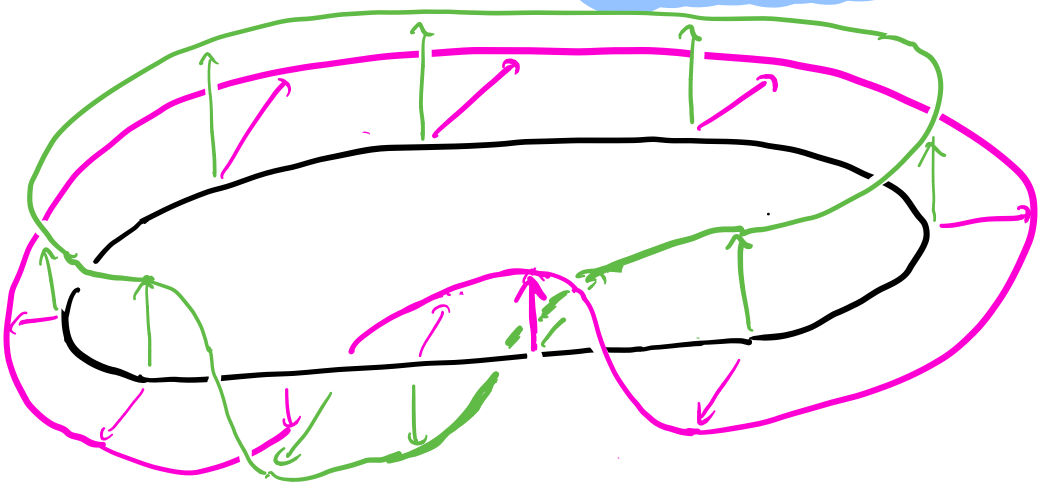
Floer theory and Hamiltonian dynamics on non-compact hypersurfaces

Homotopy classes of proper maps.

J.w. Alberto Abbondandolo



	compact mflds	non-compact manifolds.
finite dim	Part I: classical Pondryagin-Thom construction	Part II: stable Pondryagin-Thom construction for proper maps
infinite dim		Part III: Pondryagin-Thom for proper Fredholm maps

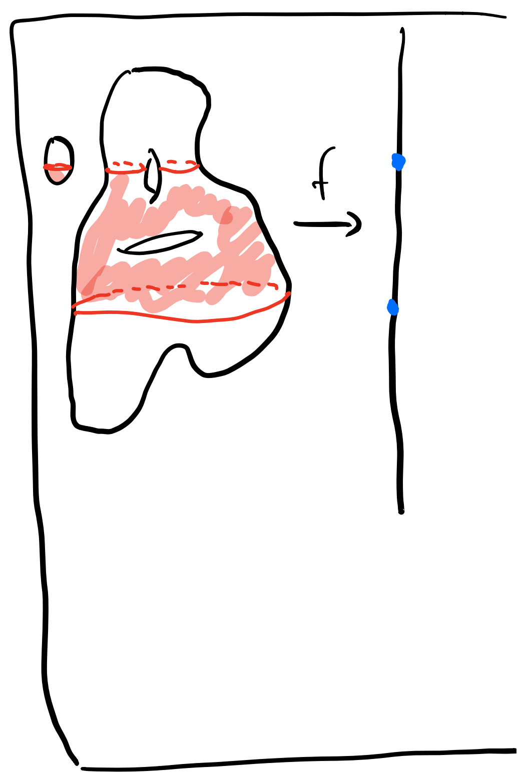
Part IV: Explicit computations

Part I : The classical PT-construction

Let $f : M^m \longrightarrow N^n$ map between closed manifolds. (f smooth N connected)

Def: $k = m - n$ is the Fredholm index

Building a homotopy invariant:

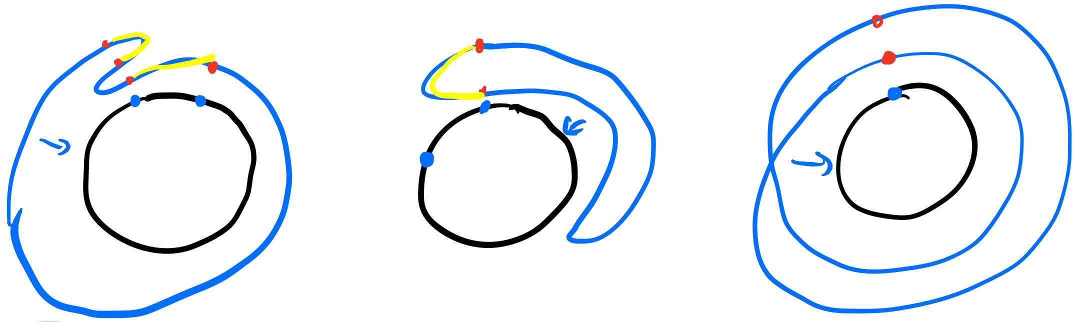


* preimage $f^{-1}(y)$ of reg value y is k-dim submfd of M
* cobordism class is independent of reg value y
* a homotopy invariant.

