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SELF-REFERENCE IN FINITE AND INFINITE PARADOXES

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Introduction

The paradoxical character of logical paradoxes has often been ascribed to self-reference. In its starkest form the traditional liar paradox reads: This sentence is not true. The fact that no consistent truth value can be ascribed to the sentence seems to arise from its referring to itself. Other paradoxes have been assumed to arise from self-reference as well. In recent years, however, it has been argued by Sorensen and Yablo that self-reference does not play an essential role in liar paradoxes or paradoxes in general. The most convincing form of this argument is based on infinite paradoxes (paradoxes consisting of an enumerable number of sentences) of the type discovered by Yablo. On the other hand Priest has argued that even in Yablo's paradox there is a hidden element of self-reference. Priest proceeds by turning the infinite Yablo paradox into a corresponding finite paradox and indicating where the self-reference arises. Sorensen shows that with every finite liar paradox there corresponds an infinite Yablo-type paradox, in which self-reference is absent.

In this paper we present a mathematical proof that for a certain class of paradoxes the following result holds: if the paradox is finite (consists of a finite number of sentences), self-reference is necessarily involved, while if the paradox is infinite, no self-reference need be involved. This result suggests how the viewpoints of those who see self-reference as essential and those who see it as inessential can be combined. Priest shows that self-reference is essential by transforming an infinite paradox into a finite one. The advocates of the opposite view show that self-reference is not essential by transforming finite paradoxes into infinite ones.

1. *Sorensen: Self-reference not essential*

That paradoxes arise from self-reference or circularity has long seemed a natural idea. Bertrand Russell designed his theory of types so as to make circularity impossible in the formation of sets. Sorensen has argued for some time that self-reference is not essential for paradoxes to arise. In a number of publications (1982, 1984, 1988) he has presented the concept of blindspot as the key to the solution to the so-called prediction paradox. The prediction paradox concerns an unpleasant event that will be unexpected to those to whom it is announced. In the case of the surprise examination, a teacher announces an unexpected examination before the end of the week. The students reason that the exam cannot be given on Saturday, as that would make the exam expected by Friday night. Once Saturday has been eliminated, Friday becomes effectively the last day of the week, so that on Thursday night the exam will be expected on Friday. This means that Friday can be eliminated. In the same way the other days can be eliminated one by one. The reasoning shows that the announcement is inconsistent and the students conclude that an unexpected exam cannot be given. The paradox arises because, when the teacher gives the exam on, say, Tuesday, it is completely unexpected. In other words, the teacher's statement is first shown to be inconsistent and is then made true.

Sorensen's solution is based on the notion of blindspot, which is defined as follows. A proposition p is an epistemic blindspot for person A if and only if p is consistent, while the statement 'A knows p' is inconsistent. An example of an epistemic blindspot is 'It is raining but Bob doesn't know this.' This is an epistemic blindspot for Bob, but not for others. A proposition is called a conditional blindspot if it is equivalent to a conditional whose consequent is an epistemic blindspot. Sorensen now claims that if a proposition is a conditional blindspot for a person A, then it is possible for A to know the proposition and it is possible for A to know the antecedent, but it is not possible for A to know both the proposition and the antecedent. Blindspots and conditional blindspots are statements with a special status. When somebody is informed of a statement that is a blindspot for him, he will have to recognize it as a blindspot and conclude that it is unknowable for him, not that it is false. Similarly, a conditional blindspot may become unknowable when somebody knowing it is informed of the antecedent.

The teacher's announcement is a conditional blindspot for the students. Once they know that the exam has not been held on Friday, their knowledge turns into a blindspot: they know that the exam will be held on Saturday and that they cannot know this. According to Sorensen's rule the announcement becomes unknowable for them once the exam has not been held on Friday. This means that the students cannot reason backwards. They cannot conclude that the assumption that the exam has not yet been held on Friday leads to a contradiction. It only leads them to conclude that they are saddled with a blindspot, which is unknowable for them.

