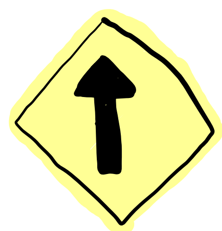
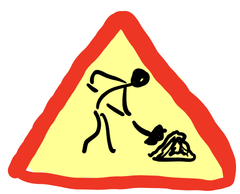
Differential topology in ∞ dimensions

j.w. Alberto Abbondandolo
Michael Jung
Lauran Toussaint.

I : How to do differential topology in ∞ dimensions.



II : ∞-dim homotopy



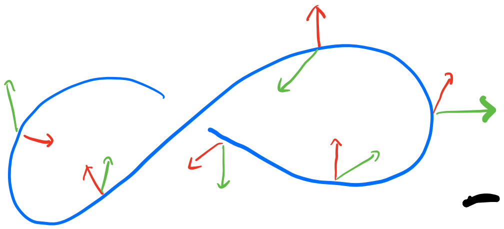
III : ∞-dim homology

Part I

How to do differential topology

in ∞- dimensions

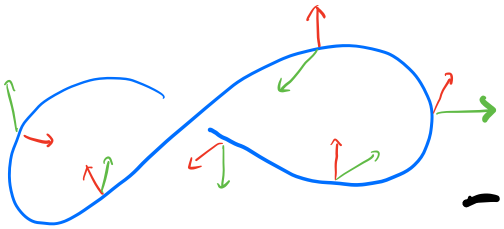
Differential topology in finite dim.



- $GL(\mathbb{R}^n)$ has complicated topology

(bundles are nontrivial)

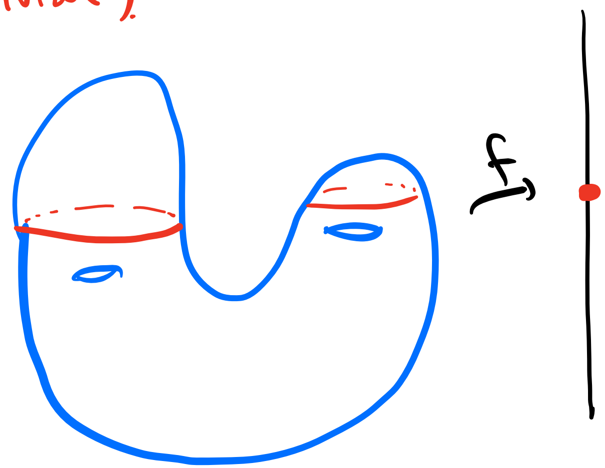
Differential topology in finite dim.



- $GL(\mathbb{R}^n)$ has complicated topology

(bundles are nontrivial)

- Typical values of mappings



$$f : M \to N$$

are regular

$\left(\begin{array}{c} f^{-1}(y) \text{ is a mfd for} \\ y \text{ regular} \end{array} \right)$

Differential topology in ∞ dim

Hilbert space $\mathbb{H} = \ell^2(\mathbb{N}) = \{ (x_1, x_2, \dots) \mid x_i \in \mathbb{R} \quad \sum_{i=1}^{n} x_i^2 < \infty \}$

Differential topology in ∞ dim

Hilbert space $\mathbb{H} = \ell^2(\mathbb{N}) = \{ (x_1, x_2, \dots) \mid x_i \in \mathbb{R} \quad \sum_{i=1}^{n} x_i^2 < \infty \}$

- $Gl(\mathbb{H})$ is contractible

(lack of rigidity)

Differential topology in ∞ dim

Hilbert space $\mathbb{H} = \ell^2(\mathbb{N}) = \{ (x_1, x_2, \dots) \mid x_i \in \mathbb{R} \quad \sum_{i=1}^{n} x_i^2 < \infty \}$

- $Gl(\mathbb{H})$ is contractible

  (lack of rigidity)

- Sard's theorem fails

  (lack of genericity)