Invariant for homotopy classes

Homotopy classes $M \to N$ $\xrightarrow[\text{of regular value}]{\text{Take preimage}}$ submflds in M up to cobordism in $M \times [0,1]$

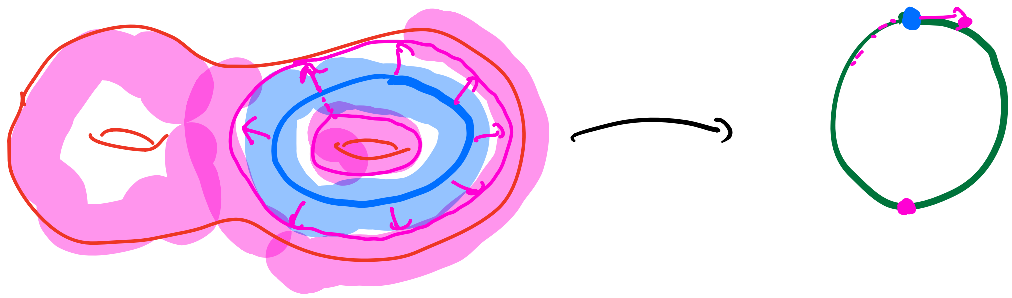
Example: mod-2 degree of maps $f: S^1 \to S^1$



Questions:

* Does every submanifold occur as the regular value of a map?
* Can we upgrade the invariant to a full invariant?

Suppose $N = S^n$



Def (Framing): A framed submfd is a submfd + choice of trivialization of normal bundle.

Thm (Pontryagin) Framed cobordism is a full invariant

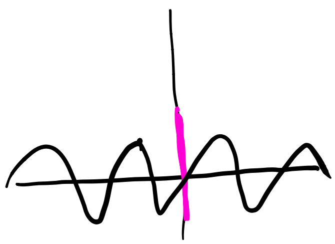
$$[M^m, S^n] \cong \Omega^{fr}_{m-n}(M).$$

Rem: If M connected and $m=n$ $\Rightarrow$

$\Omega^{fr}_0(M) \cong \mathbb{Z}$ ($\mathbb{Z}$ valued degree)

Part II: Mappings between non-compact mfds

Def: A map is proper if preimages of compact sets are compact.



not a proper map

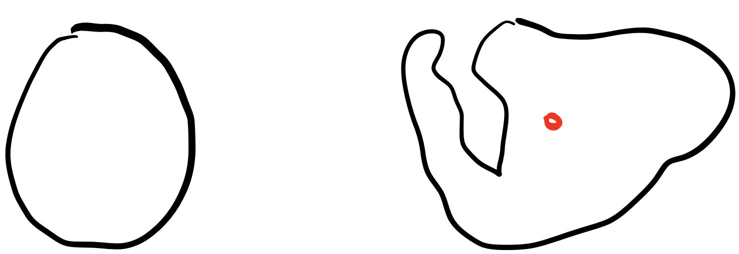
proper maps

Slogan: "A proper map sends ∞ to ∞".

Question: What is $[M, \mathbb{R}^n]_{prop}$?
(M open mfd)

Prop (R.) Let M be v-bundle over compact Hausdorff space X. Then $[M, \mathbb{R}^n]_{prop} \cong [S(M), S^{n-1}]$

Pf:



Warning (suppose X manifold)

- There exist framed submfds that are not preimages of regular values of proper maps
- There exist maps that are not proper homotopic, but whose Pontryagin mfds are framed cobordant.

However

- Stably this should not happen. (Freudenthal)

Question: Does there exist a Stable P-T construction for open manifolds?