Sorensen (1988, p. 298) claims that the self-referential approach to the solution of the paradox is mistaken. Although he does not explicitly say so, it is clear that he assumes that his blindspot solution does not involve self-reference. In a trivial way this is indeed the case. Sorensen is of the opinion that a genuine solution of a paradox should not only explain how the problem arises, but also indicate how the problem is eliminated. He achieves this by introducing his rule that blindspots are unknowable. As long as this rule applies, the paradox does not arise, and obviously there is no self-reference. The cause of the original paradox is simply taken to be the violation of this rule. Sorensen's solution involves a modification of the rules of logic (it could even be claimed that the introduction of the rule that blindspots are unknowable is ad hoc: its only function is to eliminate the paradoxes arising from blindspots). The question of how the paradox arises when the generally accepted rules of logic apply and blindspots are taken to be knowable is not discussed by Sorensen. (Later in the same book Sorensen abandons his solution and comes up with a completely different one; we have shown elsewhere (Jongeling & Koetsier 2001) that his reasons for abandoning the blindspot solution are flawed and we will ignore this alternative approach.)

We have shown earlier (Jongeling & Koetsier 1993) that when the paradox is analysed in terms of the received, unmodified rules of logical reasoning it is seen to be self-referential. The phrase 'unexpected' can be taken to mean 'not derivable from the available information'. At any moment the students have a 'store of information', and anything that cannot be derived from the store of information is unexpected if it happens. According to this interpretation the teacher's statement refers to the students' store of knowledge ('there will be an exam, and you cannot derive this from the content of your store of knowledge'). This means that, as soon as the students accept the statement and add it to their store of knowledge, a certain kind of self-reference is introduced: the statement now refers to a set, the students' store of knowledge, of which the statement itself is an element. A blindspot S of the form 'p & A does not know p' figuring in Sorensen's solution can be taken to say 'p & p is not in A's store of knowledge'. When S is accepted by A, it enters A's store of knowledge. However, p can be derived from S and enters A's store of knowledge as well. The contradiction that arises now involves self-reference because S refers to A's store of knowledge and is itself part of A's store of knowledge. (That the announcement can be made true results from the fact that the term 'unexpected' refers to the store of knowledge of the students before the exam takes place; in Sorensen's terms we can say that after the event the announcement is no longer a blindspot for the students.) From this analysis it is clear that Sorensen's claim that self-reference is not involved in the prediction paradox is incorrect. Sorensen manages to ignore the underlying cause of the paradox, because he concentrates on how to prevent the paradox from arising. If the rule that blindspots are unknowable is followed no paradox arises, and there is no self-reference. The kind of self-reference that plays a role in the prediction paradox is obviously different in character from that involved in the liar. In the liar a statement refers directly to itself; in the prediction paradox a statement refers to a set of statements of which it is itself an element.

Sorensen has also argued that self-reference is not a necessary condition for paradox to arise on the basis of an infinite paradox discovered by Yablo (1985) in which no self-reference is apparent. Yablo's paradox can be represented as follows in the form of infinitely many statements:

$S(1)$ For all $y > 1$: non- $S(y)$
$S(2)$ For all $y > 2$: non- $S(y)$
$S(3)$ For all $y > 3$: non- $S(y)$
. . . .
Etc.

Each sentence only refers to sentences later in the sequence, so that none refers directly or indirectly (via a loop) to itself. If the first statement is assumed to be false, at least one of the following statements is true. Call this statement N. As N is true, $N + 1$ is false. This means that at least one of the statements following N is true, contrary to N. If $S(1)$ is assumed to be true, take $S(1)$ to be N. No consistent ascription of truth values is possible.

Sorensen (1998) reasons that Yablo's paradox is simply a liar paradox smeared out into infinity. He argues that not only liar paradoxes, but also many other paradoxes, such as Russell's paradox, have infinitary counterparts that do not involve self-reference. He claims this suggests that paradoxes in general do not essentially involve self-reference, as it seems likely that with any paradox there corresponds a Yabloesque version that is self-reference free.

2. *Priest: Self-reference essential*

While both Yablo and Sorensen argue that Yablo's paradox demonstrates that self-reference is not essential for paradox, Priest (1997) claims that even in Yablo's paradox there is a hidden form of self-reference. We shall give the argument in the form presented by Hardy (1995), which brings out the role of finitary and infinitary procedures. If Yablo's paradox is formulated in the form of an infinite sequence of sentences in a formal system of first order logic, attempts to consistently ascribe truth values to all statements do not lead to a contradiction. This is virtually self-evident, as a formal proof that no consistent ascription of truth values is possible would have to be of finite length. Only a finite number of the sentences $S(n)$ can figure in a finite derivation. But a finite subset of the sentences $S(n)$ is not paradoxical and can be ascribed truth values consistently. Therefore no contradiction can be formally derived.

