Advanced Logic 2014–15

Exercises Set 2

A frame $\mathcal{F} = (W, R)$ is

symmetric	if	$\forall xy\,(Rxy \rightarrow Ryx)$
serial	if	$\forall x\,(\exists y\,(Rxy))$
partially functional	if	$\forall xyz\,(Rxy \wedge Rxz \rightarrow y = z)$
functional	if	serial and partially functional

1. Let $\mathcal{F} = (W, R)$ be a frame.

   (a) Show that the following implications hold:

      i. if $\mathcal{F}$ is symmetric, then $\mathcal{F} \vDash p \rightarrow \Box\Diamond p$

      ii. if $\mathcal{F}$ is serial, then $\mathcal{F} \vDash \Box p \rightarrow \Diamond p$

      iii. if $\mathcal{F}$ is serial, then $\mathcal{F} \vDash \Diamond\top$

      iv. if $\mathcal{F}$ is partially functional, then $\mathcal{F} \vDash \Diamond p \rightarrow \Box p$

      v. if $\mathcal{F}$ is functional, then $\mathcal{F} \vDash \Diamond p \leftrightarrow \Box p$

   (b) Show that the modal formulas in (a) *define* the corresponding frame property. In other words, show that the reverse implications hold as well.

2. Let $\mathcal{N}$ be the frame $(\mathbb{N}, S)$ of the natural numbers $\mathbb{N} = \{0, 1, 2, \ldots\}$ with the successor relation S defined by:

$$Smn \quad \text{if and only if} \quad n = m + 1\,,$$

   and recall the binary tree $\mathcal{B} = (\{0,1\}^*, R)$ from last week with R defined by:

$$Rst \quad \text{if and only if} \quad t = s0 \text{ or } t = s1$$

   (a) Define a valuation U on $\mathcal{N}$ such that:

$$\mathcal{B}, V, \varepsilon \leftrightarroweq \mathcal{N}, U, 0$$

   where V is a valuation on $\mathcal{B}$ such that:

$$V(p) = \{\, w \in \{0,1\}^* \mid w \text{ is of even length} \,\}\ .$$

   Show that this is the only possibility for U.

(b) Let V' be a valuation on $\mathcal{B}$ such that:

$$V'(p) = \{\, 0w \mid w \in \{0,1\}^* \,\}$$
$$V'(q) = \{\, 1w \mid w \in \{0,1\}^* \,\}\ .$$

Show that there exists no valuation U' on $\mathcal{N}$ such that:

$$\mathcal{B}, V', \varepsilon \leftrightarroweq \mathcal{N}, U', 0$$

3. Consider the following two frames $\mathcal{F}$ en $\mathcal{F}'$

Here it is to be understood that Frame $\mathcal{F}$ has infinitely many paths of finite length $1, 2, 3, \ldots$. Frame $\mathcal{F}'$ is like $\mathcal{F}$ but additionaly has one infinite path.

Argue that there cannot be a bisimulation Z between $\mathcal{F}$ and $\mathcal{F}'$ such that wZw'.

4. (a) Give an example of a formula φ and a model $\mathcal{M}$ such that neither $\mathcal{M} \models \varphi$ nor $\mathcal{M} \models \neg\varphi$.

   (b) Give an example of a formula φ, a frame $\mathcal{F}$ and two models $\mathcal{M}$ and $\mathcal{M}'$ based on $\mathcal{F}$ such that $\mathcal{M} \models \varphi$ and $\mathcal{M}' \models \neg\varphi$.

5. Consider the frame $\mathcal{N} = (\mathbb{N}, S)$ from Exercise **2** above, and the set W:

$$W = \{\overline{1}, \overline{2}\} \cup \{3n \mid n \in \mathbb{N}\}$$

Find a relation $R \subseteq W \times W$, and a bisimulation $Z \subseteq \mathbb{N} \times W$ between frames[1] $\mathcal{N}$ and $\mathcal{F} = (W, R)$ such that:

$$\{(3n, 3n) \mid n \in \mathbb{N}\} \subseteq Z$$
$$\{(3n+1, \overline{1}) \mid n \in \mathbb{N}\} \subseteq Z$$
$$\{(3n+2, \overline{2}) \mid n \in \mathbb{N}\} \subseteq Z$$

Prove that Z is indeed a bisimulation.

[1] Let B be called a bisimulation between *frames* $\mathcal{F}$ and $\mathcal{F}'$ if B fulfills the zig and zag conditions in the definition of bisimulations between models $(\mathcal{F}, V)$ and $(\mathcal{F}', V')$ (Definition 3.2.1). So we ignore the condition of 'local harmony' saying that points x and y with xBy should agree on their atomic information (for all propositional variables p, p true in x if and only if p true in y).