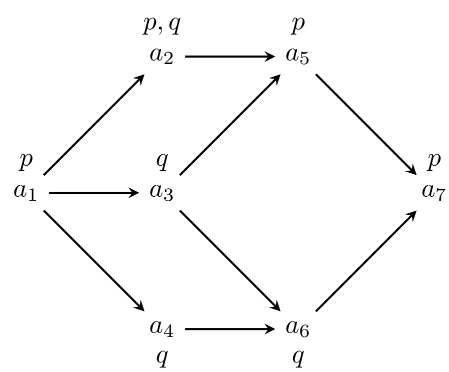
Advanced Logic 2014–15

Exercises Set 1

1. Consider the model $\mathcal{M} = (W, R, V)$ given by the following picture:



   (a) Write out the definitions of W, R and V.

   (b) Show that

   (i) $\mathcal{M}, a_1 \models \Box\Box(p \vee q)$

   (ii) $\mathcal{M}, a_2 \models \Diamond q \to \Diamond\Diamond q$

   (iii) $\mathcal{M}, a_3 \models \Diamond p \to \Box(q \to \Box(p \to \Box p))$

   (c) Show that

   (i) $\mathcal{M} \not\models p \to \Diamond p$

   (ii) $\mathcal{M} \models \Box\Box\Box\neg q$

   (iii) $\mathcal{M} \not\models q \to (\Diamond q \to \Box(q \to \Diamond q))$

   (d) Change the valuation on the frame such that in the new model $\mathcal{M}'$ it holds that: $\mathcal{M}' \models \Box p \to p$.

2. The *binary tree* is the frame $\mathcal{B} = (W, R)$ where the domain W is the set of all finite strings over the alphabet {0,1}:

$$W = \{0, 1\}^* = \{\varepsilon, 0, 1, 00, 01, 10, 11, 000, \ldots\}\,.$$

   (ε denotes the empty string) and where the transition relation $R \subseteq W \times W$ is given by:

$$Rst \quad \text{if and only if} \quad t = s0 \ \text{or} \ t = s1$$

(a) Make a drawing of the first four levels of $\mathcal{B}$.

(b) Consider a valuation V on $\mathcal{B}$ that makes p true on all strings of even length. Show that $\mathcal{B}, V \models \Box\Diamond p \to \Diamond\Box p$.

(c) Let V' be a valuation on $\mathcal{B}$ which makes the variable p true on all strings whose first letter is 0, and q on strings with first letter 1, so:

$$V'(p) = \{\, 0w \mid w \in \{0,1\}^* \,\}$$
$$V'(q) = \{\, 1w \mid w \in \{0,1\}^* \,\}$$

Use this valuation to show that the formula λ defined by:

$$\lambda = \Diamond p \wedge \Diamond q \to \Diamond(p \wedge \Diamond q) \vee \Diamond(p \wedge q) \vee \Diamond(\Diamond p \wedge q)$$

is not valid in the binary tree.

(d) Show that the formula $\Diamond\Diamond p \to \Diamond p$ is not valid in $\mathcal{B}$.

3. Prove or disprove universal validity of the following formulas:

(a) $\Box(p \to q) \to (\Diamond p \to \Diamond q)$

(b) $\Box(p \wedge q) \to (\Diamond p \wedge \Diamond q)$

(c) $\Box(p \wedge q) \to (\Box p \wedge \Box q)$

(d) $\Box p \to \Diamond p$

(e) $\Diamond p \to \Box p$

(f) $\Diamond(p \to q) \to (\Box p \to \Diamond q)$

(g) $\Box(\Box p \to p) \to \Box p$