Possible Answers to
Exam Advanced Logic

VU University Amsterdam, 26 March 2014, 15:15–18:00

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1. (a) *Define what it means that a modal formula is globally true in a model. (3 pt)*

A formula φ is globally true in a model $\mathcal{M} = (W, R, V)$, which we denote by $\mathcal{M} \models \varphi$, if φ is true in all of its points, that is,

$$\mathcal{M} \models \varphi \iff \mathcal{M}, w \models \varphi, \text{ for all } w \in W\,.$$

(b) *Define what it means that a modal formula is valid in a frame.* *(3 pt)*

A formula φ is valid in a frame $\mathcal{F} = (W, R)$, which we denote by $\mathcal{F} \models \varphi$, if φ is globally true in all models based on $\mathcal{F}$, that is,

$$\mathcal{F} \models \varphi \iff (\mathcal{F}, V) \models \varphi, \text{ for all valuations } V : \Omega \to \mathbf{2}^W\,,$$

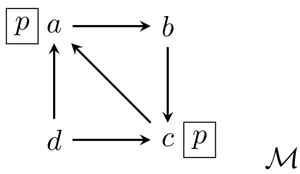
where Ω is a set of propositional variables containing the variables that occur in φ.

Consider the frame $\mathcal{F} = (W, R)$ with W and R given by

$$W = \{a, b, c, d\} \qquad R = \{(a, b), (b, c), (c, a), (d, a), (d, c)\}$$

and the model $\mathcal{M} = (\mathcal{F}, V)$ with valuation V defined by $V(p) = \{a, c\}$.

(c) *Give a graphical representation of $\mathcal{M}$.* *(2 pt)*

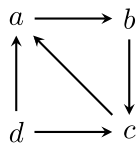


(d) *Prove that $p \to \Box\Box\Box p$ is globally true in $\mathcal{M}$, but not valid in $\mathcal{F}$.* *(4+4 pt)*

- To prove that $\mathcal{M} \models p \to \Box\Box\Box p$, we have to show $\mathcal{M}, x \models p \to \Box\Box\Box p$ for all $x \in \{a, b, c, d\}$. In points b and d the implication $p \to \Box\Box\Box p$ is trivially true because they do not satisfy p. To see that $\mathcal{M}, a \models \Box\Box\Box p$ we note that[1] $R^3[a] = \{a\}$ and $\mathcal{M}, a \models p$ (a is the only third R-successor of a), or, step by step: $\mathcal{M}, c \models \Box p$ (by $R[c] = \{a\}$ and $\mathcal{M}, a \models p$), hence $\mathcal{M}, b \models \Box\Box p$ (by $R[b] = \{c\}$), and hence $\mathcal{M}, a \models \Box\Box\Box p$ (by $R[a] = \{b\}$). Likewise we see that $\mathcal{M}, c \models \Box\Box\Box p$ since also $R^3[c] = \{c\}$ and $\mathcal{M}, c \models p$. So both a and c satisfy $\Box\Box\Box p$, and so they satisfy $p \to \Box\Box\Box p$. Thus we have seen that $\mathcal{M}, x \models p \to \Box\Box\Box p$ for all points x of the model, an we conclude $\mathcal{M} \models p \to \Box\Box\Box p$.
- To prove that $\mathcal{F} \not\models p \to \Box\Box\Box p$, we have find a valuation V' on $\mathcal{F}$ and a point $x \in W$ such that $\mathcal{F}, V', x \not\models p \to \Box\Box\Box p$.

  We take $V'(p) = \{d\}$. Then, clearly $\mathcal{F}, V', d \models p$ and $\mathcal{F}, V', d \not\models \Box\Box\Box p$ because $b \in R^3[d]$ and $\mathcal{F}, V', b \not\models p$.

