Possible Answers to

Exam Advanced Logic

VU University Amsterdam, 25 March 2015, 15:15–18:00

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March 26, 2015

1. (a) *What frame property is characterised by the formula $p \leftrightarrow \Box p$? Prove your answer.*

In the lecture we have seen that the multi-modal formula $[1]p \leftrightarrow [2]p$ characterises the class of frames (W, R_1, R_2) where $R_1 = R_2$.[1] The formula $p \leftrightarrow \Box p$ can be seen to be an instance of that formula: Consider the modal box [Id] that we interpret by $\mathrm{Id} = \{(w, w) \mid w \in W\}$, the identity relation on W. Then we have

$$\begin{aligned} \mathcal{M}, s \vDash [\mathrm{Id}]\varphi \quad &\Longleftrightarrow \quad \mathcal{M}, t \vDash \varphi \text{ for all } t \text{ with } (s,t) \in \mathrm{Id} \\ &\Longleftrightarrow \quad \mathcal{M}, s \vDash \varphi\,. \end{aligned}$$

(Also note that $\mathcal{M}, s \vDash \langle \mathrm{Id} \rangle \varphi$ iff $\mathcal{M}, s \vDash \varphi$.)

As the formula $p \leftrightarrow \Box p$ can be thought of as $[\mathrm{Id}]p \leftrightarrow [R]p$, it characterises the class of frames (W, R) where $R = \mathrm{Id}$. That is, for all frames $\mathcal{F} = (W, R)$ we have

$$\mathcal{F} \models p \leftrightarrow \Box p \quad \Longleftrightarrow \quad R = \mathrm{Id}\,.$$

Let $\mathcal{F} = (W, R)$ be a frame. We prove both implications.

[1] See the **Fact** on page 4 of the answers to exercises set 5, stating that $[1]p \to [2]p$ characterises $R_2 \subseteq R_1$.

($\Rightarrow$) Assume $\mathcal{F} \models p \leftrightarrow \Box p$. We first show $R \subseteq \mathrm{Id}$. So we consider a pair $(x, y) \in R$, and show that $(x, y) \in \mathrm{Id}$, i.e., that $x = y$. For this, consider a valuation V on $\mathcal{F}$ such that $V(p) = \{x\}$. Since $p \to \Box p$ is valid in $\mathcal{F}$, it holds in the point x of the model $(\mathcal{F}, V)$. By definition of V, $x \vDash p$ and so we get $x \vDash \Box p$. This in turn implies that $y \vDash p$, as $(x, y) \in R$. But p is true *only* in x by definition of V, and we conclude $x = y$, as desired.

Second, to see that $\mathrm{Id} \subseteq R$, we consider a point x of $\mathcal{F}$ and show $(x, x) \in R$. Define $V(p) = R[x] = \{y \mid Rxy\}$, i.e., we let p be true in all R-successors of x and only there. Then, in the model $(\mathcal{F}, V)$, we clearly have $x \vDash \Box p$. As we know that also $\Box p \to p$ holds in x (as it is valid in $\mathcal{F}$ by assumption), we obtain that $x \vDash p$. Hence $x \in V(p)$, i.e., $(x, x) \in R$.

($\Leftarrow$) Assume that $R = \mathrm{Id}$. Let V be any valuation on $\mathcal{F}$, and x any point of $(\mathcal{F}, V)$. We have show $x \vDash p \to \Box p$ and $x \vDash \Box p \to p$. For the first, assume $x \vDash p$. Then we also have $x \vDash \Box p$ since $R[x] = \{x\}$. For the second, assume $x \vDash \Box p$, then $x \vDash p$ follows from Rxx.

(b) *Show that $q \leftrightarrow \Diamond q$ characterises the same frame property. (Note that for this you do not have to know the frame property asked for in* **(a)***.)*