Consequences

- $S^\infty = \{ x \in \mathbb{H} \mid \|x\| = 1 \}$ is contractible

Consequences

- $S^\infty = \{x \in \mathbb{H} \mid \|x\| = 1\}$ is contractible

- $B = B^\infty = \{x \in \mathbb{H} \mid \|x\| \leq 1\}$ retracts to S^∞

Consequences

- $S^\infty = \{x \in \mathbb{H} \mid \|x\| = 1\}$ is contractible

- $B = B^\infty = \{x \in \mathbb{H} \mid \|x\| \leq 1\}$ retracts to S^∞

- $S^\infty \cong \mathbb{H}$

Consequences

- $S^\infty = \{x \in \mathbb{H} \mid \|x\| = 1\}$ is condractible
- $B = B^\infty = \{x \in \mathbb{H} \mid \|x\| \leq 1\}$ retracts to S^∞
- $S^\infty \cong \mathbb{H}$
- Hilberd mfds are

homotopy equivalent iff diffeomorphic

$GL(\mathbb{H})$ is path connected

- fin-dim $\begin{pmatrix} -1 & & 0 \\ & 1 & \\ 0 & & \ddots & \\ & & & 1 \end{pmatrix}$ $\begin{pmatrix} 1 & & 0 \\ & 1 & \\ 0 & & \ddots & \\ & & & 1 \end{pmatrix}$ in different components of $GL(\mathbb{R}^n)$
  (det)

$GL(\mathbb{H})$ is path connected

- fin - dim $\begin{pmatrix} -1 & & 0 \\ & 1 & \\ 0 & & \ddots & \\ & & & 1 \end{pmatrix}$ $\begin{pmatrix} 1 & & 0 \\ & 1 & \\ 0 & & \ddots & \\ & & & 1 \end{pmatrix}$ in different components of $GL(\mathbb{R}^n)$
  (det)
- ∞ - dim $\begin{pmatrix} -1 & & 0 \\ & 1 & \\ 0 & & \ddots \end{pmatrix}$ $\begin{pmatrix} 1 & & 0 \\ & 1 & \\ 0 & & \ddots \end{pmatrix}$ in same component of $Gl(\mathbb{H})$

$GL(\mathbb{H})$ is path connected

- fin-dim $\begin{pmatrix} -1 & & 0 \\ & 1 & \\ 0 & & \ddots & \\ & & & 1 \end{pmatrix}$ $\begin{pmatrix} 1 & & 0 \\ & 1 & \\ 0 & & \ddots & \\ & & & 1 \end{pmatrix}$ in different components of $GL(\mathbb{R}^n)$
  (det)

- ∞ - dim $\begin{pmatrix} -1 & & 0 \\ & 1 & \\ 0 & & \ddots \end{pmatrix}$ $\begin{pmatrix} 1 & & 0 \\ & 1 & \\ 0 & & \ddots \end{pmatrix}$ in same component of $Gl(\mathbb{H})$

$$\begin{pmatrix} \cos \pi t & \sin \pi t \\ -\sin \pi t & \cos \pi t \end{pmatrix} \qquad \begin{array}{c} t=0 \\ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{array} \qquad \begin{array}{c} t=1 \\ \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \end{array}$$

GL(H) is path connected

- fin-dim $\begin{pmatrix} -1 & & 0 \\ & 1 & \\ 0 & & \ddots & \\ & & & 1 \end{pmatrix}$ $\begin{pmatrix} 1 & & 0 \\ & 1 & \\ 0 & & \ddots & \\ & & & 1 \end{pmatrix}$ in different components of $GL(\mathbb{R}^n)$
  (det)
- ∞-dim $\begin{pmatrix} -1 & & 0 \\ & 1 & \\ 0 & & \ddots \end{pmatrix}$ $\begin{pmatrix} 1 & & 0 \\ & 1 & \\ 0 & & \ddots \end{pmatrix}$ in same component of $Gl(\mathbb{H})$

$$\begin{pmatrix} \cos \pi t & \sin \pi t \\ -\sin \pi t & \cos \pi t \end{pmatrix} \qquad \begin{array}{c} t=0 \\ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{array} \qquad \begin{array}{c} t=1 \\ \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \end{array}$$

$$\begin{pmatrix} -1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & 1 & \\ & & & & & \ddots \end{pmatrix}$$