(e) *Prove that for any formula φ, the formula $\Box\varphi \leftrightarrow \Box\Box\Box\Box\varphi$ is valid in $\mathcal{F}$.* *(8 pt)*



$\mathcal{F}$

This follows from the observation that, in the frame $\mathcal{F}$, $R^4 = R$, and the general fact that $\mathcal{G} \models [\alpha]p \leftrightarrow [\beta]p$ follows from $R_\beta = R_\alpha$, for all frames $\mathcal{G}$ with relations R_α and R_β. A more ad hoc proof goes as follows:

We show that $\mathcal{F} \models \Box p \leftrightarrow \Box\Box\Box\Box p$. Then $\mathcal{F} \models \Box\varphi \leftrightarrow \Box\Box\Box\Box\varphi$, with φ an arbitrary formula, follows since validity is closed under substitution, that is, for all frames $\mathcal{G}$, modal formulas ψ and substitutions σ, if $\mathcal{G} \models \psi$ then $\mathcal{G} \models \psi^\sigma$.

For $x \in \{a, b, c\}$ we write x' to denote the (unique) y such that Rxy; so $a' = b$, $b' = c$, and $c' = a$; clearly we have $x''' = x$. For $x \in \{a, b, c\}$ we see that

$$\mathcal{N}, x \models \Box\Box\Box\Box p \iff \mathcal{N}, x' \models \Box\Box\Box p$$

[1] Recall that we use the notation $R[x]$ for the set of R-successors of x, i.e., $R[x] = \{y \mid Rxy\}$. Moreover $R^3 = R;R;R$ where $R;S$ denotes the relational composition of R and S, that is, $R;S = \{(x,y) \mid \exists u\,((x,u) \in R \text{ and } (u,y) \in S)\}$. By the way, instead of the semi-colon ';', many people use the symbol $\circ$.

$$
\begin{aligned}
&\iff \mathcal{N}, x'' \models \Box\Box p \\
&\iff \mathcal{N}, x''' \models \Box p \\
&\iff \mathcal{N}, x \models \Box p \\
&\iff \mathcal{N}, x' \models p
\end{aligned}
$$

For d we find

$$
\begin{aligned}
\mathcal{N}, d \models \Box\Box\Box\Box p &\iff \mathcal{N}, a \models \Box\Box\Box p \text{ and } \mathcal{N}, c \models \Box\Box\Box p \\
&\iff \mathcal{N}, a \models p \text{ and } \mathcal{N}, c \models p \\
&\iff \mathcal{N}, d \models \Box p
\end{aligned}
$$

Thus we have shown that $\mathcal{F}, U, x \models \Box\Box\Box\Box p$ iff $\mathcal{F}, U, x \models \Box p$ for all $x \in \{a, b, c, d\}$, i.e., that $\mathcal{F}, U \models \Box p \leftrightarrow \Box\Box\Box\Box p$.

Since U was an arbitrary valuation, we conclude that $\mathcal{F} \models \Box p \leftrightarrow \Box\Box\Box\Box p$.

2. (a) *Let I be an arbitrary index set, and let $i, j \in I$. Prove that the formula $p \to [i]\langle j\rangle p$ characterizes the class of I-frames $\mathcal{F} = (W, \{R_k \mid k \in I\})$ that satisfy the property $R_i \subseteq R_j^{-1}$.* *(10 pt)*

The multi-modal formula $p \to [i]\langle j\rangle p$ characterizes the property $R_i \subseteq R_j^{-1}$ if

$$
\mathcal{F} \models p \to [i]\langle j\rangle p \iff R_i \subseteq R_j^{-1}
$$

for all I-frames $\mathcal{F} = (W, \{R_k \mid k \in I\})$ and $i, j \in I$. So let $\mathcal{F}$ be an arbitrary I-frame, and let $i, j \in I$. We prove the two directions:

($\Rightarrow$) By contraposition. Assume that $R_i \not\subseteq R_j^{-1}$. We prove $\mathcal{F} \not\models p \to [i]\langle j\rangle p$. By the assumption there are (not necessarily distinct) points a and b such that $R_i ab$ and $\neg R_j ba$.. In order to show that $p \to [i]\langle j\rangle p$ is not valid in $\mathcal{F}$ we have to find a valuation V on $\mathcal{F}$ and a point x such that $\mathcal{M}, x \models p$ and $\mathcal{M}, x \not\models [i]\langle j\rangle p$.