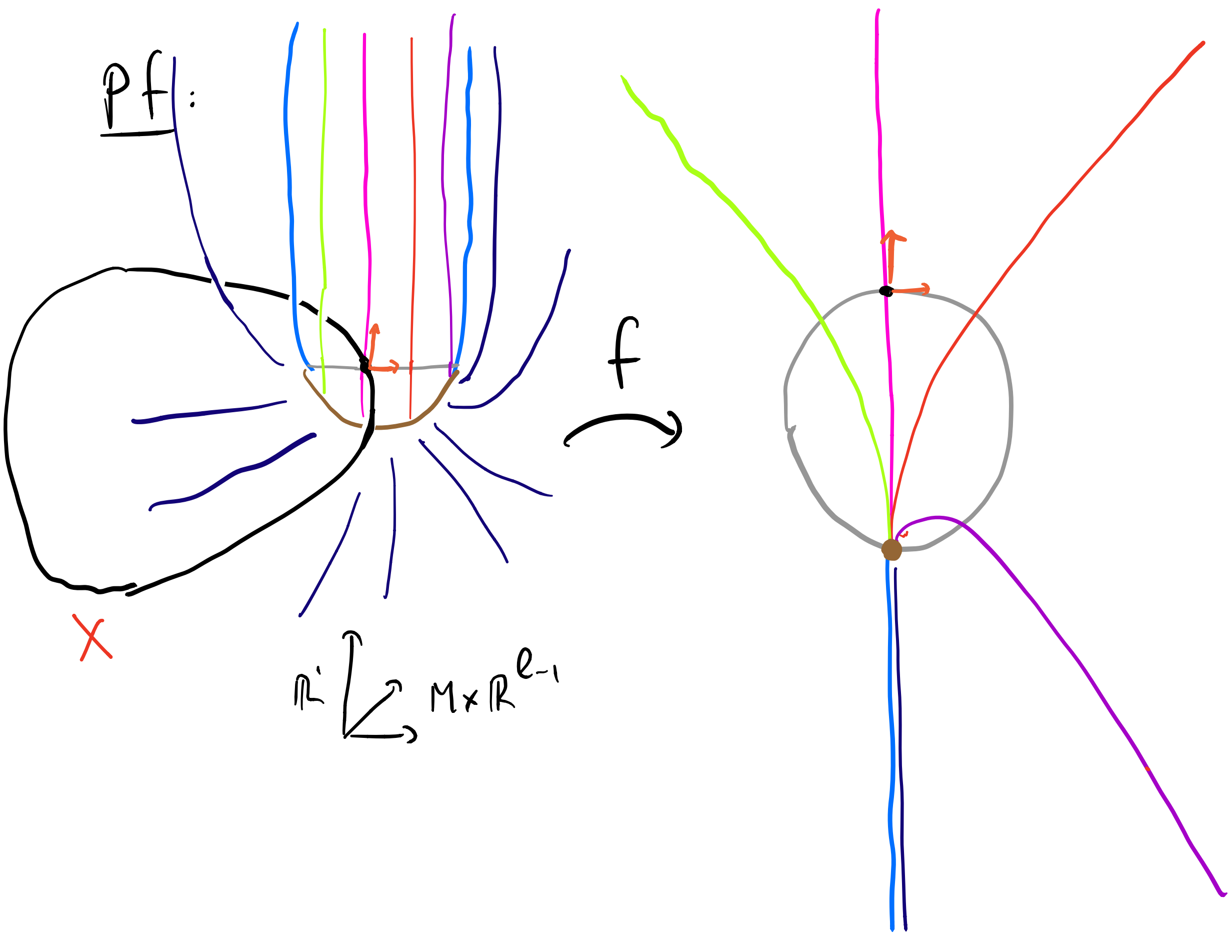
Thm ('20 Csépai): Let M be open mfd.

Then there exists an ℓ s.t.

$$[M^m \times \mathbb{R}^\ell, \mathbb{R}^{n+\ell}]_{prop} \cong \Omega^{fr}_{m-n}(M \times \mathbb{R}^\ell)$$

(And $\Omega^{fr}_{m-n}(M \times \mathbb{R}^\ell) \cong \Omega^{fr}_{m-n}(M \times \mathbb{R}^\ell \times \mathbb{R}^p)$ $\forall\, p \geq 0$)

Pf:



Part III: Mappings between ∞-dim mfds.

Problems

* Infinite dimensional mfds are non-compact.
  - proper maps work
* Sard's theorem does not hold for all smooth mappings.
  - Smale showed that Sard holds for Fredholm mappings
* Many different ∞-dim spaces
  - Work with Hilbert mfds.
    real separable

  > Examples of Hilbert manifold. X fin dim manifold
  > $M = H^1(S^1, X)$ loop space (and variants),
  > or $X \times \mathbb{H}$

* Many topological constraints dissappear in ∞-dim
  - $GL(\mathbb{H})$ is trivial (Kuiper's theorem).
  - Hilbert bundles are trivial
  - I'll get back to this.

Goal: Classify the set $\mathcal{F}_k^{prop}[M,\mathbb{H}]$ of proper homotopy classes of proper Fredholm maps

$$f: M \longrightarrow \mathbb{H}.$$

of index k.