$GL(\mathbb{H})$ is path connected

- fin-dim $\begin{pmatrix} -1 & & 0 \\ & 1 & \\ 0 & & \ddots & \\ & & & 1 \end{pmatrix}$ $\begin{pmatrix} 1 & & 0 \\ & 1 & \\ 0 & & \ddots & \\ & & & 1 \end{pmatrix}$ in different components of $GL(\mathbb{R}^n)$ (det)
- ∞-dim $\begin{pmatrix} -1 & & 0 \\ & 1 & \\ 0 & & \ddots \end{pmatrix}$ $\begin{pmatrix} 1 & & 0 \\ & 1 & \\ 0 & & \ddots \end{pmatrix}$ in same component of $Gl(\mathbb{H})$

$$\begin{array}{ccc} & t=0 & t=1 \\ \begin{pmatrix} \cos\pi t & \sin\pi t \\ -\sin\pi t & \cos\pi t \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \end{array}$$

$$\begin{pmatrix} -1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & 1 & \\ & & & & & \ddots \end{pmatrix} \sim \begin{pmatrix} -1 & & & & \\ & \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} & & & \\ & & \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} & & \\ & & & & \ddots \end{pmatrix}$$

GL(ℍ) is path connected

- fin-dim $\begin{pmatrix} -1 & & 0 \\ & 1 & \\ 0 & & \ddots & \\ & & & 1 \end{pmatrix}$ $\begin{pmatrix} 1 & & 0 \\ & 1 & \\ 0 & & \ddots & \\ & & & 1 \end{pmatrix}$ in different components of $GL(\mathbb{R}^n)$ (det)
- ∞-dim $\begin{pmatrix} -1 & 0 \\ 0 & 1 & \\ & & \ddots \end{pmatrix}$ $\begin{pmatrix} 1 & 0 \\ 0 & 1 & \\ & & \ddots \end{pmatrix}$ in same component of $Gl(\mathbb{H})$

$$\begin{array}{ccc} & t=0 & t=1 \\ \begin{pmatrix} \cos \pi t & \sin \pi t \\ -\sin \pi t & \cos \pi t \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \end{array}$$

$$\begin{pmatrix} -1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \\ & & & & & \ddots \end{pmatrix} \sim \begin{pmatrix} -1 & & \\ & \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} & \\ & & \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} \\ & & & \ddots \end{pmatrix} \sim \begin{pmatrix} -1 & & \\ & \begin{pmatrix} -1 & \\ & -1 \end{pmatrix} & \\ & & \begin{pmatrix} -1 & \\ & -1 \end{pmatrix} \\ & & & \ddots \end{pmatrix}$$

$GL(\mathbb{H})$ is path connected

- fin-dim $\begin{pmatrix} -1 & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{pmatrix}$ $\begin{pmatrix} 1 & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{pmatrix}$ in different components of $GL(\mathbb{R}^n)$
  (det)
- ∞-dim $\begin{pmatrix} -1 & & 0 \\ & 1 & \\ 0 & & \ddots \end{pmatrix}$ $\begin{pmatrix} 1 & & 0 \\ & 1 & \\ 0 & & \ddots \end{pmatrix}$ in same component of $Gl(\mathbb{H})$

$$\begin{array}{ccc} & t=0 & t=1 \\ \begin{pmatrix} \cos \pi t & \sin \pi t \\ -\sin \pi t & \cos \pi t \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \end{array}$$

$$\begin{pmatrix} -1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & 1 & \\ & & & & & \ddots \end{pmatrix} \sim \begin{pmatrix} -1 & & & & \\ & \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} & & & \\ & & \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} & & \\ & & & \ddots \end{pmatrix} \sim \begin{pmatrix} -1 & & & \\ & \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} & & \\ & & \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} & \\ & & & \ddots \end{pmatrix} \sim \begin{pmatrix} \begin{pmatrix} -1 & \\ & -1 \end{pmatrix} & & & \\ & \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} & & \\ & & \begin{pmatrix} 1 & \\ & \ddots \end{pmatrix} & \\ & & & \ddots \end{pmatrix}$$

