

ADJOINT SPACES AND FLAG VARIETIES OF p -COMPACT GROUPS

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ABSTRACT. For a compact Lie group G with maximal torus T , Pittie and Smith showed that the flag variety G/T is always a stably framed boundary. We generalize this to the category of p -compact groups, where the geometric argument is replaced by a homotopy theoretic argument showing that the class in the stable homotopy groups of spheres represented by G/T is trivial, even G -equivariantly. As an application, we consider an unstable construction of a G -space mimicking the adjoint representation sphere of G inspired by work of the second author and Kitchloo. This construction stably and G -equivariantly splits off its top cell, which is then shown to be a dualizing spectrum for G .

1. INTRODUCTION

Let G be a compact, connected Lie group of dimension d and rank r with maximal torus T . Left translation by elements of G on the tangent space $\mathfrak{g} = T_e G$ induces a framing of G . By the Pontryagin-Thom construction, G with this framing represents an element $[G]$ in the stable homotopy groups of spheres; this has been extensively studied for example in [Smi74, Woo76, Kna78, Oss82].

The following classical argument shows that the flag variety G/T , while not necessarily framed, is still a stably framed manifold: since every element in a compact Lie group is conjugate to an element in the maximal torus, the conjugation map $G \times T \rightarrow G$, $(g, t) \mapsto gtg^{-1}$ is surjective, and furthermore, it factors through $c: G/T \times T \rightarrow G$. An element $s \in T$ is called *regular* if the centralizer $C_G(s) \supseteq T$ equals T , or, equivalently, if $c|_{G/T \times \{s\}}$ is an embedding; it is a fact from Lie theory that the set of irregular elements has positive codimension in T . Thus there is a regular element s such that the derivative of c has full rank along $G/T \times \{s\}$, and by the tubular neighborhood theorem, it induces an embedding of $G/T \times U$, where U is a contractible neighborhood of s in T . Thus the framing of G can be pulled back to a stable framing of G/T .

Pittie and Smith showed in [Pit75, PS75] that G/T is always the boundary of another framed manifold M , and moreover, that M has a G -action which agrees with the standard G -action on G/T on the boundary. In terms of homotopy theory, this is saying that the class $[G/T] \in \pi_{d-r}^s$ induced by the Pontryagin-Thom construction is trivial.

The first main result of this paper generalizes this fact to p -compact groups.

Theorem 1.1. *Let G be a \mathbf{Z}/p -local, p -finite group with maximal torus T such that $\dim(G) > \dim(T)$. Then the Pontryagin-Thom construction $[G/T]: S_G \rightarrow S_T$ is G -equivariantly null-homotopic, with G acting trivially on S_T .*

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The statement of this theorem requires some explanation. A *p*-compact group [DW94] is a triple (G, BG, e) such that

- G is *p*-finite, i. e., $H_*(G; \mathbf{F}_p)$ is finite;
- BG is \mathbf{Z}/p -local, i. e. whenever $f: X \rightarrow Y$ is a mod- p homology equivalence of CW-complexes, then $f^*: [Y, BG] \rightarrow [X, BG]$ is an isomorphism;
- $e: G \rightarrow \Omega BG$ is a homotopy equivalence.

Clearly, G and e are determined by BG up to homotopy, making this definition somewhat redundant. Although a priori G is only a loop space, we will henceforth assume we have chosen a rigidification such that G is actually a topological group. This is always possible, for example by using the geometric realization of Kan's group model of the loops on a simplicial set [Kan56].

A \mathbf{Z}/p -local, *p*-finite loop space is only slightly more general than a *p*-compact group in that the latter also requires $\pi_0(G)$ to be a *p*-group. We will have no need to assume this in Theorem 1.1.

By [DW94], every *p*-compact group has a maximal torus T ; that is, there is a monomorphism $T \rightarrow G$ with $T \simeq L_p(\mathbf{S}^1)^r$ and r is maximal with this property. By definition, a *monomorphism* of *p*-compact groups is a group monomorphism $H \rightarrow G$ such that G/H is *p*-finite (see [Bau04] for this slightly nonstandard point of view). Dwyer and Wilkerson show that T is essentially unique. Since a maximal torus is always contained in the identity component of a *p*-compact group, the same works for \mathbf{Z}/p -local, *p*-finite groups.

Denote by $\mathbf{S}^0[X]$ the suspension spectrum of a space X with a disjoint base point added.

Definition ([Kle01]). Let G be a topological group. Define S_G , the *dualizing spectrum* of G , to be the spectrum of homotopy fixed points of the right action of G on its own suspension spectrum. That is, $S_G = (\mathbf{S}^0[G])^{hG^{\text{op}}}$ as left G -spectra.

In [Bau04], the first author showed that for a connected, d -dimensional *p*-compact group G , S_G is always homotopy equivalent to a \mathbf{Z}/p -local sphere of dimension d . Furthermore, there is a G -equivariant logarithm map $\mathbf{S}^0[G] \rightarrow S_G$, where G acts on the left by conjugation. If G is the \mathbf{Z}/p -localization of a connected Lie group, then S_G is canonically identified with the suspension spectrum of the one-point compactification of the Lie algebra of G . Thus we may call S_G the *adjoint (stable) sphere* of G .