Rem/Def
- M Hilbert mfd modeled on ∞-dim separable Hilbert space $\mathbb{H}$
- Fredholm ($k = \dim \ker df - \dim \operatorname{coker} df(x)$) makes dif top available.
- $\mathbb{H} \cong S^{\mathbb{H}} = \{x \in \mathbb{H} \mid \|x\| = 1\}$
- applications to non-linear elliptic PDE's and Floer theory.

Previous work: Cacciopoli, Smale, Schwarz

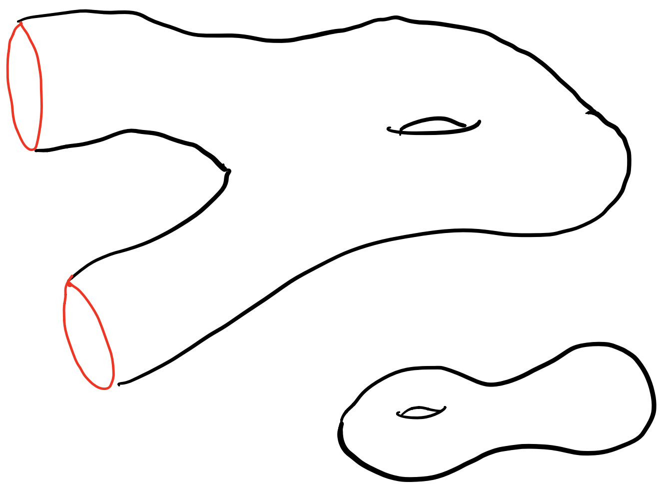
Elworthy-Tromba, Gęba, Berger,

Bauer-Furuta, ...

Thm: (Smale) Invariant

$$\mathcal{F}_n^{prop}[M,N] \longrightarrow \mathcal{N}_n(M)$$

Thm: (Abbondandolo-R.) If $M=N=H$, $n\neq 0$ image is zero.



Question: Can we introduce framings to obtain a full invariant?

$\boxed{GL(\mathbb{H}) \text{ contractible} \Rightarrow \text{Hilbert bundles are trivial}}$

* Tangent bundle of Hilb. mfd is trivial

$$TM \cong M \times \mathbb{H}$$

* Differential of Fredholm map $f: M \to N$

$$df: M \longrightarrow \Phi_k(\mathbb{H})$$

(linear Fredholm operators)

Question: When are Fredholms $f, g: M \to N$ homotopic?

Obvious conditions:

* $f, g: M \to N$ homotopic

* $df, dg: M \to \Phi_k(\mathbb{H})$ homotopic.

Thm (Elworthy-Tromba / Abbondandolo-R.)

$$\mathcal{F}_k[M, N] \cong [M, N] \times [M, \Phi_k(\mathbb{H})]$$

Framings in finite dimensions

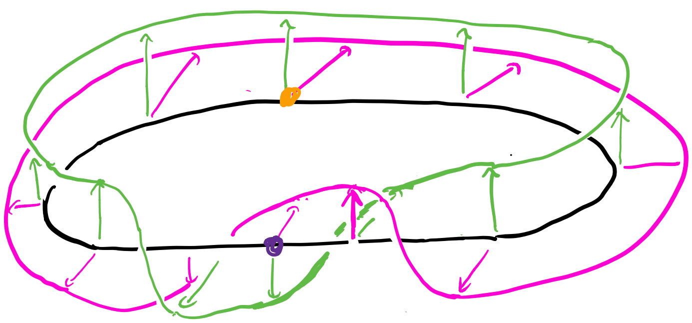
Classically: Framing of $X \subset M$ is section

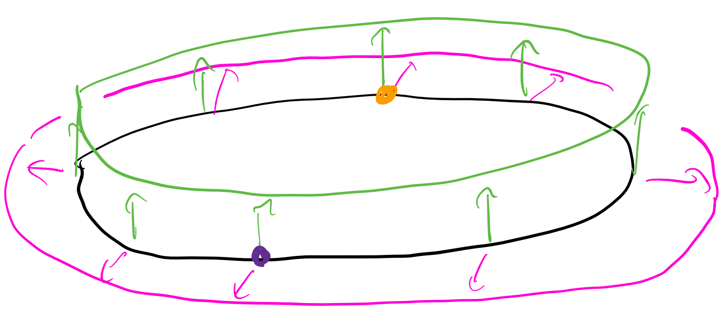
$$\tau : X \to \mathrm{Iso}(NX, \mathbb{R}^n) \quad (\text{or } \mathrm{Iso}(\mathbb{R}^n, NX))$$