In order to derive a contradiction from the ascription of truth values the paradox has to be formulated in a finite form, says Priest. This is easy, as all the statements have the same form. The sequence of statements can be replaced by

$$\text{For all } n : S(n) \Leftrightarrow \text{For all } k > n : \text{non-}\mathsf{S}(k, S)$$

in which S is the two-place satisfaction relation between numbers and predicates. The paradoxical character of this statement can be derived easily. The predicate S, however, is defined as $S(n) =$ For all $k > n$: non-$S(k)$. This means that a form of self-reference has returned, as the predicate is defined in terms of itself. (It should be noted that this is again a different form of self-reference, viz. a predicate that is defined in terms of itself, instead of a statement that refers to itself or to a set of which it is itself an element. It could be argued that here we have to do with a more dilute form of self-reference than in the other cases. No reference is involved in the strict sense of a sentence that refers to itself or to a sentence involving itself.)

Priest claims that, as the impossibility of consistently ascribing truth values to the statements of Yablo's paradox cannot be formally derived, the paradox has to be formulated differently, in finite terms. However, in phrasing Yablo's paradox not in the form of infinitely many sentences but in the form of one sentence, he changes the character of the paradox. As formulated by Yablo the paradox is ω inconsistent. A system is called ω inconsistent if not only the statements $S(1), S(2), S(3) \ldots$ but the statement 'non-[For all $n : S(n)$]' as well can all be proved. A system in which an inconsistency can be derived is called simply inconsistent. In a system that is simply consistent but ω inconsistent, formally no inconsistent statement can be derived, but an infinite set of statements can be derived that intuitively seems inconsistent. Strictly reasoning on the basis of simple consistency, as Priest seems to advocate, one would have to conclude that Yablo's paradox is not a paradox at all, because its paradoxical character cannot be derived in a finite number of steps. What Priest does is something quite different. What he says is basically: Yablo's paradox is obviously paradoxical, and as we can't demonstrate this in the form in which it is presented, we have to formulate it differently, so that the impossibility of consistently ascribing truth values can be derived in a finite number of steps. But this means that he replaces a set of statements that is simply consistent but ω inconsistent by a single statement that is simply inconsistent. This new, finite paradox involves a form of self-reference. Contrary to what Priest suggests, this does not imply that the original infinite Yablo paradox involves self-reference. The original infinite paradox is different in structure from the finite paradox by which Priest replaces it. Priest also claims that the sequence of statements constituting Yablo's paradox can only be generated with the help of a predicate that is self-referential. We think this is irrelevant. Once the sequence has been generated, it has a certain structure which is not self-referential, and that is what matters.

3. *The Golden Mean*

The positions about paradoxes of Sorensen and Priest seem totally contradictory. Sorensen claims that self-reference is never essentially involved, Priest claims that self-reference is always involved. As we have shown, Sorensen's claim is incorrect for the finite situation of the prediction paradox, but it seems quite convincing for the infinite situation of Yablo's paradox. Priest proves his case by arguing that Yablo's paradox should properly be formulated in finite form, but it is clear that the resulting finite paradox is different in structure from the original infinite paradox. We shall demonstrate a result that suggests that the two viewpoints can be combined, and that self-reference is necessarily involved in finite paradoxes, but not in infinite ones.

Our result is based on the following intuitive idea. If a statement S_1 refers to one or more other statements S_i and ascribes a certain truth value to some function of the S_i, the truth value of S_1 itself depends on the truth values of the S_i. The truth values of the S_i may in turn depend on other statements. The relations between the statements form a graph. If the graph is finite and contains no self-referential loops, it is possible, starting at S_1, to move down along the graph to the ends of the branches, to establish the truth values at the end nodes, and working one's way backward to the origin, to determine unambiguously the truth values at the nodes and finally S_1. No contradiction, no paradox can arise. (The graph need not be a tree, as there may be what we define further down as 'double-referential cycles'.) If the graph is infinite, we cannot do this; at least along some branches there are no end nodes that we can reach. In a finite graph with self-referential loops the situation is similar. Not all branches terminate in an end node. This suggests that nothing can go wrong in finite sets of statements without self-reference, but that contradictions may arise both in sets of statements with self-reference (loops) and in infinite sets (whether they contain loops or not).