In a spectacular case of shortsightedness, [Bau04] restricts its scope to connected *p*-compact groups where everything would have worked for \mathbf{Z}/p -local, *p*-finite groups G as well. In this case, S_G has the mod- p homology of a d -dimensional sphere. Similarly, the proof of the following was given in [Bau04, Cor. 24] for connected groups, but immediately generalizes.

Let DM be the Spanier-Whitehead dual of a finite CW-spectrum M .

Lemma 1.2. *Let $H < G$ be a monomorphism of \mathbf{Z}/p -local, *p*-finite groups. Then there is a relative G -equivariant duality weak equivalence*

$$G_+ \wedge_H S_H \simeq D \left(\mathbf{S}^0[G/H] \right) \wedge S_G.$$

For any space X , there is a canonical map $\epsilon: \mathbf{S}^0[X] \rightarrow \mathbf{S}^0$ given by applying the functor $\mathbf{S}^0[-]$ to $X \rightarrow *$. If $T < G$ is a sub-torus in a \mathbf{Z}/p -local, *p*-finite group then

there is a stable G -equivariant map

$$(1.3) \quad [G/T]: S_G \xrightarrow{\text{id} \wedge D\epsilon} S_G \wedge D(\mathbf{S}^0[G/T]) \\ \underset{\text{Lemma 1.2}}{\simeq} G_+ \wedge_T S_T \simeq \mathbf{S}^0[G/T] \wedge S_T \xrightarrow{\epsilon} S_T$$

where the homotopy equivalence on the right hand side holds because S_T has a homotopy trivial T -action as T is homotopy abelian. The first map is studied in [Bau04]. This is the map referred to in Theorem 1.1; it generalizes the Pontryagin-Thom construction.

In the second part of this paper, as an application of Theorem 1.1, we study the relationship between two notions of adjoint objects of p -compact groups. It is an interesting question to ask whether the action of G on S_G actually comes from an unstable action of G on \mathbf{S}^d . We will not be able to answer this question in this paper. However, there is an alternative, unstable construction of an adjoint object for a connected p -compact group G inspired by the following:

Theorem 1.4 ([CK02, Mit88]). *Let G be a semisimple, connected Lie group of rank r . There exist subgroups $G_I < G$ for every $I \subsetneq \{1, \dots, r\}$ and a homeomorphism of G -spaces*

$$A_G := \Sigma \text{hocolim}_{I \subsetneq \{1, \dots, r\}} G/G_I \rightarrow \mathfrak{g} \cup \{\infty\}$$

to the one-point compactification of the Lie algebra \mathfrak{g} of G .

In the second part of this paper, we define a G -space A_G for every connected p -compact group G and show:

Theorem 1.5. *For any connected p -compact group G , there is a G -equivariant splitting $\mathbf{S}^0[A_G] \simeq S_G \vee R$ for some finite G -spectrum R .*

This result links the two notions of adjoint objects together. Thus stably, the adjoint sphere is a wedge summand of the adjoint space.

Unfortunately, A_G is in general not a sphere.

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2. THE STABLE p -COMPLETE SPLITTING OF COMPLEX PROJECTIVE SPACE

2.1. Stable splittings from homotopy idempotents. Let p be a prime. We denote by L_p the localization functor on topological spaces with respect to mod- p homology, which coincides with p -completion on nilpotent spaces [BK72]. Let $S = L_p \mathbf{S}^1$ be the p -complete 1-sphere, and set $P = \mathbf{S}^0[BS]$. It is a classical result that

$$(2.1) \quad P \simeq \bigvee_{s=0}^{p-2} P_s$$

for certain $(2i - 1)$ -connected spectra P_i . In this section, we will investigate this splitting and its compatibility with certain transfer maps.

Let X be a spectrum, $e \in [X, X]$ and define

$$eX = \text{hocolim}\{X \xrightarrow{e} X \xrightarrow{e} \dots\}.$$

If e is idempotent, this is a homotopy theoretic analog of the image of e . Any such idempotent e yields a stable splitting $X \simeq eX \vee (1 - e)X$. If $\{e_1, \dots, e_n\}$ are a complete set of orthogonal idempotents (this means that each e_i is idempotent, $e_i e_j \simeq *$, and $\text{id}_X \simeq e_1 + \dots + e_n$), then they induce a splitting $X \simeq e_1 X \vee \dots \vee e_n X$.

Example 2.2. Let p be an odd prime. Denote by $\psi: P \rightarrow P$ the map induced by multiplication with a $(p - 1)$ st root of unity ζ . Define $e_s: P \rightarrow P$ by

$$e_s = \frac{1}{p-1} \left(\sum_{i=0}^{p-2} \zeta^{-is} \psi^i \right).$$

It is straightforward to check that $\{e_0, \dots, e_{p-2}\}$ are a complete set of orthogonal idempotents in $[P, P]$. They induce the splitting (2.1) by defining $P_s = e_s P$.