The two formulas are equivalent (hence they are equivalent in the frame class of **(a)** which is actually all we need). The reason is that validity is closed under taking substitution instances, and the formulas are (equivalent to) substitution instances of each other. Let C be the class asked for under **(a)**. We have:

$$\begin{aligned} \mathcal{F} \in \mathsf{C} &\iff \mathcal{F} \models p \leftrightarrow \Box p & (1) \\ &\iff \mathcal{F} \models \neg p \leftrightarrow \Box \neg p & (2) \\ &\iff \mathcal{F} \models p \leftrightarrow \neg \Box \neg p & (3) \\ &\iff \mathcal{F} \models p \leftrightarrow \Diamond p & (4) \end{aligned}$$

where steps are justified as follows: Step (1) was shown under **(a)**. The implication $\Rightarrow$ of (2) follows from the fact that validity is preserved under taking substitution instances, (i.e., $\mathcal{F} \vDash \varphi$ implies $\mathcal{F} \vDash \varphi^\sigma$ for all substitutions σ); here we substitute $\neg p$ for p. The converse implication of (2) uses the same result and the same substitution, and additionally the fact that we may replace equivalents by equivalents without affecting validity (or satisfiability); here we replace $\neg\neg p$ by p. Step (3) uses the equivalence $\varphi \leftrightarrow \psi \equiv \neg\varphi \leftrightarrow \neg\psi$, and finally step (4) uses $\neg\Box\varphi \equiv \Diamond\neg\varphi$.

2. **(a)** *Show that the formula* $\Box\Diamond p \to \Box\Diamond\Box\Diamond p$ *is valid on all transitive frames.*

Let $\mathcal{M}$ be some model on a transitive frame (W, R), let x be some point in $\mathcal{M}$, and assume $x \models \Box\Diamond p$. To see $x \models \Box\Diamond\Box\Diamond p$, consider a point y with Rxy. By the assumption, we have $y \models \Diamond p$. This means that $z \models p$ for some z with Ryz. We now show that $z \models \Box\Diamond p$ (then we have $y \models \Diamond\Box\Diamond p$ as desired). For this, we let u be an arbitrary successor of z. Now $u \models \Diamond p$ follows from the assumption that $x \models \Box\Diamond p$ plus the fact that, due to transitivity, u is a successor of x too.

(b) *Is the formula* $\Box\Diamond p \to \Box\Diamond\Box\Diamond p$ only *valid on transitive frames? Prove your answer.*

No, the formula $\psi = \Box\Diamond p \to \Box\Diamond\Box\Diamond p$ can also be valid on frames that are not transitive. For example, consider the non-transitive frame $\mathcal{F} = (\{a, b, c\}, \{(a, b), (b, c)\})$, and let V be an arbitrary valuation on $\mathcal{F}$. We show that the formula holds in all points of the model $(\mathcal{F}, V)$. First, since c has no successors, it holds that $c \models \Box\varphi$ for any formula φ; in particular $c \models \Box\Diamond\Box\Diamond p$ and $c \models \Box\Diamond p$. By the latter, we get that $b \models \Diamond\Box\Diamond p$ and so $a \models \Box\Diamond\Box\Diamond p$. So we have seen that both c and a satisfy the consequent of ψ, whence ψ holds in both. To see that it also holds in b, we note that $b \not\models \Box\Diamond p$, the reason being again that c is blind and so $c \not\models \Diamond\varphi$ for all φ, in particular $c \not\models \Diamond p$.

3. *System S5 is the extension of system K with the truth axiom (if something is known, it is true), the axiom of positive introspection, and the axiom of negative introspection.*

(a) *Show that* $p \to K\neg K\neg p$ *is a theorem of S5.*

1.	$Kp \to p$	(truth axiom)
2.	$\neg Kp \to K\neg Kp$	(axiom of negative introspection)
3.	$K\neg p \to \neg p$	(substitution instance of 1)
4.	$p \to \neg K\neg p$	(by propositional logic from 3)
5.	$\neg K\neg p \to K\neg K\neg p$	(substitution instance of 2)
6.	$p \to K\neg K\neg p$	(by propositional logic from 4 and 5)

(b) *Show that the following rule is admissible in S5:*

$$\frac{\varphi \to K\psi}{\neg K\neg\varphi \to \psi}$$

A rule $\dfrac{\alpha_1\ \alpha_2\ \cdots\ \alpha_n}{\beta}$ is *admissible* in a proof system H if β is derivable in H whenever α_i is, for every i with $1 \leq i \leq n$.

