.

Clearly all $Z$-related points have the same atomic information.

To see that $Z$ satisfies the zig-condition, we check for every $Z$-link $(x, x') \in Z$, that for every step $R_i^{\mathcal{A}}xy$ , there exists a step $R_i^{\mathcal{B}}x'y'$ and $(y, y') \in Z$; we indicate this by $\begin{pmatrix} x & x' \\ y & y' \end{pmatrix}_i$. The 10 diagrams we thus find are:

$$\begin{pmatrix} k & u \\ \ell & v' \end{pmatrix}_a \begin{pmatrix} k & u \\ m & w \end{pmatrix}_a \begin{pmatrix} \ell & v \\ m & w \end{pmatrix}_b \begin{pmatrix} \ell & v' \\ m & w' \end{pmatrix}_b \begin{pmatrix} m & w \\ m & w' \end{pmatrix}_b \begin{pmatrix} m & w \\ k & u \end{pmatrix}_c \begin{pmatrix} m & w \\ \ell & v' \end{pmatrix}_c \begin{pmatrix} m & w' \\ m & w \end{pmatrix}_b \begin{pmatrix} m & w' \\ k & u \end{pmatrix}_c \begin{pmatrix} m & w' \\ \ell & v \end{pmatrix}_c$$

Checking the zag-condition proceeds in a similar way.

**(b)** *Let $\widehat{\mathcal{A}}$ be the PDL-extension of model $\mathcal{A}$. Compute the transition relation $\widehat{R}_\alpha$ corresponding to the PDL-program $\alpha = \mathsf{while}\ \neg p\ \mathsf{do}\ a \cup bc$.* *(8 pt)*

We know that $\mathsf{while}\ \varphi\ \mathsf{do}\ \pi$ is syntax for $(\varphi? \,;\, \pi)^* \,;\, \neg\varphi?$. So, in order to compute $\widehat{R}_\alpha$ we compute the transition relation corresponding to the program $(\neg p?;(a\cup bc))^*;\neg\neg p?$, in a compositional way as follows; here, we use a second abbreviation

$$\beta = \neg p? \,;\, (a \cup bc)$$

so that $\alpha = \beta^* \,;\, p?$ (of course the test $\neg\neg p?$ is equivalent to $p?$ (i.e., have the same transition relation in all PDL models)).

$$\begin{aligned}
R_a &= \{(k,\ell),(k,m)\}\\
R_b &= \{(\ell,m),(m,m)\}\\
R_c &= \{(m,k),(m,\ell)\}\\
\widehat{R}_{bc} = R_b; R_c &= \{(\ell,k),(\ell,\ell),(m,k),(m,l)\}\\
\widehat{R}_{a\cup bc} = R_a \cup \widehat{R}_{bc} &= \{(k,l),(k,m),(\ell,k),(\ell,\ell),(m,k),(m,l)\}\\
\widehat{R}_{\neg\neg p?} = \widehat{R}_{p?} = \{(x,x) \mid x \vDash p\} &= \{(\ell,\ell)\}\\
\widehat{R}_{\neg p?} = \{(x,x) \mid x \nvDash p\} &= \{(k,k),(m,m)\}\\
\widehat{R}_\beta = \widehat{R}_{\neg p?}; \widehat{R}_{a\cup bc} &= \{(k,l),(k,m),(m,k),(m,l)\}\\
\widehat{R}_{\beta^*} = (\widehat{R}_\beta)^* &= \{(k,k),(k,l),(k,m),(\ell,\ell),(m,k),(m,l),(m,m)\}\\
\widehat{R}_\alpha = \widehat{R}_{\beta^*}; \widehat{R}_{p?} &= \{(k,\ell),(\ell,\ell),(m,\ell)\}
\end{aligned}$$

**(c)** *Determine whether the PDL-formula $[\alpha]p \rightarrow \langle b\rangle\top$ globally holds in $\widehat{\mathcal{A}}$. Prove your answer.* *(4 pt)*

All states in the model have precisely one $\widehat{R}_\alpha$-successor, namely state $\ell$ where $p$ holds. Hence $[\alpha]p$ is true at all states. However not all states can do a $R_b$-step: state $k$ is blind with respect to $R_b$. So we find that $k \nvDash \langle b\rangle\top$, and so, in combination with $k \vDash [\alpha]p$, we obtain $k \nvDash [\alpha]p \rightarrow \langle b\rangle\top$. Hence $[\alpha]p \rightarrow \langle b\rangle\top$ is not globally valid in $\widehat{\mathcal{A}}$.

