Advanced Logic 2014–15

Exercises Set 6

1. Prove the following statements by a deduction in the indicated system:
   (a) $\vdash_K Kp \vee Kq \rightarrow K(p \vee q)$
   (b) $\vdash_T p \rightarrow \neg K \neg p$
   (c) $\vdash_T Kp \rightarrow \neg K \neg p$

2. Show that the following deduction rule is admissible in system K:

$$\frac{\varphi \rightarrow \psi}{K\varphi \rightarrow K\psi}$$

3. (a) Give an S5-derivation of $\neg K \neg Kp \rightarrow p$.
   (b) Show that $\neg K \neg Kp \rightarrow p$ is not derivable in S4.

4. Show that $\nvdash_{S5} K_1 K_2 p \rightarrow K_2 K_1 p$.

5. Let R be a binary relation. R is *euclidean* if $\forall xyz\,(Rxy \wedge Rxz \rightarrow Ryz)$. Prove:
   (a) If R is reflexive and euclidean then R is symmetric.
   (b) If R is euclidean and symmetric then R is transitive.
   (c) If R is symmetric and transitive, then R is euclidean.
   (d) R is reflexive and euclidean if and only if R is an equivalence relation.

6. In this exercise you show that A2, the axiom of positive introspection, is derivable in the system K extended with axioms A1 (veridicality) and A3 (negative introspection); call this system Q.
   (a) $\vdash_Q Kp \rightarrow \neg K \neg Kp$
   (b) $\vdash_Q \neg K \neg Kp \rightarrow K \neg K \neg Kp$
   (c) $\vdash_Q K \neg K \neg Kp \rightarrow KKp$
   (d) $\vdash_Q$ A2

7. Consider a distributed system with processors $1, \ldots, n$ connected via a communication network. We denote by s_i the *local* state of processor i at a specific moment. We define the *global* state of the system to be the tuple $(s_1, \ldots, s_n)$.

   Epistemic relations R_i are now defined als follows. We assume that every processor i 'knows' in what state it is, but does not know the states of the other processors. Thus the epistemic alternatives of processor i, that is, the global states he considers possible, are precisely those global states whose i-th element corresponds to his current local state s_i.

   More precisely, we define a distributed epistemic frame $\mathcal{F}$ by:

$$\begin{aligned} \mathcal{F} &= (S, \{R_i\}_{1 \leq i \leq n}) \\ S &= \{(s_1, \ldots, s_n) \mid s_i \text{ is a local state of processor } i, \text{ for all } 1 \leq i \leq n\} \\ R_i &= \{(\vec{s}, \vec{t}) \mid s_i = t_i\} \end{aligned}$$

   (a) Show that the relations R_i are equivalence relations, and conclude that the logic S5 is valid in $\mathcal{F}$.

   (b) Show that $\mathcal{F} \vDash Cp \leftrightarrow EEp$.

8. Show that the axiom $Cp \rightarrow CEp$ is valid in all epistemic frames.

9. Prove that if the formula $p \rightarrow Ep$ is true in all points of an epistemic model $\mathcal{M}$, then so is $p \rightarrow Cp$.

10. Let Equiv be the class of epistemic n-frames $\mathcal{F} = (W, \{R_1, \ldots, R_n\})$ for the logic $S5^+$, that is, every R_i is an equivalence relation, and relations R_E and R_C are defined by $R_E = R_1 \cup \ldots \cup R_n$ and $R_C = (R_E)^+$.

    Prove or disprove validity in Equiv of the following formulas:

$$\begin{aligned} K_1(p \wedge q) &\rightarrow K_1 p \wedge K_1 K_1 q \\ K_2 K_1 p &\rightarrow K_3 K_1 p \\ K_1 K_2 p &\rightarrow K_1 p \\ Ep &\rightarrow Cp \\ EEEp &\rightarrow Cp \\ Cp &\rightarrow EEEp \end{aligned}$$