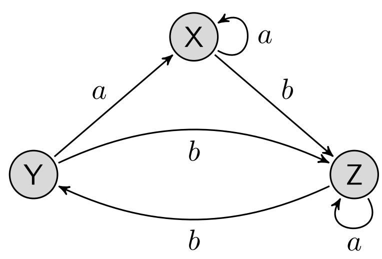
We will employ the completeness theorem for the proof system *S5* (but a syntactic proof surely is possible too):

$$\vdash_{S5} \xi \quad \Longleftrightarrow \quad \mathsf{Equiv} \vDash \xi\,, \text{for all epistemic formulae } \xi,$$

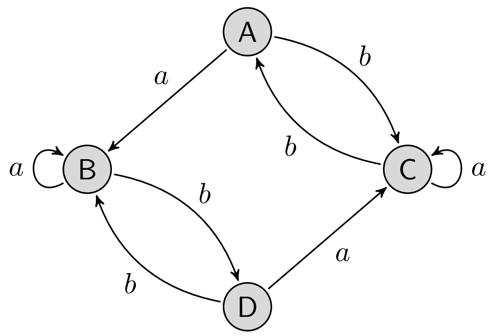
where Equiv denotes the class of epistemic n-frames $\mathcal{F} = (W, \{R_1, R_2, \ldots, R_n\})$ such that R_i is an equivalence relation for every i with $1 \leq i \leq n$. Here we only have to reason about the knowledge of one agent and we use R to interpret the box K (i.e., $\mathcal{M}, x \vDash K\gamma$ iff $\mathcal{M}, y \vDash \gamma$ for all y with Rxy).

Let φ and ψ be some epistemic formulas, and abbreviate $\alpha = \varphi \to K\psi$ and $\beta = \neg K \neg\varphi \to \psi$. Assume $\vdash_{S5} \alpha$. Then, by completeness for *S5*, we know that $\mathsf{Equiv} \models \alpha$. It suffices to show that $\mathsf{Equiv} \vDash \beta$, because then by the completeness theorem for *S5* (but now in the other direction), we obtain that $\vdash_{S5} \beta$. Let $\mathcal{F}$ be some frame in the class Equiv, let V be some valuation on $\mathcal{F}$, let s be a state in the model $(\mathcal{F}, V)$, and assume $s \vDash \neg K\neg\varphi$ (goal: $s \vDash \psi$). Then $s \nvDash K\neg\varphi$ and so there must be a state t such that Rst and $t \nvDash \neg\varphi$, i.e., $t \vDash \varphi$. Now as we know that $\mathsf{Equiv} \vDash \alpha$, also $t \vDash \alpha$. It follows that $t \vDash K\psi$. Moreover, as we have Rts (by symmetry of R), we get $s \vDash \psi$, as desired.

4. *Consider the following* $\{a, b\}$*-frames* $\mathcal{F}_1$ *and* $\mathcal{F}_2$ *:*



$\mathcal{F}_1$



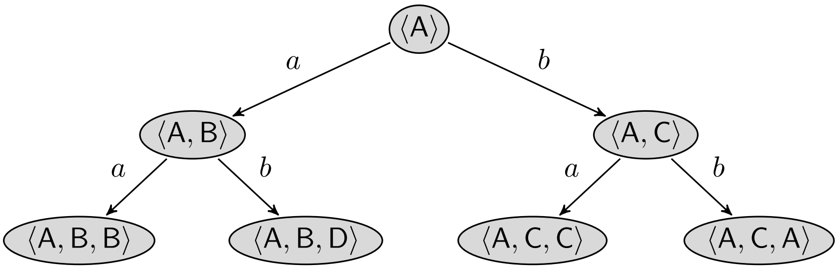
$\mathcal{F}_2$

(a) *Explain why in all models on these frames, the meanings of* $\langle i\rangle\varphi$ *and* $[i]\varphi$ *coincide for* $i \in \{a, b\}$*. That is, explain why in all models* $\mathcal{M}$ *on a frame* $\mathcal{F} \in \{\mathcal{F}_1, \mathcal{F}_2\}$ *for each label* $i \in \{a, b\}$*, for all states* s*, and for all formulas* φ *it holds:* $\mathcal{M}, s \vDash [i]\varphi \iff \mathcal{M}, s \vDash \langle i\rangle\varphi$*. (An informal proof suffices.)*