**4.** *System $T$ is the extension of the minimal modal logic $K$ with the axiom of veridicality (if something is known, it is true). System $S4$ extends $T$ with the axiom of positive introspection; $S5$ extends $S4$ with the axiom of negative introspection. Assume there are $n \geq 2$ agents.*

**(a)** *Prove or disprove the following epistemic claims:*

(i) $\vdash_T K_1K_2p \to K_2\neg K_1\neg p$ *(5 pt)*

We give a derivation in system $T$:

$$
\begin{array}{rll}
1. & K_1p \to p & \text{(Axiom A1)} \\
2. & K_1\neg p \to \neg p & \text{(Substitution, 1)} \\
3. & (A \to \neg B) \to (B \to \neg A) & \text{(Tautology)} \\
4. & (K_1\neg p \to \neg p) \to (p \to \neg K_1\neg p) & \text{(Substitution, 3)} \\
5. & p \to \neg K_1\neg p & \text{(Modus Ponens, 4, 2)} \\
6. & K_2(p \to \neg K_1\neg p) & \text{(Necessation, 5)} \\
7. & K_2(p \to q) \to (K_2p \to K_2q) & \text{(Distribution Axiom)} \\
8. & K_2(p \to \neg K_1\neg p) \to (K_2p \to K_2\neg K_1\neg p) & \text{(Substitution, 7)} \\
9. & K_2p \to K_2\neg K_1\neg p & \text{(Modus Ponens, 8, 6)} \\
10. & K_1K_2p \to K_2p & \text{(Substitution, 1)} \\
11. & (A \to B) \to ((B \to C) \to (A \to C)) & \text{(Tautology)} \\
12. & 10 \to (9 \to (K_1K_2p \to K_2\neg K_1\neg p)) & \text{(Subtitution, 11)} \\
13. & 9 \to (K_1K_2p \to K_2\neg K_1\neg p) & \text{(Modus Ponens, 12, 10)} \\
14. & K_1K_2p \to K_2\neg K_1\neg p & \text{(Modus Ponens, 13, 9)}
\end{array}
$$

(Alternatively (and much easier) we could have shown that the formula is valid in all frames $(W, \{R_1, R_2, \ldots, R_n\})$ where the $R_i$ are reflexive, and use the completeness theorem for system $T$ to conclude that the formula is derivable in $T$.)

(ii) $\vdash_{S4} \neg K_1K_1p \to K_1\neg K_1p$ *(5 pt)*

By the completeness theorem for system $S4$, we know that $\vdash_{S4} \varphi$ if and only if $\mathfrak{F} \models \varphi$ for all frames $\mathfrak{F}$ that are reflexive and transitive. Hence, to show that $\neg K_1K_1p \to K_1\neg K_1p$ is *not* derivable in S4 it suffices to give a reflexive, transitive frame $\mathfrak{F} = (W, R_1)$ where the formula is *not* valid (in turn proved by giving a valuation $V$ and a point $x$ of the frame such that $\mathfrak{F}, V, x \not\models \varphi$). Consider the frame $\mathfrak{F} = (W, R_1)$ with $W = \{a, b\}$ and $R_1 = \{(a, a), (a, b), (b, b)\}$ which is clearly reflexive and transitive. Take the valuation $V$ given by $V(p) = \{b\}$. Then, in the model $(\mathfrak{F}, V)$, we have that $a \not\models p$, and hence $a \not\models K_1p$ since $R_1aa$. Then also $a \not\models K_1K_1p$, by $a \not\models K_1p$ and $R_1aa$. Hence $a \models \neg K_1K_1p$. On the other hand, we have $b \models p$, and so $b \models K_1p$ since $b$ is the only $R_1$-successor of $b$. Hence $b \not\models \neg K_1p$, and therefore $a \not\models K_1\neg K_1p$ follows from $R_1ab$. Combining $a \models \neg K_1K_1p$ with $a \not\models K_1\neg K_1p$, we obtain $a \not\models \neg K_1K_1p \to K_1\neg K_1p$.