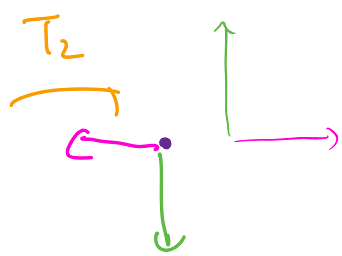
given τ_1, τ_2 framings, form

$$\sigma : X \to GL(\mathbb{R}^n) \text{ by}$$

$$\sigma(x) = \tau_2(x)\tau_1^{-1}(x).$$







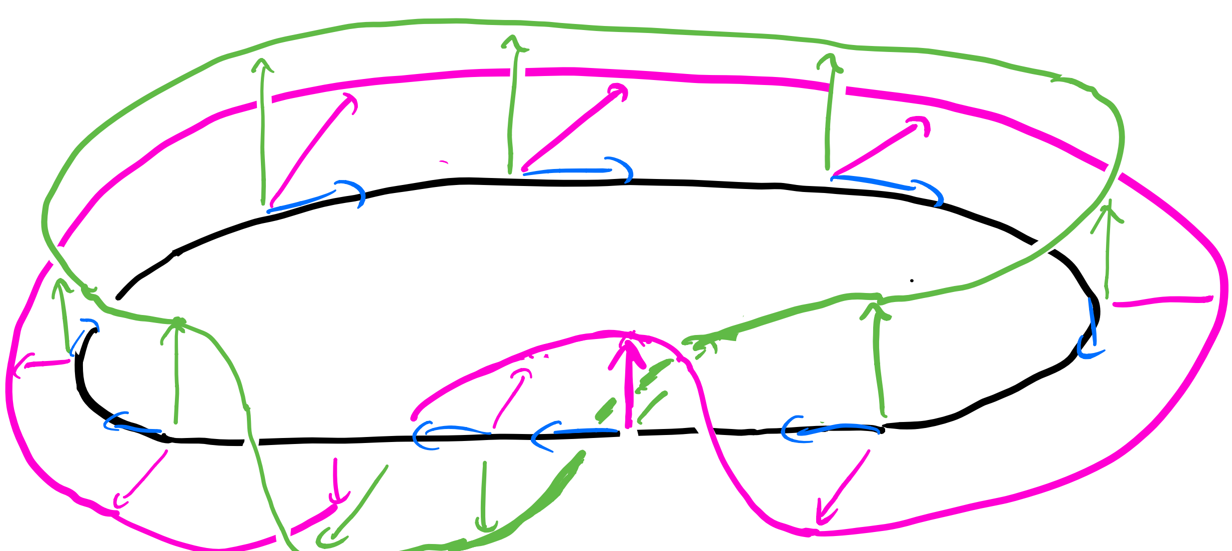
The obvious extension to ∞ dim gives no new info as $GL(H)$ is contractible.

$*$ in fin dim T extends to map

$$\tilde{T} : M \longrightarrow Hom(TM, \mathbb{R}^n)$$

where for $x \in X$ $\ker \tilde{T}(x) = T_x X$, or

$$0 \to T_x X \to T_x M \xrightarrow{\tilde{T}(x)} \mathbb{R}^n \to 0, \quad \forall x \in X.$$



Framings in ∞-dim.

• **Def** (Abbondandolo - R.).

let M be a Hilbert mfd, X^k submanifold

a framing of X is a map

$$A : M \longrightarrow \Phi_k(\mathbb{H})$$

This space has interesting topology

s.t.

$$0 \to T_x X \longrightarrow \mathbb{H} \xrightarrow{A(x)} \mathbb{H} \longrightarrow 0. \quad \forall x \in X$$

Thm (Abbondandolo - R.)

$$\mathcal{F}_k^{prop}[M, \mathbb{H}] \cong \Omega_k^{fr}(M)$$

Part IV: Explicit computations

Thm: $k<0$

$$\mathcal{F}_n^{prop}[M,\mathbb{H}] \simeq \Omega_n^{fr}(M) \cong [M, \Phi_0(\mathbb{H})] \cong \widetilde{K}(M)$$

M fin dim homotopy type.

Def: $f: M \to \mathbb{H}$ is

orientable if $(\pi_1)_* df : \pi_1(M) \to \pi_1(\Phi_0(\mathbb{H})) \overset{\shortparallel}{} \mathbb{Z}_2$

is trivial.

spin M simply connected and

$(\pi_2)_* df : \pi_2(M) \to \pi_2(\Phi_1(\mathbb{H})) \overset{\shortparallel}{} \mathbb{Z}_2$

is trivial.