Setting $H_*(P) = \mathbf{Z}_p\{x_j\}$ with $|x_j| = 2j$, we have that $(e_i)_*: H_*(P) \rightarrow H_*(P)$ is given by

$$(2.3) \quad (e_i)_*(x_j) = \begin{cases} x_j; & j \equiv i \pmod{p-1} \\ 0; & \text{otherwise.} \end{cases}$$

2.2. Transfers as splittings. Let $1 \rightarrow H \xrightarrow{i} G \rightarrow W \rightarrow 1$ be an extension of compact Lie groups. Then associated to the fibration $W \rightarrow BH \rightarrow BG$ there are two versions of functorial stable transfer maps [BG75, BG76]:

- (1) The Becker-Gottlieb transfer $\bar{\tau}: \mathbf{S}^0[BG] \rightarrow \mathbf{S}^0[BH]$
- (2) The stable Umkehr map $\tau: BG^{\mathfrak{g}} \rightarrow BH^{\mathfrak{h}}$ of Thom spaces of the adjoint representation of the Lie groups.

Both versions can be generalized to a setting where the groups involved are not Lie groups but only \mathbf{Z}/p -local and p -finite [Dwy96, Bau04]. For such a group G , $BG^{\mathfrak{g}}$ is defined to be the homotopy orbit spectrum of G acting on the dualizing spectrum S_G ; since $H_*(S_G) = H_*(\mathbf{S}^d; \mathbf{Z}_p)$, we have a (possibly twisted) Thom isomorphism between $H_*(BG)$ and $H_*(BG^{\mathfrak{g}})$.

Note that $\bar{\tau}$ factors through τ in the following way:

$$(2.4) \quad \mathbf{S}^0[BG] \xrightarrow{\tau'} BH^{\nu} \xrightarrow{\text{comult.}} BH^{\nu} \wedge_{BG} \mathbf{S}^0[BH] \\ \xrightarrow{\text{id} \wedge \Delta} BH^{\nu} \wedge_{BG} \mathbf{S}^0[BH] \wedge_{BG} \mathbf{S}^0[BH] \xrightarrow{\text{eval} \wedge \text{id}} \mathbf{S}^0[BH]$$

where $\nu = \mathfrak{h} - i^* \mathfrak{g}$ is the normal fibration along the fibers of $BH \rightarrow BG$, τ' is τ twisted by $-\mathfrak{g}$, and the right hand side evaluation map is defined by identifying BH^{ν} with the fiberwise Spanier-Whitehead dual of BH over BG .

Proposition 2.5. *Let $W = C_l$ be a finite cyclic group acting freely on S , with $l \mid p - 1$. Denote by $N = S \rtimes W$ the semidirect product with respect to this action. Then the Becker-Gottlieb transfer map $\bar{\tau}: \mathbf{S}^0[BN] \rightarrow P$ factors through $fP \rightarrow P$ for some idempotent $f: P \rightarrow P$ which induces the same map in homology as $e_0 + e_1 + \dots + e_{p-1-l}$, and the induced map $\mathbf{S}^0[BN] \rightarrow fP$ is a mod- p homology equivalence.*

Proof. Since $p \nmid |W|$, the Serre spectral sequence associated to the group extension $S \xrightarrow{i} N \rightarrow W$ is concentrated on the vertical axis and shows that

$$H^*(BN; \mathbf{Z}_p) \cong H^*(BS; \mathbf{Z}_p)^W \cong \mathbf{Z}_p[z^l] \hookrightarrow \mathbf{Z}_p[z] \cong H^*(BS; \mathbf{Z}_p).$$

In this case, the Becker-Gottlieb transfer is nothing but the usual transfer for finite coverings, therefore $i \circ \bar{\tau}$ is multiplication by $|W| = l \in \mathbf{Z}_p^\times$. Setting $I = l^{-1}i: P \rightarrow \mathbf{S}^0[L_p BN]$, we thus get orthogonal idempotents in $[P, P]$:

$$f = \bar{\tau} \circ I \quad \text{and} \quad e = \text{id}_P - f.$$

Clearly, $e \circ \bar{\tau} \simeq *$, thus $\bar{\tau}$ factors through fP and induces an isomorphism $\mathbf{S}^0[L_p BN] \rightarrow fP$, in particular a mod- p homology isomorphism between $\mathbf{S}^0[BN]$ and fP . The computation of the homology of BN together with (2.3) implies that $f_* = (e_0 + e_l + \cdots + e_{p-1-l})_*$. \square

Corollary 2.6. *Let S, N, W be as above. Then the stable Umkehr map*

$$BN^n \rightarrow BS^s \simeq \Sigma P$$

factors through $\Sigma fP \rightarrow \Sigma P$ for some $f: P \rightarrow P$ which induces the same morphism in homology as $\sum_{i=0}^{\frac{p-1}{l}} e_{(i+1)l-1}$. The induced map $BN^n \rightarrow \Sigma fP$ is a mod- p homology equivalence.

