Advanced Logic 2014–15

Exercises Set 4

Recall the truth definition of the until operator U, in a model $\mathcal{M} = (W, R, V)$, and point w of $\mathcal{M}$:

$$\begin{aligned} &\mathcal{M}, w \vDash \varphi U \psi \quad \text{if and only if} \\ &\quad \exists u\,(Rwu \wedge \mathcal{M}, u \vDash \psi \wedge \forall v\,(Rwv \wedge Rvu \to \mathcal{M}, v \vDash \varphi)) \end{aligned}$$

1. Consider the temporal frame $(\mathbb{N}, <)$ of the natural numbers. The semantics of the so-called 'next-time'-operator $\otimes$ is given by the following truth definition in a model on the frame $(\mathbb{N}, <)$:

$$(\mathbb{N}<, V), n \vDash \otimes\varphi \quad \text{if and only if} \quad (\mathbb{N}, <, V), n+1 \vDash \varphi$$

   (a) Define $\otimes$ in terms of U.

   (b) Suppose you formalize a measuring system where you use proposition letters p and t for "the pressure is measured" and "the temperature is measured", respectively. How can we model that the measurement of pressure and temperature are alternated day by day? Specify the semantic notion you use: validity in $(\mathbb{N}, <)$, in a specific model on $(\mathbb{N}, <)$ (which one?), or in a specific point of a specific model (which?).

   (c) Consider the model $\mathcal{M} = ((\mathbb{N}, <), V)$ on $(\mathbb{N}, <)$ with $V : \textsc{Var} \to \mathcal{P}(\mathbb{N})$ defined by:

$$V(p) = \{\, 3n \mid n \in \mathbb{N} \,\}$$

   Show that (by giving a bisimulation) for all formulas φ of the basic modal language and $n \in \mathbb{N}$:

$$\begin{aligned} \mathcal{M}, n \vDash \varphi &\iff \mathcal{M}, n+3 \vDash \varphi \\ \mathcal{M}, 1 \vDash \varphi &\iff \mathcal{M}, 2 \vDash \varphi \end{aligned}$$

   (d) Prove that the next-time operator cannot be defined in the basic modal language.

2. Formulate the box version of "Modal Decomposition" (MLOM, p. 41), i.e., give necessary and sufficient conditions for validity of a modal sequent of the form

$$\vec{p}, \Box\varphi_1, \ldots, \Box\varphi_k \implies \Box\psi_1, \ldots, \Box\psi_m, \vec{q}$$

3. Use semantic tableaux (MLOM, pp. 42–45) to find a counterexample against universal validity of $\Diamond p \land \Diamond q \to \Diamond(p \land q)$.

4. Let $I = \{a, b, c\}$ and consider the I-frame $\mathcal{F} = (W, \{R_a, R_b, R_c\})$ where:

$$\begin{aligned} W &= \{w_1, w_2, w_3, w_4\} & R_b &= \{(w_2, w_3)\} \\ R_a &= \{(w_1, w_2), (w_3, w_3)\} & R_c &= \{(w_2, w_4), (w_4, w_1)\} \end{aligned}$$

   and let $\mathcal{M} = (\mathcal{F}, V)$ with V the valuation on $\mathcal{F}$ defined by:

$$V(p) = \{w_3\} \quad \text{and} \quad V(q) = \{w_1\} .$$

   (a) Draw the model $\mathcal{M}$ as a graph.

   (b) Prove or disprove:

      i. $\mathcal{M} \vDash [a]p$

      ii. $\mathcal{M} \vDash [a](p \lor (\langle b\rangle\top \land \langle c\rangle\top))$

      iii. $\mathcal{F} \vDash [a](\langle a\rangle\top \lor (\langle b\rangle\top \land \langle c\rangle\top))$

5. The class of $\{F, P\}$-frames (T, R_F, R_P) for which R_F and R_P are each other's inverse[1] can be defined by a modal formula over $\{F, P\}$:

   (a) Prove that $(T, R_F, R_P) \vDash q \to [P]\langle F\rangle q$ if and only if $R_P tu \implies R_F ut$.

   (b) Prove that $(T, R_F, R_P) \vDash q \to [F]\langle P\rangle q$ if and only if $R_F tu \implies R_P ut$.

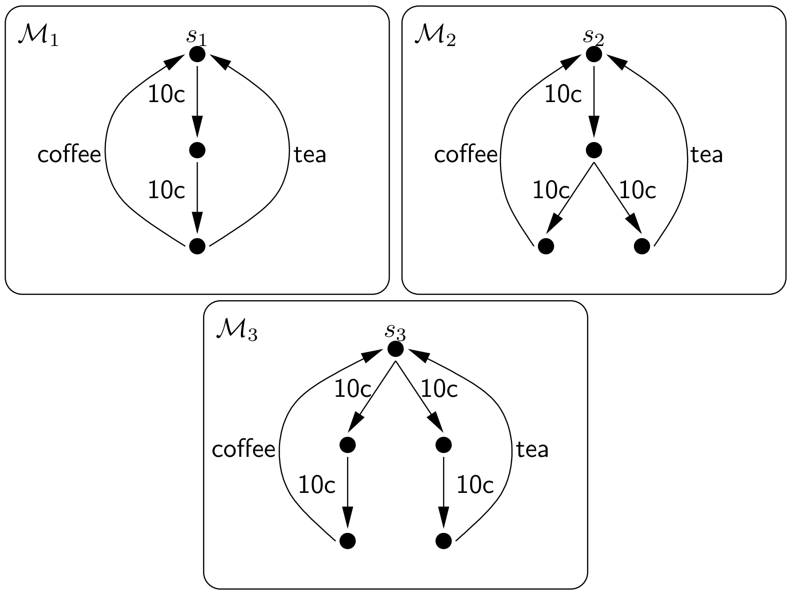
   (c) Conclude that the formula

$$(q \to [P]\langle F\rangle q) \land (q \to [F]\langle P\rangle q)$$

   defines the class of $\{F, P\}$-frames where $R_P = {R_F}^{-1}$.

[1] The inverse of a relation R is the relation $R^{-1} = \{\, (x, y) \mid (y, x) \in R \,\}$.

6. Let $I = \{\mathsf{10c}, \mathsf{coffee}, \mathsf{tea}\}$ and consider the I-models $\mathcal{M}_1$, $\mathcal{M}_2$, $\mathcal{M}_3$:



Give distinguishing formulas (over I) for the processes s_1, s_2 and s_3.