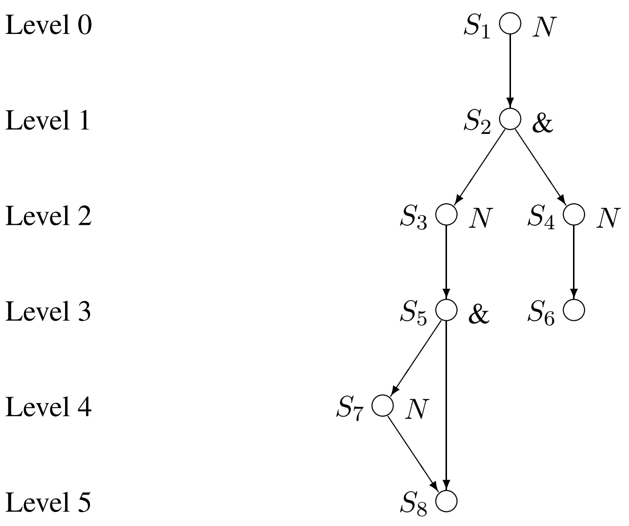
In this context a paradox is a set of statements to which no truth values can be assigned consistently. This implies that we do not consider paradoxes like Buridan's Sophism 8 (Goldstein 1999, p. 286), which has the form: S_1 says S_2 is false, and S_2 says S_1 is false. These statements are 'paradoxical' in the sense that truth values can be assigned in two different ways (S_1 is true and S_2 is false, and the other way round) and that there is no reason to prefer either. This is of course a curious situation, but it is not a paradox in the sense in which we use the term in this paper.

4. *Finite liars require self-reference*

We use a language suitable to handle classical propositional logic. We restrict ourselves to propositions that are either finite or infinite conjunctions or disjunctions of finite expressions. First we will consider finite propositions. In Section 5 we will also consider infinite propositions. A graphical representation of the referential pattern sometimes facilitates its understanding. Let us consider an example.

Example 1
(S_1) non-S_2
(S_2) S_3 & S_4
(S_3) non-S_5
(S_4) non-S_6
(S_5) S_7 & S_8
(S_6)
(S_7) non-S_8
(S_8)

Example 1 can be graphically represented as follows.



Example 1
(To the right of each node the type of proposition is indicated)

If, as in Example 1, there is no self-reference, we can name the propositions in such a way that, for all n, S_n only refers to propositions S_k with $k > n$. It is often convenient to distinguish, within the sequence, *levels* 0, 1, 2, 3, etc. Sentences that are not referred to by other propositions constitute level 0. Sentences that are only referred to from level 0 constitute level 1. In general: propositions that are referred to from level n propositions and not from propositions that do not belong to the levels 0 through n, constitute level $(n + 1)$. This means that if there is no self-reference and the levels are defined in this way, the propositions at a particular level only refer to higher-level propositions. (NB Note that higher-level propositions are further downward in the graphical representations.) If there is self-reference in a sequence of propositions the notion of level is no longer meaningful, although we can still use graphical representations. In general, if there is no self-reference, the truth value of a proposition is determined by the truth values of all higher-level propositions. This yields the following general result: If we have a finite sequence of finite propositions without (direct or indirect) self-reference,

$$(S_1)E_1(S_2, S_3, S_4, \ldots)$$
$$(S_2)E_2(S_3, S_4, S_5, \ldots)$$
$$.$$
$$(S_{n-1})E_{n-1}(S_n)$$
$$(S_n),$$

no paradox can arise. Here $E_i(S_j, \ldots S_{j+k})$ denotes a proposition defined in terms of the propositions S_j through S_{j+k}.

Proof. Because the number of propositions is finite, the number of levels will be finite. This means that we can assign arbitrary truth values to all atomic propositions, including the propositions at the highest level, which are necessarily all atomic. By moving along the levels from the highest to level 0, we establish that the sequence S_1 through S_n can be satisfied and cannot be paradoxical.

From this we conclude by contraposition: *Liar-paradoxes consisting of a finite number of finite propositions must necessarily contain direct or indirect self-reference.*

In the next section we will generalize this result.