We choose V to be such that p holds in a only, $V(p) = \{a\}$. Then in the model $\mathcal{M} = (\mathcal{F}, V)$ we have $\mathcal{M}, b \not\models \langle j\rangle p$ since $\neg R_j ba$. Hence, due to $R_i ab$, also $\mathcal{M}, a \not\models [i]\langle j\rangle p$. We conclude that $\mathcal{M}, a \not\models p \to [i]\langle j\rangle p$. Hence $\mathcal{F} \not\models p \to [i]\langle j\rangle p$.

($\Leftarrow$) Assume $R_i \subseteq R_j^{-1}$, that is, $R_i uv$ implies $R_j vu$, for all points u and v.

We have to show that $p \to [i]\langle j\rangle p$ is valid in $\mathcal{F}$. Let V be an arbitrary valuation on $\mathcal{F}$, x an arbitrary point in the model $\mathcal{M} = (\mathcal{F}, V)$, and assume $\mathcal{M}, x \vDash p$. In order to show $\mathcal{M}, x \vDash [i]\langle j\rangle p$, we consider an arbitrary R_i-successor y of x, $R_i xy$, and prove $\mathcal{M}, y \vDash \langle j\rangle p$. By the assumption $R_i \subseteq R_j^{-1}$ we know that $R_j yx$. Hence, since we have $\mathcal{M}, x \vDash p$, it follows that $\mathcal{M}, y \vDash \langle j\rangle p$.

(b) *Use the result of the previous question to show that the formula $\langle i\rangle[j]p \to p$ also characterizes the frame property $R_i \subseteq R_j^{-1}$.* *(7 pt)*

We reason as follows

$$
\begin{aligned}
\mathcal{F} \vDash \langle i\rangle[j]p \to p &\iff \mathcal{F} \vDash \langle i\rangle[j]\neg p \to \neg p && (1)\\
&\iff \mathcal{F} \vDash p \to \neg\langle i\rangle[j]\neg p && (2)\\
&\iff \mathcal{F} \vDash p \to [i]\neg[j]\neg p && (3)\\
&\iff \mathcal{F} \vDash p \to [i]\langle j\rangle p && (4)\\
&\iff R_i \subseteq R_j^{-1} && (5)
\end{aligned}
$$

where the steps are justified as follows:

(1) The direction $\Rightarrow$ follows from the fact that validity is closed under substitution; here we substitute $\neg p$ for p. The direction $\Leftarrow$ uses additionally that we may replace subformulas by equivalent subformulas; so from $\mathcal{F} \vDash \langle i\rangle[j]\neg p \to \neg p$ we infer $\mathcal{F} \vDash \langle i\rangle[j]\neg\neg p \to \neg\neg p$ and then replace $\neg\neg p$ by p.

(2) These are equivalent since one formula is the contraposition of the other.

(3) $\neg\langle i\rangle[j]\neg p$ is equivalent to $[i]\neg[j]\neg p$.

(4) $\neg[j]\neg p$ is equivalent to $\langle j\rangle p$.

(5) By the result proven in **2.(a)**.

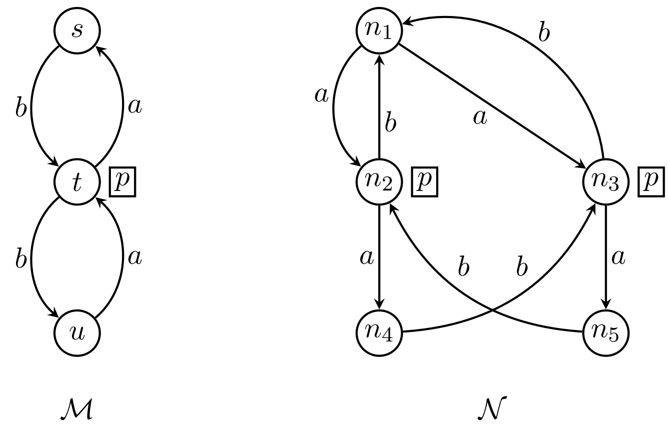
(c) *Are the formulas $p \to [i]\langle j\rangle p$ and $\langle i\rangle[j]p \to p$ equivalent? Prove your answer.* *(8 pt)*

No, they are not. Clearly, *inside* the class of I-frames with the property $R_i \subseteq R_j^{-1}$ they are equivalent, as we just have shown that they are both valid in that class. However, *outside* this class they need not be equivalent, as we show by the following counterexample.