Note: $[M, \Phi_0(\mathbb{H})] = [M, \Phi_0(\mathbb{H})]_{or} \sqcup [M, \Phi_0]_{no}$

$[M, \Phi_1(\mathbb{H})] = [M, \Phi_1(\mathbb{H})]_{sp} \sqcup [M, \Phi_1]_{ns}$

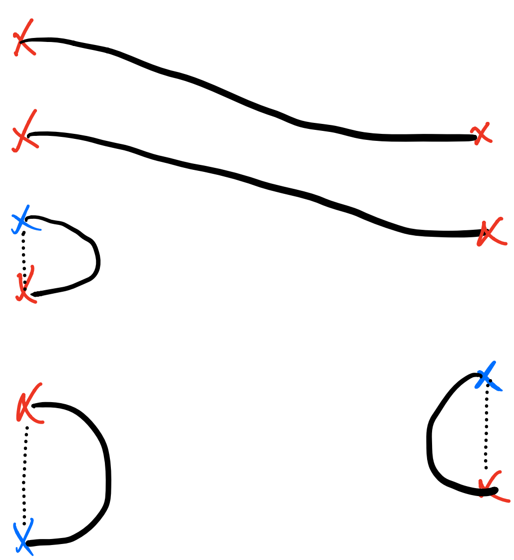
Thm (Abbondandolo - R.)

$$\mathcal{F}_0^{prop}[M, H] \cong \left([M, \Phi_0(H)]_{o2} \times \mathbb{N}_0\right) \sqcup \left([M, \Phi_0(H)]_{no} \times \mathbb{Z}_2\right)$$

if f orientable $\quad f \longmapsto ([df], |\deg|(f))$

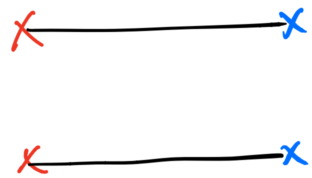
if f non-orientable $\quad f \longmapsto ([df], \deg_2(f) = \# f^{-1}(y) \bmod 2)$

(isomorphism given by absolute value of Fitzpatrick - Pejsachowicz - Rabier degree in orientable case / Caccioppoli - Smale degree in non-orientable case)



$[0,1] \times M.$

Remark:



Explicit generators

Let $M = \mathbb{H} \cong \mathbb{C} \times \mathbb{H}$.

$$f_n(z,x) = (z^n, x) \qquad n>0$$

$$f_0(a+bi, x) = (a^2+b^2 i, x)$$

Also define $f_{-n}(z,x) = (\bar{z}^n, x) \qquad n>0$

> In finite dim f_n & f_{-n} are not proper homotopic. In infinite dim f_n & f_{-n} are proper Fredholm homotopic.

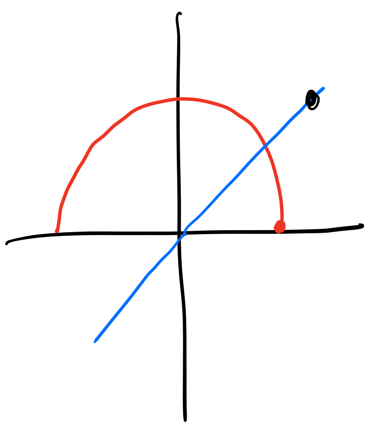
Example of non-orientable map

$$\begin{array}{c}\gamma \\ \downarrow \\ \mathbb{RP}^1\end{array} \quad \text{Tautological bundle.}$$