Proof. This follows from a similarly simple homological consideration. The S -fibration n is not orientable, thus we have a twisted Thom isomorphism

$$\tilde{H}^{n+1}(BN^n) \cong H^n(BN; \mathcal{H}^1(S; \mathbf{Z}_p))$$

where $\pi_1(BN) = \mathbf{Z}/l$ acts on $H^1(S; \mathbf{Z}_p) \cong \mathbf{Z}_p$ by multiplication by an l th root of unity. Thus

$$H^i(BN^n; \mathbf{Z}_p) = \begin{cases} \mathbf{Z}_p; & i \equiv -1 \pmod{l} \\ 0; & \text{otherwise.} \end{cases}$$

The factorization (2.4) of $\bar{\tau}$ through τ

$$\begin{aligned} \mathbf{S}^0[BN] &\xrightarrow{\tau'} BS^\nu \xrightarrow{\text{comult.}} BS^\nu \wedge_{BN} \mathbf{S}^0[BS] \\ &\xrightarrow{\text{id} \wedge \Delta} BS^\nu \wedge_{BN} \mathbf{S}^0[BS] \wedge_{BN} \mathbf{S}^0[BS] \xrightarrow{\text{eval} \wedge \text{id}} \mathbf{S}^0[BS] \end{aligned}$$

simplifies considerably since $i^* \nu$ is the trivial 1-dimensional fibration over BS , and the composition of the three right hand side maps is an equivalence.

In Prop. 2.5 it was shown that $I \circ \bar{\tau} = \text{id}_{\mathbf{S}^0[L_p BN]}$, thus the same holds after twisting with n :

$$\text{id}_{BN^n}: L_p BN^n \xrightarrow{\tau} BS^s \rightarrow BS^{i^*n} \xrightarrow{I^n} L_p BN^n.$$

If we denote the composition $BS^s \rightarrow BS^{i^*n} \xrightarrow{I^n} L_p BN^n$ by I , overriding its previous meaning, the argument now proceeds as in Prop. 2.5. Using the computation of $H^*(BN^n; \mathbf{Z}_p)$, we find that $L_p BN^n \simeq (\tau \circ I)P$, and

$$(\tau \circ I)_* = \sum_{i=0}^{\frac{p-1}{l}} (e_{(i+1)l-1})_*$$

\square

3. FRAMING p -COMPACT FLAG VARIETIES

Before proving Theorem 1.1, we need an alternative description of the Pontryagin-Thom construction (1.3) on G/T .

Lemma 3.1. *The map $[G/T]$ is G -equivariantly homotopic to the map*

$$S_G \xrightarrow{\text{incl}} BG^{\mathfrak{g}} \xrightarrow{\tau} BT^{\mathfrak{t}} \simeq \Sigma^r \mathbf{S}^0[BT] \xrightarrow{\Sigma^r \epsilon} \mathbf{S}^r,$$

where $BG^{\mathfrak{g}}$, $BT^{\mathfrak{t}}$, and τ are as in Section 2.2, and all spectra except S_G have a trivial G -action.

Proof. Applying homotopy G -orbits to (1.3), we get a G -equivariant diagram

$$\begin{array}{ccccccc} S_G & \longrightarrow & S_G \wedge D(\mathbf{S}^0[G/T]) & \xrightarrow{\sim} & \mathbf{S}^0[G/T] \wedge S_T & \longrightarrow & S_T \\ \downarrow \text{incl} & & & & \downarrow & & \parallel \\ BG^{\mathfrak{g}} & \xrightarrow{\tau} & BT^{\mathfrak{t}} & \xrightarrow{\sim} & \mathbf{S}^0[BT] \wedge S_T & \longrightarrow & S_T \end{array}$$

which is commutative by the definition of τ [Bau04, Def. 25]. \square

In the proof of Theorem 1.1, certain special subgroups will play an important role. In order to define them we need to recall certain facts about the Weyl group of a p -compact group.

Dwyer and Wilkerson showed in their ground-breaking paper [DW94] that given any connected p -compact group G with maximal torus T , there is an associated *Weyl group* $W(G)$, which is defined as the group of components of the homotopy discrete space of automorphisms of the fibration $BT \rightarrow BG$. This generalizes the notion of Weyl groups of compact Lie groups; they are canonically subgroups of $GL(H_1(T)) = GL_r(\mathbf{Z}_p)$, and they are so-called finite complex reflection groups. This means that they are generated by elements (called reflections or, more classically, pseudo-reflections) that fix hyperplanes in \mathbf{Z}_p^r . The complete classification of complex reflection groups over \mathbf{C} is classical and due to Shephard and Todd [ST54], the refinement to the p -adics is due to Clark and Ewing [CE74].

Call a reflection $s \in W$ *primitive* if there is no reflection $s' \in W$ of strictly larger order such that $s = (s')^k$ for some k .