5. *Infinite liars without self-reference*

We will consider infinite sequences of propositions without self-reference and we will derive four necessary conditions for paradox. First of all, from the result of Section 4 we can immediately draw the conclusion that

(i) *a necessary condition for a paradox without self-reference is that the sequence of propositions is infinite.*
(ii) *a necessary condition for a paradox without self-reference is that the sequence of propositions is such that if we remove for arbitrary k all propositions belonging to the levels 0 through k from the sequence, the remaining sequence is still paradoxical.*
The proof is easy. Suppose after such a removal the remaining sequence can be satisfied. Then the original sequence can also be satisfied, because the truth value assignment for the reduced sequence can be extended to a truth value assignment for the whole sequence by 'moving upwards in the graph'.
(iii) *a necessary condition for a paradox without self-reference is that there exists in the sequence of propositions an infinite referential path. This means that in the graphical representation there exists an infinite path starting at level 0 and reaching, for all levels k, levels lower in the graph.*
Proof. Suppose we have a paradox without self-reference and all paths starting at level 0 are finite. Then they all end in atomic statements. By systematically assigning arbitrary truth values to all atomic statements, the truth values of all the other propositions are uniquely determined. Contradiction. So at least one path must be infinite.
(iv) *a necessary condition for a paradox without self-reference is that the sequence of propositions contains, for all levels k, infinitely many infinite propositions on levels lower in the graph.*
We first prove a lemma: an infinite sequence of finite propositions without self-reference can never be paradoxical. The proof can also be given by means of the compactness theorem for propositional logic.
Consider an infinite sequence of finite propositions that starts as follows:

(S_1) non-S_2
(S_2) S_3 & S_4
(S_3) non-S_4
(S_4) non-S_5
(S_5) S_6 & S_7 & S_8
Etc.

Let V_n be the set of all possible truth value assignments that satisfy S_1 through S_k, where k is the largest index that occurs in the definitions of S_1 through S_n. This means that, for $n = 1$, we consider the definitions of the propositions S_1 and S_2. In the example V_2 is the set of all possible truth value assignments that satisfy S_1 through S_4, because S_1 refers to S_2 while S_2 refers to S_3 and S_4, so that the largest index that is referred to is 4. In the example we have $V_1 = \{01, 10\}$, $V_2 = \{1001, 1010, 1000, 0111\}$, $V_3 = \{1001, 1010\}$, $V_4 = \{10010, 10101\}$, etc. (The elements of V_n are represented as binary sequences; so 10010 in V_4 corresponds to the assignment true, not true, not true, true, not true to respectively the propositions S_1 through S_5.) The sets V_n are all non-empty (because of the result of Section 4) and they are finite (because the propositions are finite). Moreover, clearly all elements of a set V_n are extensions of, or equal to, certain elements of V_{n-1}.

For all k there is at least one element r in V_k that is such that for all $n > k$ this element is identical with the initial part of some element of V_n. In other words: for all k there is at least one element r in V_k that can be extended indefinitely far. Why? Suppose such an element does *not* exist. Then for all elements ρ of the finite set V_k there would be a number h such that ρ can be extended to an element of V_h, but not to an element of V_{h+1}. Because V_k is finite this would imply that the elements of V_k would not allow extension beyond a certain length, which would contradict the fact that arbitrary long finite sequences of propositions possess truth value assignments.

We can now show that the sequence of propositions cannot be paradoxical. We choose an element e_1 from V_1 that allows arbitrarily long extensions. Then we choose from V_2 an element e_2 that is an extension of e_1 and allows infinitely long extensions. Then we choose from V_3 an element e_3 that is an extension of e_2 and allows infinitely long extensions, etc. Step by step we extend the truth value assignment and because the process can be repeated indefinitely this proves the existence of an infinite sequence of truth values that satisfies the infinite sequence of propositions. N.B. In a formal treatment we would need a weak axiom of choice.

This lemma actually generalises the result from Section 4. We now also have: *Liar-paradoxes consisting of a infinite number of finite propositions necessarily contain direct or indirect self-reference.*

Proof of condition (iv): From the lemma we can immediately draw the conclusion that if an infinite sequence without self-reference is a paradox it contains at least one infinite proposition. Necessary condition (ii) implies that the number of infinite propositions must be infinite.