$$\cong S^1 \times \mathbb{H}$$

$$f: \gamma \times \mathbb{H} \longrightarrow \mathbb{R}^2 \times \mathbb{H}$$

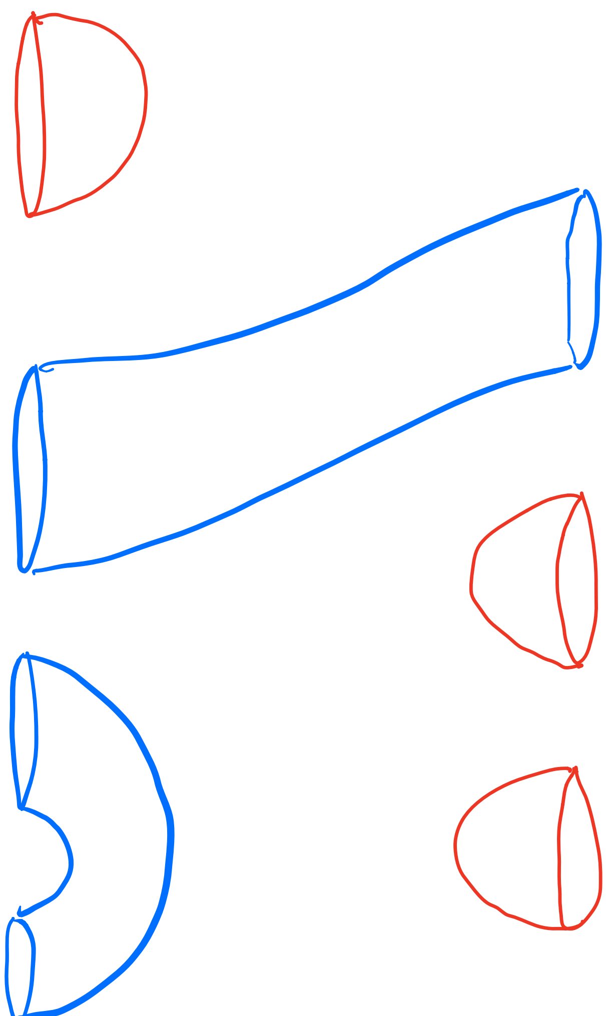
$$f((\ell, p), x) = (p, x)$$



Thm (Abbondandolo - R.) Suppose M simply-connected

$$\mathcal{F}_1^{prop}[M, \mathbb{H}] \cong \left([M, \tilde{\Phi}_1(\mathbb{H})]_{sp} \times \mathbb{Z}_2\right) \sqcup [M, \Phi]_{ns}$$

iso given by new numerical invariant. τ, which uses intersection theory in $\Phi_1(\mathbb{H})$



$[0,1] \times M$ (M spin)

Example $M = \mathbb{H}$ $[\mathbb{H}, \Phi_1(\mathbb{H})] = [\mathbb{H}, \Phi_1(\mathbb{H})]_{sp} = *$

$$\mathcal{F}_1^{prop}[\mathbb{H}, \mathbb{H}] \cong \mathbb{Z}_2$$

What are respresentatives?

$M = \mathbb{C}^2 \times \mathbb{H}$ $N = \mathbb{C} \times \mathbb{R} \times \mathbb{H}$

$$f(z_1, z_2, x) = (2 z_1 \overline{z_2}, |z_1|^2 - |z_2|^2, x)$$

$$g(z_1, z_2, x) = (z_1, |z_2|^2, x)$$

(Preimage of $(0, 1, 0)$ are both circles, $df, dg : M \to \Phi_1(\mathbb{H})$ homotopic, but not proper Fredholm homotopic)

Thank you for your Attention

Appendix: Extension of Fredholm maps

Thm M, N Hilb m/ds $\quad U \subset \overline{U} \subset V$

$f : V \longrightarrow N$ Fredholm index n

1) $\exists\ g : M \to N \quad g|_U = f.$

2) $\exists\ A : M \to \Phi_n(H) \quad A|_V = df.$

Then there exists $\bar{f} : M \to N$ s.t. $\bar{f}|_U = f$

$\bar{f} \sim g$ rel U

$d\bar{f} \sim A$ rel U.

Appendix: Extension of proper Fredholm maps

let $U \subset \bar{U} \subset V \subset \bar{V} \subset W \subset M$.

st $f: W \to \mathbb{H}$ Fredholm of index n.

1) $\exists A: M \to \Phi_n(\mathbb{H})$ $A|_W = df$

2) $f|\bar{V}$ is proper

3) $\exists z \in \mathbb{H} \setminus f(\partial U)$

Then there exist a proper Fredholm $\bar{f}: M \to \mathbb{H}$

s.t. $\bar{f}|_U = f|U$ and

1) $d\bar{f} \sim A$ rel U

2) $\exists Z \subset \mathbb{H}$ nbd of z s.t.

$\bar{f}^{-1}(Z) = f^{-1}(Z) \cap U$.

Key lemma.

M Hilbert mfd $f: M \to \mathbb{H}$

Fredholm $\quad U \subset \bar{U} \subset V \subset \bar{V} \subset M$

1) $f|_{\bar{V}}$ proper

2) $\exists \; z \in \mathbb{H} \setminus f(\partial U)$

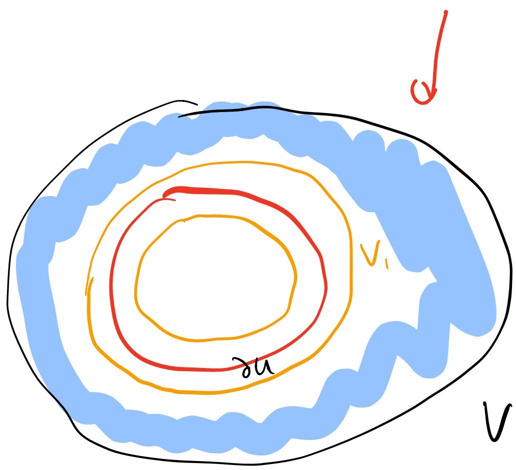
$\exists \; \bar{f}$ st $\bar{f}|_U \sim f|_U$