Denote by $s \in W$ a primitive reflection of minimal order $l > 1$. Let $T^s < T$ be the fixed point subtorus under s . Since s is primitive,

$$\langle s \rangle = \{w \in W \mid w|_{T^s} = \text{id}_{T^s}\}.$$

Definition. Given connected p -compact group G and a primitive reflection $s \in W(G)$ of minimal order $l > 1$, define C_s to be the centralizer of T^s in G .

Since G is connected, so is the subgroup C_s [DW95, Lemma 7.8]. Furthermore, C_s has maximal rank because $T < C_s$ by definition, and the inclusion $C_s < G$ induces the inclusion of Weyl groups $\langle s \rangle < W$ [DW95, Thm. 7.6]. Since the Weyl group of C_s is \mathbf{Z}/l , the quotient of C_s by its p -compact center, $C_s/Z(C_s)$, can have rank at most 1. By the (almost trivial) classification of rank-1 p -compact groups, we find that its rank is equal to 1 and

$$(3.2) \quad C_s \cong \left(L_p(\mathbf{S}^1)^{r-1} \times L_p \mathbf{S}^{2l-1} \right) / \Gamma,$$

where $L_p \mathbf{S}^{2l-1}$ is simply $L_p \mathrm{SU}(2)$ for $l = 2$, and the Sullivan group given by

$$L_p \mathbf{S}^{2l-1} = \Omega L_p \left(L_p (BS^1)_{h\mathbf{Z}/l} \right)$$

for p odd, and Γ is a finite central subgroup.

Proof of Thm. 1.1. By Lemma 3.1, showing equivariant null-homotopy is equivalent to showing that the map

$$h(G/T): BG^{\mathfrak{g}} \xrightarrow{\tau} BT^t \xrightarrow{\mathrm{proj}} \mathbf{S}^r$$

is null. Note that for any given subgroup $H < G$ of maximal rank, there is a factorization of τ through $BH^{\mathfrak{h}}$. In particular, we may assume that G is connected. By the dimension hypothesis of the theorem, $W(G)$ is nontrivial. If $H = C_s$ is the subgroup associated to a primitive reflection $s \in W(G)$ of minimal order $l > 1$, then the map $h(C_s/T)$ is the $(r-1)$ -fold suspension of $h(L_p \mathbf{S}^{2l-1}/S)$ by (3.2). Therefore, it is enough to prove the theorem for those p -compact groups C_s .

We distinguish two cases.

First suppose that $l = 2$. By the classification of complex reflection groups [ST54], and with the terminology of that paper, this is always the case except when W is a product of any number of groups from the list

$$\{G_4, G_5, G_{16}, G_{18}, G_{20}, G_{25}, G_{32}\}.$$

This comment is only meant to intimidate the reader and is insubstantial for what follows.

In this case, the map $h(L_p \mathrm{SU}(2)/S)$ is null by Pittie [Pit75, PS75] since the spaces involved are Lie groups, thus $h(G/T) \simeq *$.

Now suppose that $l > 2$. This forces $p > 2$ as well, and since $\langle s \rangle$ acts faithfully on some line in $H^1(T; \mathbf{Z}_p)$ while fixing the complementary hyperplane, we must have that it acts by an l th root of unity, and thus $l \mid p-1$. The proof is finished if we can show that

$$h(L_p \mathbf{S}^{2l-1}/S) = 0$$

where S is the 1-dimensional maximal torus in the Sullivan group $G' := L_p \mathbf{S}^{2l-1}$. To see this, note that the inclusion $S \rightarrow G'$ factors through the maximal torus normalizer $N_{G'}(S) \cong S \times \mathbf{Z}/l$, and thus

$$h(G'/S): BG'^{\mathfrak{g}'} \xrightarrow{\tau_1} BN^n \xrightarrow{\tau_2} \Sigma \mathbf{S}^0[BS] \rightarrow \mathbf{S}^1.$$

If $P \simeq \bigvee_{i=0}^{p-2} e_i P$ is any stable splitting of the p -completed complex projective space $P = \mathbf{S}^0[BS]$ induced by idempotents e_i as in the previous section, then the rightmost projection map clearly factors through $e_0 P$, which is the part containing the bottom cell. Since $p > 2$, Corollary 2.6 shows that there is an idempotent $f \in [P, P]$ such that $\tau \simeq f \circ \tau$ and $\tau \circ e_0 = e_0 \circ \tau = 0$, proving the theorem. \square

4. THE ADJOINT REPRESENTATION

Let G be a d -dimensional connected p -compact group with maximal torus T of rank r . Choose a set $\{s_1, \dots, s_{r'}\}$ of generating reflections of $W = W(G)$ with r' minimal. The classification of pseudo-reflection groups [ST54, CE74] implies that for G semisimple, most of the time $r = r'$, but there are cases where $r' = r + 1$.