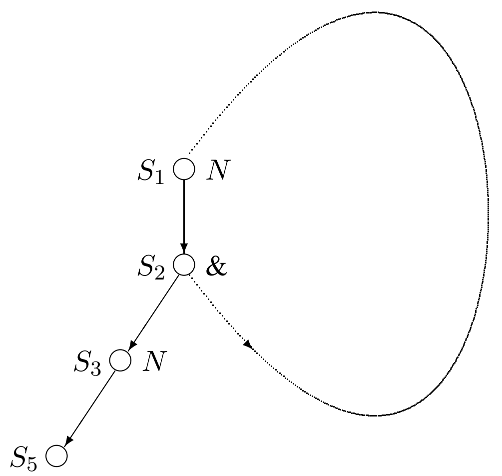
The result we have derived applies only to a certain class of liar paradoxes, viz paradoxes that can be formulated in terms of the restricted language of propositional logic that we have used here. This means that it only concerns the first type of self-reference we have come across, involving statements that directly or indirectly refer to their own truth value. We should not be surprised, however, if our result is valid more generally.

6. *The flow of truth values and double references*

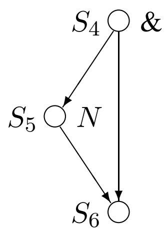
In this section we consider a simplified language, in which all propositions are either conjunctions of other propositions and contain no negations, or consist of a once-negated proposition. Self-reference in the graphs of Examples 2 and 3 corresponds to the occurrence of cycles that return to the starting point when the direction of the arrows is followed. We will call them self-referential cycles. Self-referential cycles with an odd number of negations always limit the possible truth value assignments. It is clear that an isolated cycle with an odd number of negations is necessarily paradoxical. However, in Example 2 the result of the presence of the cycle is that S_3 can only have the truth value 0, and then S_4 must necessarily have truth value 1. Example 3 shows there is another way to limit possible truth value assignments. There is an (indirect) double reference from S_4 to S_6. The result is that S_4 must necessarily have truth value 0. This indirect double reference to a proposition from a higher-level proposition corresponds to a different type of cycle in the graph, viz. one that leads to the same point via two different routes. We will call such cycles double-referential cycles. Double-referential cycles with an odd number of negations also limit the possible truth value assignments. (We discuss double references only in the context of the simplified language. In graphical representations of sequences of simplified-language propositions double references appear as separate links. In more complex languages a proposition may refer to another proposition in more than one way, which does not show up in the corresponding graph.)

When we identify the S_4 in Example 2 with the S_4 in Example 3, that is when we paste the two graphs together, the result is paradoxical. The self-referential cycle forces S_4 to possess truth value 1, while the double-referential cycle forces that same truth value to be 0. Although it is not difficult to construct finite liar paradoxes in this way, one has to be careful. A paradoxical cycle does not necessarily remain paradoxical when another graph is pasted on to it, if as a result of the pasting a new conjunction is introduced.

This example shows that a paradox can be produced by splicing a self-referential graph on to a double-referential graph. Is it also possible to produce a paradox by splicing two double-referential graphs? The answer is: no. In a double-referential graph the element whose truth value is forced is at (what is in the figure) the top of the graph. It is easily possible to construct two double-referential graphs that have different truth values at their top elements. However, by identifying these elements so as to paste the two graphs, we change the character of the top element: we introduce a new conjunction and as a result the expected paradox does not materialise. In self-referential graphs, the element whose truth value is forced can be situated at the bottom, and in that case the graph can be pasted on to another one without a change in the character of the linking element, and as a result a combination of a self-referential graph with a double-referential graph or of two self-referential graphs can be paradoxical.



Example 2



Example 3

Although double-referential cycles with an odd number of negations can play an interesting role in finite paradoxes, they can produce paradox only in combination with self-referential graphs. In finite paradoxes, self-reference is absolutely essential. In infinite paradoxes the situation is different. For paradoxes formulated in the simplified language used in this section we can formulate another condition that highlights the role of double references.
(v) *a necessary condition for a paradox without self-reference is that the sequence of propositions contains infinitely many double references, which occur at infinitely many levels.* The proof runs as follows. Suppose we have a paradox without self-reference and there are no double references at all. In this case we can construct a truth value assignment starting from an arbitrary assignment on level 0 and from that assignment derive a possible assignment on level 1, from that one for level 2, etc. The conclusion is that the sequence of propositions can be satisfied, which contradicts the assumption that the sequence is paradoxical. This means that there must be at least one double reference. However, because of necessary condition (ii) the existence of one double reference implies the existence of infinitely many double references.