First we recall the definition of equivalence of modal formulas: Two formulas φ and ψ are *equivalent*, which we denote by $\varphi \equiv \psi$, if $\varphi \leftrightarrow \psi$ is universally valid. In other words, $\varphi \equiv \psi$ when $\mathcal{M}, x \vDash \varphi$ iff $\mathcal{M}, x \vDash \psi$ for all models $\mathcal{M}$ and all points x of $\mathcal{M}$.

Now consider the following model $\mathcal{M} = (\{a\}, \{R_1, R_2\}, V)$ with $R_1 = \{(a, a)\}$, $R_2 = \varnothing$, and $V(p) = \varnothing$. Then we have $\mathcal{M}, a \vDash p \to [1]\langle 2\rangle p$ because of $\mathcal{M}, a \nvDash p$. On the other hand $\mathcal{M}, a \vDash [2]p$ by $R_2 = \varnothing$, and so, by $R_1 aa$, we have $\mathcal{M}, a \vDash \langle 1\rangle[2]p$. In combination with $\mathcal{M}, a \nvDash p$ this gives $\mathcal{M}, a \nvDash \langle 1\rangle[2]p \to p$. We conclude that $(p \to [1]\langle 2\rangle p) \not\equiv (\langle 1\rangle[2]p \to p)$.

3. *Consider the $\{a, b\}$-models $\mathcal{M}$ and $\mathcal{N}$ defined by:*



(a) *Define model $\mathcal{M}$ by means of set notation.* *(2 pt)*

$$\begin{aligned} \mathcal{M} &= (W^{\mathcal{M}}, R_a^{\mathcal{M}}, R_b^{\mathcal{M}}, V^{\mathcal{M}}) \\ W^{\mathcal{M}} &= \{s, t, u\} \\ R_a^{\mathcal{M}} &= \{(t, s), (u, t)\} \\ R_b^{\mathcal{M}} &= \{(s, t), (t, u)\} \\ V^{\mathcal{M}}(p) &= \{t\} \end{aligned}$$

(Likewise we will use $\mathcal{N} = (W^{\mathcal{N}}, R_a^{\mathcal{N}}, R_b^{\mathcal{N}}, V^{\mathcal{N}})$.)

(b) *Is there a modal formula that distinguishes state n_3 in model $\mathcal{N}$ from state t in model $\mathcal{M}$? Prove your answer.* *(10 pt)*

No, there is no such formula. We show that t and n_3 are bisimilar, and we know that bisimilar states have the same modal theory: if pointed models

$\mathcal{X}, x$ and $\mathcal{X}', x'$ are bisimilar then, for all modal formulas φ, it holds that $\mathcal{X}, x \vDash \varphi$ if and only if $\mathcal{X}', x' \vDash \varphi$.

Define the relation $G \subseteq W^{\mathcal{M}} \times W^{\mathcal{N}}$ by

$$G := \{(s, n_4), (s, n_5), (t, n_2), (t, n_3), (u, n_1)\}\,.$$

We show that G is a bisimulation:

- First of all, we notice that G satisfies the requirement of atomic harmony: for all $(x, x') \in G$ and all propositional variables q we have $\mathcal{M}, x \vDash q$ iff $\mathcal{N}, x' \vDash q$.
- To verify the zig-condition of G, for every pair $(x, x') \in G$, for every $i \in \{a, b\}$, and for every $y \in W^{\mathcal{M}}$ with $R_i^{\mathcal{M}} xy$, we have to find a point $y' \in W^{\mathcal{N}}$ such that $R_i^{\mathcal{N}} x'y'$ and $(y, y') \in G$. This we indicate by $\begin{array}{c|c} x & x' \\ \hline y & \textcolor{red}{y'} \end{array} i$.

$$\begin{array}{c|c} s & n_4 \\ \hline t & \textcolor{red}{n_3} \end{array} b \quad \begin{array}{c|c} s & n_5 \\ \hline t & \textcolor{red}{n_2} \end{array} b \quad \begin{array}{c|c} t & n_2 \\ \hline s & \textcolor{red}{n_4} \end{array} a \quad \begin{array}{c|c} t & n_2 \\ \hline u & \textcolor{red}{n_1} \end{array} b \quad \begin{array}{c|c} t & n_3 \\ \hline s & \textcolor{red}{n_5} \end{array} a \quad \begin{array}{c|c} t & n_3 \\ \hline u & \textcolor{red}{n_1} \end{array} b \quad \begin{array}{c|c} u & n_1 \\ \hline t & \textcolor{red}{n_3} \end{array} a$$