The reason is that, in both frames, every state precisely has one outgoing arrow labelled a, and precisely one labelled b. In other words, for $i \in \{a, b\}$, the relations $R_i^{\mathcal{F}_1}$ and $R_i^{\mathcal{F}_2}$ are serial and deterministic. A relation $R \subseteq W \times W$ is called *serial* when $\forall x \, \exists y \, Rxy$; it is called *deterministic* when $\forall xyz \, (Rxy \wedge Rxz \to z = y)$; so for every state x, the set $R[x] = \{y \mid Rxy\}$ of i-successors of x is a singleton set. Clearly, when quantifying over a singleton set, the existential and the universal quantifier have the same meaning. Let $\mathcal{M} = (W, R, V)$ be a model such that R is serial and deterministic; then:

$$\begin{aligned} \mathcal{M}, s \vDash \Box\varphi &\iff \forall t \in R[s].\, \mathcal{M}, t \vDash \varphi \\ &\iff \exists t \in R[s].\, \mathcal{M}, t \vDash \varphi \\ &\iff \mathcal{M}, s \vDash \Diamond\varphi\,. \end{aligned}$$

(b) *Draw the first three levels of the tree unravelling (or unfolding) of $\mathcal{F}_2$ taking state* A *as root node.*



Now consider the models $\mathcal{M}_1 = (\mathcal{F}_1, V_1)$ and $\mathcal{M}_2 = (\mathcal{F}_2, V_2)$ where $V_1(p) = \{\mathsf{Z}\}$ and $V_2(p) = \{\mathsf{C}, \mathsf{D}\}$, and $V_1(q) = V_2(q) = \varnothing$ for all propositional variables $q \neq p$.

(c) *If possible, give a multi-modal formula over the index set $\{a, b\}$ that distinguishes state* Y *of $\mathcal{M}_1$ from state* A *of $\mathcal{M}_2$. Otherwise, prove that there is no such formula.*

There is no modal formula distinguishing state Y from A. As we will show, states Y and A are bisimilar, and we know that bisimilar states have equal modal theories, i.e., for all states s, t, if $s \,\underline{\leftrightarrow}\, t$, then for all modal formulas φ, $[s \vDash \varphi$ iff $t \vDash \varphi]$. We use $R_a^{\mathcal{M}_1}$, etc., for the transition relation with label a in model $\mathcal{M}_1$, etc.

We define a relation $G \subseteq \{\mathsf{X}, \mathsf{Y}, \mathsf{Z}\} \times \{\mathsf{A}, \mathsf{B}, \mathsf{C}, \mathsf{D}\}$ by

$$G = \{\, (\mathsf{X}, \mathsf{A}), (\mathsf{X}, \mathsf{B}), (\mathsf{Y}, \mathsf{A}), (\mathsf{Y}, \mathsf{B}), (\mathsf{Z}, \mathsf{C}), (\mathsf{Z}, \mathsf{D}) \,\}\,.$$

Next we show that G is a bisimulation.[2]

For all pairs $(x, x') \in G$, we have that $x \in V_1(p)$ if and only if $x' \in V_2(p)$, thus the condition of 'atomic harmony' is fulfilled.

To verify the zig-condition of G, for every pair $(x, x') \in G$, for every label $i \in \{a, b\}$ and for every y of $\mathcal{M}_1$ with $R_i^{\mathcal{M}_1} xy$, we have to find a state y' of $\mathcal{M}_2$ such that $R_i^{\mathcal{M}_2} x'y'$ and $(y, y') \in G$. This we indicate by $\begin{array}{c|c} x & x' \\ \hline y & y' \end{array} i$. Due to the nature of the models $\mathcal{M}_1$ and $\mathcal{M}_2$ (every state has precisely one outgoing arrow labelled a, and precisely one labelled b), not only is there only one i-step possible from x ($i = a, b$), there is also precisely one candidate witness y'. For the zag-condition the situation is analogous. What is more, the diagrams that we have to give to verify that G fulfills the zag-condition are exactly those that we gave for 'zig'. Thus, the following $2 \times |G| = 12$ diagrams verify both 'zig' and 'zag':

$$\begin{array}{c|c} \mathsf{X} & \mathsf{A} \\ \hline \mathsf{X} & \mathsf{B} \end{array} a \qquad \begin{array}{c|c} \mathsf{X} & \mathsf{A} \\ \hline \mathsf{Z} & \mathsf{C} \end{array} b \qquad \begin{array}{c|c} \mathsf{X} & \mathsf{B} \\ \hline \mathsf{X} & \mathsf{B} \end{array} a \qquad \begin{array}{c|c} \mathsf{X} & \mathsf{B} \\ \hline \mathsf{Z} & \mathsf{D} \end{array} b$$

