Advanced Logic 2014–15

Exercises Set 2

A frame $\mathcal{F} = (W, R)$ is $\begin{array}{ccc} symmetric & \text{if} & \forall xy (Rxy \to Ryx) \\ serial & \text{if} & \forall x (\exists y (Rxy)) \\ partially functional & \text{if} & \forall xyz (Rxy \land Rxz \to y = z) \\ functional & \text{if} & \text{serial and partially functional} \end{array}$

1. Let $\mathcal{F} = (W, R)$ be a frame.

- (a) Show that the following implications hold:
 - i. if \mathcal{F} is symmetric, then $\mathcal{F} \vDash p \to \Box \Diamond p$
 - ii. if \mathcal{F} is serial, then $\mathcal{F} \vDash \Box p \rightarrow \Diamond p$
 - iii. if $\mathcal F$ is serial, then $\mathcal F \vDash \Diamond \top$
 - iv. if \mathcal{F} is partially functional, then $\mathcal{F} \vDash \Diamond p \rightarrow \Box p$
 - v. if \mathcal{F} is functional, then $\mathcal{F} \vDash \Diamond p \leftrightarrow \Box p$
- (b) Show that the modal formulas in (a) *define* the corresponding frame property. In other words, show that the reverse implications hold as well.
- 2. Let \mathcal{N} be the frame (\mathbb{N}, S) of the natural numbers $\mathbb{N} = \{0, 1, 2, ...\}$ with the successor relation S defined by:

Smn if and only if n = m + 1,

and recall the binary tree $\mathcal{B} = (\{0, 1\}^*, R)$ from last week with R defined by:

Rst if and only if t = s0 or t = s1

(a) Define a valuation U on \mathcal{N} such that:

 $\mathcal{B}, V, \varepsilon \leftrightarrow \mathcal{N}, U, 0$

where V is a valuation on \mathcal{B} such that:

 $V(p) = \{ w \in \{0, 1\}^* \mid w \text{ is of even length } \}.$

Show that this is the only possibility for U.

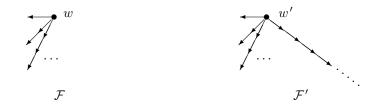
(b) Let V' be a valuation on \mathcal{B} such that:

$$V'(p) = \{ 0w \mid w \in \{0,1\}^* \}$$
$$V'(q) = \{ 1w \mid w \in \{0,1\}^* \}$$

Show that there exists no valuation U' on \mathcal{N} such that:

$$\mathcal{B}, V', \varepsilon \leftrightarrow \mathcal{N}, U', 0$$

3. Consider the following two frames \mathcal{F} en \mathcal{F}'



Here it is to be understood that Frame \mathcal{F} has infinitely many paths of finite length 1, 2, 3, Frame \mathcal{F}' is like \mathcal{F} but additionally has one infinite path. Argue that there cannot be a bisimulation Z between \mathcal{F} and \mathcal{F}' such that wZw'.

- 4. (a) Give an example of a formula φ and a model \mathcal{M} such that neither $\mathcal{M} \models \varphi$ nor $\mathcal{M} \models \neg \varphi$.
 - (b) Give an example of a formula φ , a frame \mathcal{F} and two models \mathcal{M} and \mathcal{M}' based on \mathcal{F} such that $\mathcal{M} \models \varphi$ and $\mathcal{M}' \models \neg \varphi$.
- 5. Consider the frame $\mathcal{N} = (\mathbb{N}, S)$ from Exercise 2 above, and the set W:

$$W = \{\overline{1}, \overline{2}\} \cup \{3n \mid n \in \mathbb{N}\}\$$

Find a relation $R \subseteq W \times W$, and a bisimulation $Z \subseteq \mathbb{N} \times W$ between frames¹ \mathcal{N} and $\mathcal{F} = (W, R)$ such that:

$$\{(3n, 3n) \mid n \in \mathbb{N}\} \subseteq Z$$
$$\{(3n+1, \overline{1}) \mid n \in \mathbb{N}\} \subseteq Z$$
$$\{(3n+2, \overline{2}) \mid n \in \mathbb{N}\} \subseteq Z$$

Prove that Z is indeed a bisimulation.

¹Let *B* be called a bisimulation between *frames* \mathcal{F} and \mathcal{F}' if *B* fulfills the zig and zag conditions in the definition of bisimulations between models (\mathcal{F}, V) and (\mathcal{F}', V') (Definition 3.2.1). So we ignore the condition of 'local harmony' saying that points *x* and *y* with *xBy* should agree on their atomic information (for all propositional variables *p*, *p* true in *x* if and only if *p* true in *y*).