- Similarly for diagrams showing the zag condition (when a step $R_i^{\mathcal{N}} x'y'$ has to be matched by a step $R_i^{\mathcal{M}} xy$) we write $\begin{array}{c|c} x & x' \\ \hline \textcolor{red}{y} & y' \end{array} i$.

$$\begin{array}{c|c} s & n_4 \\ \hline \textcolor{red}{t} & n_3 \end{array} b \quad \begin{array}{c|c} s & n_5 \\ \hline \textcolor{red}{t} & n_2 \end{array} b \quad \begin{array}{c|c} s & n_5 \\ \hline \textcolor{red}{t} & n_2 \end{array} b \quad \begin{array}{c|c} t & n_2 \\ \hline \textcolor{red}{s} & n_4 \end{array} a \quad \begin{array}{c|c} t & n_2 \\ \hline \textcolor{red}{u} & n_1 \end{array} b \quad \begin{array}{c|c} t & n_3 \\ \hline \textcolor{red}{s} & n_5 \end{array} a \quad \begin{array}{c|c} t & n_3 \\ \hline \textcolor{red}{u} & n_1 \end{array} b \quad \begin{array}{c|c} u & n_1 \\ \hline \textcolor{red}{t} & n_2 \end{array} a \quad \begin{array}{c|c} u & n_1 \\ \hline \textcolor{red}{t} & n_3 \end{array} a$$

  (One diagram more than for zig due to two outgoing a-steps from n_1.)

(c) *Let $\widehat{\mathcal{N}}$ be the PDL-extension of model $\mathcal{N}$. Compute the transition relation $\widehat{R}_\pi$ corresponding to the PDL-program $\pi =$ if p then ba else ab.* *(8 pt)*

In PDL syntax we have (if p then ba else ab) $= (p?; ba) \cup (\neg p?; ab)$. We compute the transition relations of the component programs:

$$\begin{aligned}
\widehat{R}_a &= R_a = \{(n_1, n_2), (n_1, n_3), (n_2, n_4), (n_3, n_5)\} \\
\widehat{R}_b &= R_b = \{(n_2, n_1), (n_3, n_1), (n_4, n_3), (n_5, n_2)\} \\
\widehat{R}_{ba} &= \widehat{R}_b; \widehat{R}_a = \{(n_2, n_2), (n_2, n_3), (n_3, n_2), (n_3, n_3), (n_4, n_5), (n_5, n_4)\} \\
\widehat{R}_{p?} &= \{(x, x) \mid \mathcal{N}, x \vDash p\} = \{(n_2, n_2), (n_3, n_3)\} \\
\widehat{R}_{p?ba} &= \widehat{R}_{p?}; \widehat{R}_{ba} = \{(n_2, n_2), (n_2, n_3), (n_3, n_2), (n_3, n_3)\}
\end{aligned}$$

$$\widehat{R}_{ab} = \widehat{R}_a ; \widehat{R}_b = \{(n_1,n_1),(n_2,n_3),(n_3,n_2)\}$$
$$\widehat{R}_{\neg p?} = \{(x,x) \mid \mathcal{N}, x \vDash \neg p\} = \{(n_1,n_1),(n_4,n_4),(n_5,n_5)\}$$
$$\widehat{R}_{\neg p?ab} = \widehat{R}_{\neg p?} ; \widehat{R}_{ab} = \{(n_1,n_1)\}$$
$$\widehat{R}_{\pi} = \widehat{R}_{p?ba} \cup \widehat{R}_{\neg p?ab} = \{(n_1,n_1),(n_2,n_2),(n_2,n_3),(n_3,n_2),(n_3,n_3)\}$$

(d) *Determine whether the PDL-formula $[b]\bot \to ([\pi]p \to \bot)$ globally holds in $\widehat{\mathcal{N}}$. Prove your answer.* *(6 pt)*