(iii) $\vdash_{S5} \neg K_2K_2p \to K_2\neg K_2p$ *(5 pt)*

We give a derivation in $S5$:

| | | |
|---|---|---|
| 1. | $K_2p \to K_2K_2p$ | (Axiom A2) |
| 2. | $(A \to B) \to (\neg B \to \neg A)$ | (Tautology) |
| 3. | $(K_2p \to K_2K_2p) \to (\neg K_2K_2p \to \neg K_2p)$ | (Substitution, 2) |
| 4. | $\neg K_2K_2p \to \neg K_2p$ | (Modus Ponens, 3, 1) |
| 5. | $\neg K_2p \to K_2\neg K_2p$ | (Axiom A3) |
| 6. | $(A \to B) \to ((B \to C) \to (A \to C))$ | (Tautology) |
| 7. | $4 \to (5 \to (\neg K_2K_2p \to K_2\neg K_2p))$ | (Substitution, 8) |
| 8. | $5 \to (\neg K_2K_2p \to K_2\neg K_2p)$ | (Modus Ponens, 7, 4) |
| 9. | $\neg K_2K_2p \to K_2\neg K_2p$ | (Modus Ponens, 8, 5) |

(Alternatively we could have shown that the formula is valid in all frames $(W, \{R_1, R_2, \ldots, R_n\})$, where the $R_i$ are equivalence relations, and use the completeness theorem for system $S5$ to conclude that the formula is derivable in $S5$.)

**(b)** *Show that validity of the axiom $Cp \to ECp$ in an epistemic frame forces that the frame has the property $R_E; R_C \subseteq R_C$. (Recall that $R_E; R_C$ denotes the relational composition of $R_E$ and $R_C$.)*[2] *(8 pt)*

Let $\mathfrak{F}$ be any frame with relations $R_E$ and $R_C$; let $W$ denote its domain. Contrapositively, assume that $R_E; R_C \not\subseteq R_C$, so that there are points $x, z \in W$ such that $(x, z) \in (R_E; R_C)$ and $(x, z) \notin R_C$. The assumption $(x, z) \in (R_E; R_C)$ means that $(x, y) \in R_E$ and $(y, z) \in R_C$, for some $y \in W$. Our goal is to prove that $Cp \to ECp$ is not valid in $\mathfrak{F}$, and to this end we want a model on $\mathfrak{F}$ so that $x \vDash Cp$ and $x \nvDash ECp$. In other words, we need a valuation $V$ that makes $p$ true in all $R_C$-successors of $x$, and at the same time we need one $R_E$-successor of $x$ where $Cp$ fails; for the latter we naturally choose point $y$ where $Cp$ fails to hold if we define $V$ such that $z \notin V(p)$. The two demands we have for $V$ are not conflicting because we know $(x, z) \notin R_C$. Thus we define the desired valuation $V$ on $\mathfrak{F}$ by putting $V(p) = \{s \in W \mid (x, s) \in R_C\}$.[3] And

[2] This question is not phrased clear enough. The intention was to ask to prove that $\mathfrak{F} \vDash Cp \to ECp$ implies $R_E; R_C \subseteq R_C$ in all frames $\mathfrak{F}$ with relations $R_E$ and $R_C$ used for interpreting the box modalities $E$ and $C$, *without any further information on $R_E$ and $R_C$*. However, one may argue that the definition of epistemic frames includes that $R_C = (R_E)^+$, and then $R_E; R_C \subseteq R_C$ follows trivially. The question is thus better rephrased into: *Let $I$ be an index set. Show that for all $i, j \in I$, validity of the axiom $[j]p \to [i][j]p$ in an $I$-frame $\mathfrak{F} = (W, \{R_k \mid k \in I\})$ forces that $\mathfrak{F}$ has the property $R_i; R_j \subseteq R_j$.*

[3] This is the minimal valuation that works; a maximal one would be $W \setminus \{z\}$ so that $p$ holds everywhere except for point $z$.

we check that in the model $(\mathfrak{F}, V)$ we indeed have $x \vDash Cp$ and $x \nvDash ECp$, due to $y \nvDash Cp$ and $(x, y) \in R_E$, with $y \nvDash Cp$ in turn due to $z \nvDash p$ and $(y, z) \in R_C$.