Example 4.1 (The group no. 7). Let $p \equiv 1 \pmod{12}$. Let G_7 be the finite group generated by the reflection s of order 2 and the two reflections t, u of order 3, where $s, t, u \in GL_2(\mathbf{Z}_p)$ are given by

$$s = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad t = \frac{1}{\sqrt{2}} \begin{pmatrix} -\zeta & \zeta^7 \\ -\zeta & -\zeta^7 \end{pmatrix}, \quad u = \frac{1}{\sqrt{2}} \begin{pmatrix} -\zeta^7 & -\zeta^7 \\ \zeta & -\zeta \end{pmatrix}.$$

Here ζ is a 24th primitive root of unity. Note that although possibly $\zeta \notin \mathbf{Z}_p$, $\frac{1}{\sqrt{2}}\zeta \in \mathbf{Z}_p$. In Shephard and Todd's classification, this is the restriction to \mathbf{Z}_p of the complex pseudo-reflection group no. 7. They show that even over \mathbf{C} , G_7 cannot be generated by two reflections. The associated p -compact group is given by

$$\Omega L_p((BT^2)_{hG_7}).$$

If G is not semisimple (i. e. it contains a nontrivial normal torus subgroup), then r' may be smaller than r . Set $\kappa = r + 1 - r' \geq 0$.

Let $\mathcal{I}_{r'}$ be the set of proper subsets of $\{1, \dots, r'\}$, and for $I \subseteq \{1, \dots, r'\}$, let T_I be the fixed point subtorus $T^{\langle s_i | i \in I \rangle}$ and $C_I = C_G(T_I)$ be the centralizer in G , which is connected by [DW95, Lemma 7.8].

Definition. Let G be a connected p -compact group. Define the *adjoint space* A_G by the homotopy colimit

$$A_G = \Sigma^\kappa \operatorname{hocolim}_{I \in \mathcal{I}_{r'}} G/C_I$$

with the induced left G -action, and the trivial G -action on the suspension coordinates.

Theorem 1.4 shows that if G is a the p -completion of a connected, semisimple Lie group (in this case $r = r'$ and $\kappa = 1$), then A_G is a d -dimensional sphere G -equivariantly homotopy equivalent to $\mathfrak{g} \cup \{\infty\}$. This holds more generally: if G is a connected, compact Lie group with maximal normal torus T^k then

$$A_G \cong \Sigma^k A_{G/T^k} = (\mathfrak{t} \cup \{\infty\}) \wedge (\mathfrak{g}/\mathfrak{t} \cup \{\infty\}) = \mathfrak{g} \cup \{\infty\}.$$

Lemma 4.2. Let \mathcal{I}_r be the poset category of proper subsets of $\{1, \dots, r\}$ and

$$F, G: \mathcal{I}_k \rightarrow \{\text{finite CW-complexes or finite CW-spectra}\}.$$

be two functors. Then

- (1) If F has the property that $\dim F(\emptyset) > \dim F(I)$ for every $I \neq \emptyset$, then

$$\dim \operatorname{hocolim} F = \dim F(\emptyset) + k - 1.$$

- (2) If $f: F \rightarrow G$ is a natural transformation of two such functors such that

$$f_*(\emptyset): H_{\dim F(\emptyset)}(F(\emptyset)) \xrightarrow{\cong} H_{\dim G(\emptyset)}(G(\emptyset)),$$

then f induces an isomorphism

$$\operatorname{hocolim} f_*: H_{\dim \operatorname{hocolim} F}(\operatorname{hocolim} F) \rightarrow H_{\dim \operatorname{hocolim} G}(\operatorname{hocolim} G).$$

- (3) Let $F: \mathcal{I}_r \rightarrow \mathbf{Top}$ be the functor given by $F(\emptyset) = \mathbf{S}^n$, $F(I) = *$ for $I \neq \emptyset$. Then $\operatorname{hocolim}_{\mathcal{I}_r} F \simeq \mathbf{S}^{n+r-1}$.

Proof. The first two assertions follow from the Mayer-Vietoris spectral sequence [BK72, Chapter XII.5],

$$E_{p,q}^1 = \bigoplus_{I \in \mathcal{I}_k, |I|=k-1-p} H_q(F(I)) \implies H_{p+q}(\text{hocolim } F),$$

along with the observation that under the dimension assumptions of (1), $E_{p,q}^1 = 0$ for $q \geq \dim F(\emptyset)$ except for $E_{k-1, \dim F(\emptyset)}^1 = H_{\dim F(\emptyset)}(F(\emptyset))$. In particular, this group cannot support a nonzero differential and thus

$$H_i(F(\emptyset)) \cong H_{i+k-1}(\text{hocolim } F) \quad \text{for } i \geq \dim F(\emptyset).$$

The third one is an immediate consequence of the Mayer-Vietoris spectral sequence. \square

Corollary 4.3. *For any connected p -compact group G , A_G is a d -dimensional G -space.*

Proof. This follows from Lemma 4.2. Indeed, since any C_I ($I \neq \emptyset$) is connected and has the nontrivial Weyl group W_I , its dimension is greater than $\dim T$. So the condition

$$\dim F(\emptyset) = \dim G/T > \dim F(I)$$

is satisfied, and

$$\dim \text{hocolim } F = d - r + r' - 1 = d - \kappa.$$

\square

As mentioned at the end of the introduction, for p -compact groups G , A_G is not usually a sphere, as the following example illustrates.