Yes, $\widehat{\mathcal{N}} \vDash [b]\bot \to ([\pi]p \to \bot)$ holds. To see this, we only have to consider the point n_1, as this is the only point that is blind with respect to the relation R_b; in all other points $x \neq n_1$ we have $\mathcal{N}, x \nvDash [b]\bot$ and so the implication $[b]\bot \to ([\pi]p \to \bot)$ is trivially true there. So $\widehat{\mathcal{N}}, n_1 \vDash [b]\bot$. Now for $\widehat{\mathcal{N}}, n_1 \vDash [\pi]p \to \bot$ to hold we have to verify that $\widehat{\mathcal{N}}, n_1 \nvDash [\pi]p$, i.e., we need an R_π-successor of n_1 where p does not hold. Indeed, we have $(n_1,n_1) \in R_\pi$ and $\widehat{\mathcal{N}}, n_1 \nvDash p$ and so $\widehat{\mathcal{N}}, n_1 \nvDash [\pi]p$.

4. *System T is the extension of the minimal modal logic K with the axiom of veridicality (if something is known, it is true). System S4 extends T with the axiom of positive introspection; S5 extends S4 with the axiom of negative introspection.*

Prove or disprove the following epistemic claims (you may use completeness theorems):

(a) $\vdash_T p \to \neg K \neg p$ *(5 pt)*

1.	$Kp \to p$	(veridicality)
2.	$K\neg p \to \neg p$	(subst. instance of 1)
3.	$(a \to \neg b) \to (b \to \neg a)$	(tautology)
4.	$(K\neg p \to \neg p) \to (p \to \neg K \neg p)$	(subst. instance of 3)
5.	$p \to \neg K \neg p$	(modus ponens, 4, 2)

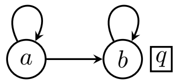
(b) $\vdash_{S4} q \vee K\neg Kq$ *(5 pt)*

We show that $q \vee K\neg Kq$ cannot be derived in *S4* by applying the completeness theorem for *S4* which states

$$\vdash_{S4} \varphi \iff \mathsf{Refl} \cap \mathsf{Trans} \vDash \varphi .$$

That is, we give a concrete model $\mathcal{M} = (W, R, V)$ with R reflexive and transitive, and a point x such that both $\mathcal{M}, x \nvDash q$ and $\mathcal{M}, x \nvDash K\neg Kq$.

Consider the following model based on a reflexive an transitive frame:



In this model we have that $a \not\models q$ and $a \not\models K\neg Kq$. The latter holds since $b \models Kq$, so $b \not\models \neg Kq$, and so $a \not\models K\neg Kq$. Hence $a \not\models q \vee K\neg Kq$, and $\mathsf{Refl} \cap \mathsf{Trans} \not\models q \vee K\neg Kq$. By the above stated completeness theorem for *S4* we thus obtain $\nvdash_{S4}\ q \vee K\neg Kq$

(c) $\vdash_{S5} \neg KKp \rightarrow K\neg Kp$ *(5 pt)*

We give a derivation in *S5*:

1.	$Kp \rightarrow KKp$	(positive introspection)
2.	$(A \rightarrow B) \rightarrow (\neg B \rightarrow \neg A)$	(tautology)
3.	$(Kp \rightarrow KKp) \rightarrow (\neg KKp \rightarrow \neg Kp)$	(subst. instance of 1)
4.	$\neg KKp \rightarrow \neg Kp$	(modus ponens, 3, 1)
5.	$\neg Kp \rightarrow K\neg Kp$	(negative introspection)
6.	$(A \rightarrow B) \rightarrow ((B \rightarrow C) \rightarrow (A \rightarrow C))$	(tautology)
7.	$4 \rightarrow (5 \rightarrow (\neg KKp \rightarrow K\neg Kp))$	(subst. instance of 8)
8.	$5 \rightarrow (\neg KKp \rightarrow K\neg Kp)$	(modus ponens, 7, 4)
9.	$\neg KKp \rightarrow K\neg Kp$	(modus ponens, 8, 5)

(Alternatively we can show that the formula is valid in all frames (W, R), where R is an equivalence relation, and use the completeness theorem for system *S5* to conclude that the formula is derivable in *S5*.)