Yablo's paradox can be formulated in the simplified language in the following way.

(S_1) S'_2 & S'_3 & $S'_4 \dots$
(S'_2) non-S_2
(S_2) S'_3 & S'_4 & $S'_5 \dots$
(S'_3) non-S_3
Etc.

Some of the paradoxes discussed in the following section (e.g. example 5) cannot be formulated in the simplified language.

7. *Yablo's paradox*

The referential pattern of Yablo's paradox clearly satisfies the necessary conditions of Sections 5 and 6. The five conditions derived in the preceding sections are not sufficient for paradox. If the proposition "The following propositions are all true" is repeated ω times, the conditions (i) through (v) are all satisfied, but we have no paradox. So what brings about Yablo's paradox?

The infinite sequence of propositions that constitutes Yablo's paradox is a limit sequence of a sequence of sequences Seq 1, Seq 2, Seq 3, ..., Seq n, etc. with Seq $n = [S_1, S_2, \ldots, S_n]$. The informal notion of limit that we use here can easily be made precise. Clearly for all n, Seq n can be satisfied, while the limit sequence cannot! From this point of view Yablo's paradox and similar paradoxes exhibit a particular discontinuity phenomenon. The way in which the multiple references do the job becomes clearer if we look at the way in which Seq n can be satisfied. Independently of what truth value we assign to S_n the truth values of S_1 through S_{n-2} are necessarily 0 while the truth values of S_{n-1} and S_n must be opposite. The referential structure of Seq n very much limits the possible assignment of truth values. All truth values necessarily become 0 except for one unique truth value 1 that must occur in one of the last two propositions. In a way the referential structure in Yablo's sequence of propositions still implies a true proposition among the last two propositions, while at the same time there are no last propositions. This rather heuristic analysis finds some support in what happens in the following example. By defining the limit somewhat differently we get a sequence of ordinal number $\omega + 1$.

Example 4
(S_1) The following propositions are all not true
(S_2) The following propositions are all not true
.
.
(S_n) The following propositions are all not true
Etc.
(S_ω) The following propositions are all not true

Example 4 constitutes no paradox: We can assign truth value 1 to the last proposition and truth value 0 to all the others. It is easy to construct other sequences that possess a similar discontinuity at infinity.

Example 5
(S_1) Among the following propositions there are at least 3 propositions not true.
(S_2) Among the following propositions there are at least 3 propositions not true.
.
.
(S_n) Among the following propositions there are at least 3 propositions not true.
Etc.

Seq n is such that the truth values of S_1 through S_{n-3} are necessarily 1. The truth values of S_{n-2} through S_n are necessarily all zero. For the limit sequence there exists no truth value assignment.
Remarkable is also:

Example 6
(S_1) For all k greater than 1 of which the decimal representation of k ends in 1, S_k is not true
(S_2) For all k greater than 2 of which the decimal representation of k ends in 2, S_k is not true
.
.
(S_n) For all k greater than n of which the decimal representation of k ends in the decimal representation of n, S_k is not true
Etc.

In Example 6 the referential structure of Seq n is such that all possible truth value assignments to S_1 through S_n have for growing n necessarily an initial segment of zeroes that grows indefinitely in length, while there is also necessarily a last segment consisting (partially) of ones.

8. *Conclusion*

Until now the discussion on the question of whether self-reference is essentially involved in paradoxes has not led to any definite conclusions. Sorensen claims that self-reference is not essential for paradox to arise. His argument concerning the finite prediction paradox is incorrect, and in the case of liar paradoxes his claim is refuted by the theorem we have proved. For infinite paradoxes his stance is based on Yablo's paradox and is more convincing. Priest on the other hand argues that even Yablo's paradox involves self-reference. He derives his result by replacing the original infinite Yablo paradox by an analogous finite one, which does involve self-reference. We have argued that Yablo's paradox is an independent infinite paradox, which involves ω-inconsistency and is not self-referential. The mathematical result that we have derived shows that finite liar paradoxes necessarily involve self-reference, while infinite ones need not involve self-reference, although they have to satisfy a number of other conditions. This theorem suggests how Sorensen and Priest can reach their contradictory conclusions. Sorensen bases (part of) his argument on the infinite Yablo paradox and then generalises to all paradoxes, while Priest transforms Yablo's paradox into an analogous finite one.

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