$GL(\mathbb{H})$ is path connected

- fin-dim $\begin{pmatrix} -1 & & 0 \\ & 1 & \\ 0 & & \ddots & \\ & & & 1 \end{pmatrix}$ $\begin{pmatrix} 1 & & 0 \\ & 1 & \\ 0 & & \ddots & \\ & & & 1 \end{pmatrix}$ in different components of $GL(\mathbb{R}^n)$
  (det)

- ∞ - dim $\begin{pmatrix} -1 & & 0 \\ & 1 & \\ 0 & & \ddots \end{pmatrix}$ $\begin{pmatrix} 1 & & 0 \\ & 1 & \\ 0 & & \ddots \end{pmatrix}$ in same component of $Gl(\mathbb{H})$

$$\begin{pmatrix} \cos \pi t & \sin \pi t \\ -\sin \pi t & \cos \pi t \end{pmatrix} \qquad \begin{array}{c} t=0 \\ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{array} \qquad \begin{array}{c} t=1 \\ \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \end{array}$$

$$\begin{pmatrix} -1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \\ & & & & & \ddots \end{pmatrix} \sim \begin{pmatrix} -1 & & & \\ & \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} & & \\ & & \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} & \\ & & & \ddots \end{pmatrix} \sim \begin{pmatrix} -1 & & & \\ & \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} & & \\ & & \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} & \\ & & & \ddots \end{pmatrix} \sim \begin{pmatrix} \begin{pmatrix} -1 & \\ & -1 \end{pmatrix} & & & \\ & \begin{pmatrix} -1 & \\ & -1 \end{pmatrix} & & \\ & & \begin{pmatrix} -1 & \\ & 1 \end{pmatrix} & \\ & & & \ddots \end{pmatrix} \sim \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \\ & & & & & \ddots \end{pmatrix}$$

$GL(\mathbb{H})$ is path connected

- fin-dim $\begin{pmatrix} -1 & & 0 \\ & 1 & \\ 0 & & \ddots & \\ & & & 1 \end{pmatrix}$ $\begin{pmatrix} 1 & & 0 \\ & 1 & \\ 0 & & \ddots & \\ & & & 1 \end{pmatrix}$ in different components of $GL(\mathbb{R}^n)$ (det)
- ∞-dim $\begin{pmatrix} -1 & & 0 \\ & 1 & \\ 0 & & \ddots \end{pmatrix}$ $\begin{pmatrix} 1 & & 0 \\ & 1 & \\ 0 & & \ddots \end{pmatrix}$ in same component of $GL(\mathbb{H})$

$$\begin{pmatrix} \cos \pi t & \sin \pi t \\ -\sin \pi t & \cos \pi t \end{pmatrix} \qquad \begin{array}{c} t=0 \\ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{array} \qquad \begin{array}{c} t=1 \\ \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \end{array}$$

$$\begin{pmatrix} -1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & \ddots \end{pmatrix} \sim \begin{pmatrix} -1 & & \\ & \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} & \\ & & \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} \\ & & & \ddots \end{pmatrix} \sim \begin{pmatrix} -1 & & \\ & \begin{pmatrix} -1 & \\ & -1 \end{pmatrix} & \\ & & \begin{pmatrix} -1 & \\ & -1 \end{pmatrix} \\ & & & \ddots \end{pmatrix} \sim \begin{pmatrix} \begin{pmatrix} -1 & \\ & -1 \end{pmatrix} & & \\ & \begin{pmatrix} -1 & \\ & -1 \end{pmatrix} & \\ & & \begin{pmatrix} -1 & \\ & 1 \end{pmatrix} \\ & & & \ddots \end{pmatrix} \sim \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & \ddots \end{pmatrix}$$

$GL(\mathbb{H})$ is contractible.