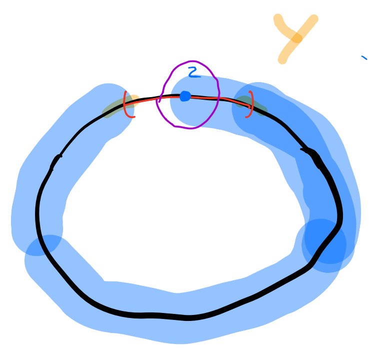
and $\exists \; Z \ni z$ st

$$\bar{f}^{-1}(z) = f^{-1}(z) \cap U$$

STEP 1 see $f: M \to S^{n+1}$

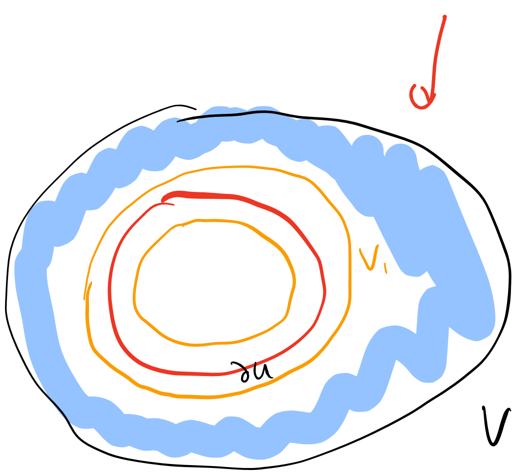
f not proper on blue.

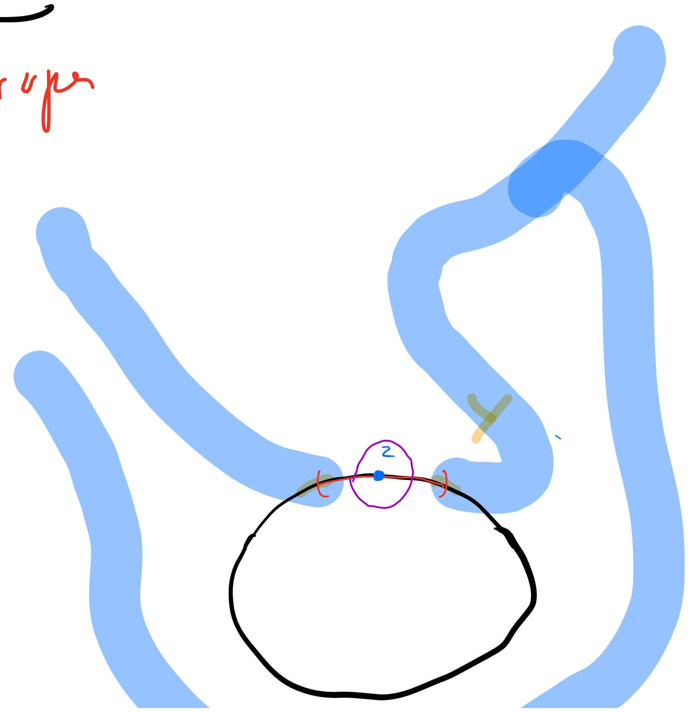




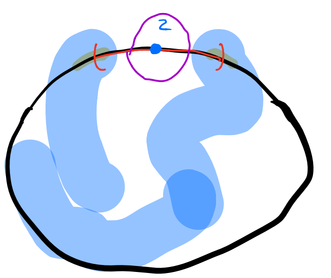
Step 1: make proper

f not proper

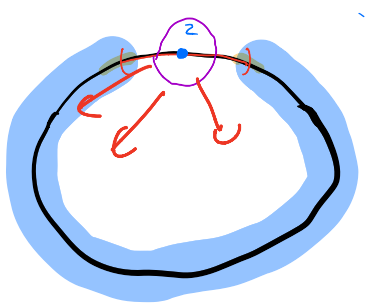




Step 2 use involution $\mathbb{H} \to \mathbb{H}$ swapping inside and outside.
(This does not exist in fin dim!)



Step 3 Project.



Step 4 see this as a map $\mathbb{M} \to \mathbb{H}$.