Example 4.4. Let $p \geq 5$ be a prime, and let $G = \mathbf{S}^{2p-3}$ be the Sullivan sphere, whose group structure is given by $BG = L_p(BS_{hC_{p-1}})$, where $C_{p-1} \subseteq \mathbf{Z}_p^\times$ acts on $BS = K(\mathbf{Z}_p, 2)$ by multiplication on \mathbf{Z}_p . Clearly, G has rank 1, and \mathcal{I}_1 consists only of a point, thus $A_G = \Sigma G/T \simeq L_p \Sigma CP^{p-2}$. Since $p \geq 5$, this is not a sphere.

For the proof of Theorem 1.5 we need a preparatory result.

Proposition 4.5. *Let P be a p -compact subgroup of maximal rank in a p -compact group G . Denote by T a maximal torus of P (and thus also of G). Then the following composition is G -equivariantly null-homotopic:*

$$f_{G,P}: S_G \wedge DS_T \rightarrow \mathbf{S}^0[G/T] \rightarrow \mathbf{S}^0[G/P].$$

The second map is the canonical projection, whereas the first map is given by using the duality isomorphism

$$\begin{aligned} S_G \wedge DS_T &\rightarrow S_G \wedge D(\mathbf{S}^0[G/T]) \wedge DS_T \xrightarrow{\sim} G_+ \wedge_T S_T \wedge DS_T \\ &\simeq \mathbf{S}^0[G/T] \wedge S_T \wedge DS_T \xrightarrow{\text{id} \wedge \text{ev}} \mathbf{S}^0[G/T]. \end{aligned}$$

Proof. In [Bau04, Cor. 24] it was shown that the relative duality isomorphism from Lemma 1.2 is natural in the sense that the following diagram commutes:

$$\begin{array}{ccc} (\mathbf{S}^0[G])^{hP^{\text{op}}} & \xrightarrow{\sim} & G_+ \wedge_P S_P \xrightarrow{\sim} D(\mathbf{S}^0[G/P]) \wedge S_G \\ \downarrow \text{res} & & \downarrow D(\text{proj}) \wedge \text{id} \\ (\mathbf{S}^0[G])^{hT^{\text{op}}} & \xrightarrow{\sim} & G_+ \wedge_T S_T \xrightarrow{\sim} D(\mathbf{S}^0[G/T]) \wedge S_G \end{array}$$

Taking duals and smashing with DS_G , we find that the map of the proposition is the left hand column in the diagram

$$\begin{array}{ccc}
S_G \wedge D(G_+ \wedge_P S_P) & \xrightarrow{\sim} & \mathbf{S}^0[G/P] \\
\uparrow & & \uparrow \\
S_G \wedge D(G_+ \wedge_T S_T) & \xrightarrow{\sim} & \mathbf{S}^0[G/T] \\
\uparrow \sim & & \\
S_G \wedge D(\mathbf{S}^0[G/T]) \wedge DS_T & & \\
\uparrow & & \\
S_G \wedge DS_T & &
\end{array}$$

Thus we need to show that the composition

$$G_+ \wedge_P S_P \rightarrow G_+ \wedge_T S_T \simeq \mathbf{S}^0[G/T] \wedge S_T \rightarrow S_T$$

is G -equivariantly trivial, or equivalently, that

$$S_P \rightarrow P_+ \wedge_T S_T \rightarrow S_T$$

is P -equivariantly trivial. But this map is exactly the homotopy class represented by $[P/T]$, thus the assertion follows from Theorem 1.1. \square

Proof of Thm. 1.5. Let G be a connected p -compact group whose Weyl group is generated by a minimal set of r' reflections. Let $F, A: \mathcal{I}_{r'} \rightarrow \text{Top}$ be the functors given by $F(\emptyset) = S_G \wedge DS_T$, $F(I) = *$ for $I \neq \emptyset$, and $A(I) = G/C_I$. Note that, since G is connected, $C_G(T) = T$ [DW94, Proposition 9.1] and $A(\emptyset) = G/T$. There is a map $\Phi: F \rightarrow A$ of $\mathcal{I}_{r'}$ -diagrams in the homotopy category of G -spectra which is fully described by defining

$$\Phi(\emptyset) = f_{G,T}: F(\emptyset) = S_G \wedge DS_T \rightarrow \mathbf{S}^0[G/T]$$

as the map given in Prop. 4.5. The strategy of the proof is to obtain a functor $F: \mathcal{I}_{r'} \rightarrow \text{Top}$ such that $F(\emptyset) = S_G \wedge DS_T$, $F(I) \simeq *$ for $I \neq \emptyset$, and a map of $\mathcal{I}_{r'}$ -diagrams $\Phi: F \rightarrow A$ in the category of G -spectra such that $\Phi(\emptyset) = f_{G,T}$. From this we get a G -equivariant map

$$S_G \simeq \mathbf{S}^k \wedge \Sigma^{r'-1} S_G \wedge DS_T \simeq \mathbf{S}^k \wedge \text{hocolim}_{\mathcal{I}_{r'}} F \rightarrow \Sigma^k \text{hocolim}_{\mathcal{I}_{r'}} \mathbf{S}^0[G/C_I] \simeq \mathbf{S}^0[A_G],$$

which will give us the splitting.

