Advanced Logic 2014–15

Some useful propositional tautologies

We list some propositional tautologies that may be of use in giving derivations in one of the Hilbert-style derivation systems K, T, $S4$, and $S5$.

$$\neg\neg p \leftrightarrow p$$
$$(p \to \bot) \leftrightarrow \neg p$$
$$(p \to q) \leftrightarrow (\neg q \to \neg p)$$

$$p \to (q \to (p \wedge q))$$
$$(p \wedge q) \to p$$
$$(p \wedge q) \to q$$
$$\neg(p \wedge q) \leftrightarrow (\neg p \vee \neg q)$$

$$p \to (p \vee q)$$
$$q \to (p \vee q)$$
$$((p \vee q) \to r) \leftrightarrow ((p \to r) \wedge (q \to r))$$
$$\neg(p \vee q) \leftrightarrow (\neg p \wedge \neg q)$$

$$(p \to q) \leftrightarrow (\neg p \vee q)$$
$$((p \to q) \wedge (q \to r)) \to (p \to r)$$
$$(p \to (q \to r)) \leftrightarrow ((p \wedge q) \to r)$$
$$\neg(p \to q) \leftrightarrow (p \wedge \neg q)$$

Also useful may be the following rule, for any propositional tautology $(\varphi_1 \wedge \cdots \wedge \varphi_n) \to \psi$, and substitution σ:

$$\frac{\varphi_1^\sigma \ \ \ldots \ \ \varphi_n^\sigma}{\psi^\sigma}\,\text{PROP}$$

which we show to be admissible below.

A rule of the form

$$\frac{\alpha_1 \ \ldots \ \alpha_n}{\beta}$$

is called *admissible* in a Hilbert system H, if $\vdash_H \alpha_1, \ldots, \vdash_H \alpha_n$ implies $\vdash_H \beta$.

We show that PROP is an admissible rule in system K (and hence also in the extensions T, S4, and S5). Let $(\varphi_1 \wedge \cdots \wedge \varphi_n) \rightarrow \psi$ be a propositional tautology, and let σ be an arbitrary substitution mapping proposition letters to modal formulas. Assume that the substitution instances φ_i^σ $(1 \leq i \leq n)$ are provable in H. Then the following derivation shows $\vdash_H \psi^\sigma$:

1.	φ_1^σ	(provable by assumption)
2.	φ_2^σ	(provable by assumption)
$\vdots$	$\vdots$	$\vdots$
n.	φ_n^σ	(provable by assumption)
$n+1$.	$\varphi_1 \rightarrow (\varphi_2 \rightarrow (\cdots \rightarrow (\varphi_n \rightarrow \psi) \cdots))$	(tautology[1])
$n+2$.	$\varphi_1^\sigma \rightarrow (\varphi_2^\sigma \rightarrow (\cdots \rightarrow (\varphi_n^\sigma \rightarrow \psi^\sigma) \cdots))$	(substitution, $n+1$)
$n+3$.	$\varphi_2^\sigma \rightarrow (\cdots \rightarrow (\varphi_n^\sigma \rightarrow \psi^\sigma) \cdots)$	(modus ponens, $n+2$, 1)
$\vdots$	$\vdots$	$\vdots$
$2n+2$.	ψ^σ	(modus ponens, $2n+1$, n)

Thus the PROP rule is admissible and can be used like any other rule. For example applications, see the answers to the exercises set 6.

[1]Note that $\varphi_1 \rightarrow (\varphi_2 \rightarrow (\cdots \rightarrow (\varphi_n \rightarrow \psi) \cdots))$ is equivalent to $(\varphi_1 \wedge \cdots \wedge \varphi_n) \rightarrow \psi$, which was assumed to be a propositional tautology.