$$\begin{array}{c|c} \mathsf{Y} & \mathsf{A} \\ \hline \mathsf{X} & \mathsf{B} \end{array} a \qquad \begin{array}{c|c} \mathsf{Y} & \mathsf{A} \\ \hline \mathsf{Z} & \mathsf{C} \end{array} b \qquad \begin{array}{c|c} \mathsf{Y} & \mathsf{B} \\ \hline \mathsf{X} & \mathsf{B} \end{array} a \qquad \begin{array}{c|c} \mathsf{Y} & \mathsf{B} \\ \hline \mathsf{Z} & \mathsf{D} \end{array} b$$

$$\begin{array}{c|c} \mathsf{Z} & \mathsf{C} \\ \hline \mathsf{X} & \mathsf{C} \end{array} a \qquad \begin{array}{c|c} \mathsf{Z} & \mathsf{C} \\ \hline \mathsf{Z} & \mathsf{A} \end{array} b \qquad \begin{array}{c|c} \mathsf{Z} & \mathsf{D} \\ \hline \mathsf{X} & \mathsf{C} \end{array} a \qquad \begin{array}{c|c} \mathsf{Z} & \mathsf{D} \\ \hline \mathsf{Z} & \mathsf{B} \end{array} b$$

The following question is about propositional dynamic logic (PDL).

(d) *Prove that $p \leftrightarrow [a^*(bb)^*]p$ is globally true in the PDL-extension of $\mathcal{M}_2$.*

Write just R_α for the relation interpreting a PDL-program α in $\widehat{\mathcal{M}_1}$, the PDL-extension of $\mathcal{M}_1 = (\{\mathsf{X}, \mathsf{Y}, \mathsf{Z}\}, R_a, R_b, V)$. We first compute the relation $R_{a^*(bb)^*}$ corresponding to the PDL-program $\pi = a^*(bb)^*$, as follows:

$$\begin{aligned}
R_a &= \{(\mathsf{X},\mathsf{X}), (\mathsf{Y},\mathsf{X}), (\mathsf{Z},\mathsf{Z})\} \\
R_{a^*} &= R_a^* = \{(\mathsf{X},\mathsf{X}), (\mathsf{Y},\mathsf{Y}), (\mathsf{Z},\mathsf{Z}), (\mathsf{Y},\mathsf{X})\} \\
R_b &= \{(\mathsf{X},\mathsf{Z}), (\mathsf{Y},\mathsf{Z}), (\mathsf{Z},\mathsf{Y})\} \\
R_{bb} &= R_b \circ R_b = \{(\mathsf{X},\mathsf{Y}), (\mathsf{Y},\mathsf{Y}), (\mathsf{Z},\mathsf{Z})\} \\
R_{(bb)^*} &= R_{bb}^* = \{(\mathsf{X},\mathsf{X}), (\mathsf{Y},\mathsf{Y}), (\mathsf{Z},\mathsf{Z}), (\mathsf{X},\mathsf{Y})\} \\
R_\pi &= R_a^* \circ R_{bb}^* = \{(\mathsf{X},\mathsf{X}), (\mathsf{Y},\mathsf{Y}), (\mathsf{Z},\mathsf{Z}), (\mathsf{Y},\mathsf{X}), (\mathsf{X},\mathsf{Y})\}\,.
\end{aligned}$$

[2] We note that the relation $G \setminus \{(\mathsf{X}, \mathsf{A})\}$ is a bisimulation too.

Next we show that both $p \to [\pi]p$ and $[\pi]p \to p$ are true in all states of the PDL-model $\widehat{\mathcal{M}_1}$: For the first, let $s \in \{\mathsf{X}, \mathsf{Y}, \mathsf{Z}\}$, and assume $s \vDash p$. Then $s = \mathsf{Z}$ and since $R_\pi[\mathsf{Z}] = \{\mathsf{Z}\}$ we have $\mathsf{Z} \vDash [\pi]p$. For the second, let $s \in \{\mathsf{X}, \mathsf{Y}, \mathsf{Z}\}$, and assume $s \vDash [\pi]p$. That is, $R_\pi[s] \subseteq V(p)$. This only holds for $s = \mathsf{Z}$, and we have $\mathsf{Z} \vDash p$.