We will proceed by induction on the number of generating reflections r' . If $r' = 1$ then A_G is $\mathbf{S}^k \wedge G/T$ and $\Phi(\emptyset) = \mathbf{S}^k \wedge f_{G,T}$. We can construct the functor F and the natural transformation Φ step by step. Fix a subset I of cardinality k , and assume that F and Φ have been defined for all vertices in the diagram corresponding to I' with $|I'| < k$.

Let $\mathcal{P}(I)$ be the poset category of all proper subsets of I . Since F and Φ are defined over $\mathcal{P}(I)$ by induction hypothesis, we can consider $\text{hocolim}_{\mathcal{P}(I)} F \simeq \Sigma^{k-1} S_G \wedge DS_T \rightarrow \mathbf{S}^0[G/C_I]$. It is enough to show that this map is G -equivariantly nullhomotopic. Then, we can fix a null-homotopy and extend the map to the cone

of $\text{hocolim}_{\mathcal{P}(I)} F$. Finally, we define $F(I) = C(\text{hocolim}_{\mathcal{P}(I)} F)$ and $\Phi(I)$ is the corresponding extension.

Note that $S_G \wedge DS_T \rightarrow \mathbf{S}^0[G/T]$ factors through $G_+ \wedge_{C_I} S_{C_I} \wedge DS_T$. By induction, we know there is a map

$$\Sigma^{k-1} S_{C_I} \wedge DS_T \rightarrow \mathbf{S}^0[\text{hocolim}_{J \in \mathcal{I}_k} C_I/C_J],$$

which splits the top cell. We get a factorization

$$(4.6) \quad \Sigma^{k-1} S_G \wedge DS_T \rightarrow \Sigma^{k-1} G_+ \wedge_{C_I} S_{C_I} \wedge DS_T \\ \rightarrow G_+ \wedge_{C_I} \mathbf{S}^0[\text{hocolim}_{J \in \mathcal{I}_k} C_I/C_J] \rightarrow G_+ \wedge_{C_I} \mathbf{S}^0.$$

It thus suffices to show that in the \mathcal{I}_k -diagram

$$\begin{array}{ccc} S_{C_I} \wedge DS_T \equiv \equiv \equiv \{*\}_{J \in \mathcal{I}_k - \{\emptyset\}} & \xrightarrow{\text{hocolim}} & \Sigma^{k-1} S_{C_I} \wedge DS_T \\ \downarrow & & \downarrow \\ \mathbf{S}^0[C_I/T] \equiv \equiv \equiv \{\mathbf{S}^0[C_I/C_J]\}_{J \in \mathcal{I}_k - \{\emptyset\}} & \xrightarrow{\text{hocolim}} & \mathbf{S}^0[\text{hocolim}_{J \in \mathcal{I}_k} C_I/C_J] \\ \downarrow & & \downarrow \\ \mathbf{S}^0 \equiv \equiv \equiv \{\mathbf{S}^0\}_{I \in \mathcal{I}_k - \{\emptyset\}} & \xrightarrow{\text{hocolim}} & \mathbf{S}^0 \end{array}$$

the right hand side composition $\Sigma^{k-1} S_{C_I} \wedge DS_T \rightarrow \mathbf{S}^0$ is C_I -equivariantly null-homotopic. In the latter diagram, it makes no difference whether the centralizers are taken in C_I or in G . But by Theorem 1.1, the left hand column is already null-homotopic, thus, as a colimit of null-homotopic maps over a contractible diagram, so is the right hand column. \square

Conclusion and questions. In this paper, we have compared two imperfect notions of adjoint representations of a p -compact group G . One (S_G) is a sphere, but has a G -action only stably; the other (A_G) is an unstable G -space, but fails to be a sphere. The question remains whether there is an unstable G -sphere whose suspension spectrum is S_G . It might even be true that A_G splits off its top cell after only one suspension, yielding a solution to this problem in the cases where the Weyl group of the rank- r group G is generated by r reflections.

There are also a number of interesting open questions about the flag variety G/T of a p -compact groups:

- By the classification of p -compact groups, $H^*(G/T; \mathbf{Z}_p)$ is torsion free and generated in degree 2. Can this be seen directly?
- Is there a manifold M such that $L_p M \simeq G/T$, analogous to smoothings of G [BKNP04, BP06]? Is it a boundary of a manifold?
- If such a manifold M exists, can it be given a complex structure?

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