Sard's theorem fails (Banach space example)

- $G : C^0([0,1]) \longrightarrow C^0([0,1])$

  $G(f) = f^3 \qquad dG(f)h = 3f^2 h$

Sard's theorem fails (Banach space example)

- $G : C^0([0,1]) \longrightarrow C^0([0,1])$

  $G(f) = f^3 \qquad dG(f)h = 3f^2h$

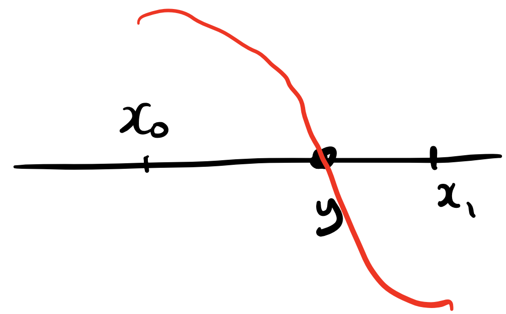
- f is critical point if it has a zero.

Sard's theorem fails (Banach space example)

- $G : C^0([0,1]) \longrightarrow C^0([0,1])$

  $G(f) = f^3 \qquad dG(f)h = 3f^2 h$

- f is critical point if it has a zero.



- if $f(x_0) > 0$ and $f(x_1) < 0$

  $\exists \; y$ s.t. $f(y) = 0$ (IVT)

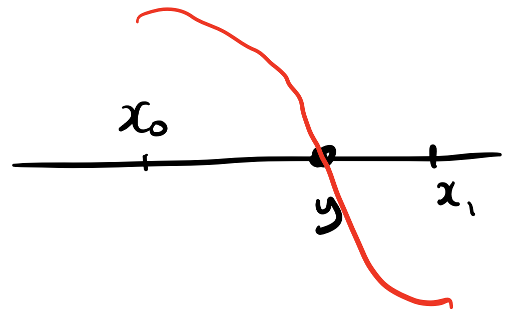
  The set of such f is open and is mapped to itself by G

Sard's theorem fails (Banach space example)

- $G : C^0([0,1]) \longrightarrow C^0([0,1])$

  $G(f) = f^3$ $\qquad dG(f)h = 3f^2 h$

- f is critical point if it has a zero.



- if $f(x_0) > 0$ and $f(x_1) < 0$

  $\exists\ y$ s.t. $f(y) = 0$ (IVT)

  The set of such f is open and is mapped to itself by G

- The set of critial values contains an open set.

A good adjective for differential topology

- M, N connected Hilbert mfds.

- $f : M \to N$ is **Fredholm** if linearization $df_x : T_x M \to T_{f(x)} N$ is **Fredholm**:
  $\dim \ker df_x < \infty$ & $\dim \operatorname{coker} df_x < \infty$.
  $\operatorname{coker} df_x = T_{f(x)}N / \operatorname{im} df_x$.

A good adjective for differential topology

- M, N connected Hilbert mfds.

- $f : M \longrightarrow N$ is Fredholm if linearization $df_x : T_x M \longrightarrow T_{f(x)} N$ is Fredholm: $\dim \operatorname{Ker} df_x < \infty$ & $\dim \operatorname{coker} df_x < \infty$.
  $\operatorname{coker} df_x = T_{f(x)}N / \operatorname{im} df_x$.

- $\operatorname{ind} f = \dim \operatorname{Ker} df_x - \dim \operatorname{coker} df_x$ Fredholm index (indep of x)

A good adjective for differential topology

- M, N connected Hilbert mfds.
- $f : M \to N$ is Fredholm if linearization $df_x : T_x M \to T_{f(x)} N$ is Fredholm:
  $\dim \operatorname{Ker} df_x < \infty$ & $\dim \operatorname{coker} df_x < \infty$.
  $\operatorname{coker} df_x = T_{f(x)}N / \operatorname{im} df_x$.
- $\operatorname{ind} f = \dim \operatorname{Ker} df_x - \dim \operatorname{coker} df_x$ Fredholm index (indep of x)
- In fin dim $f : X^m \to Y^n$ always Fredholm

  $\operatorname{ind} f = m - n$

A good adjective for differential topology

- M, N connected Hilbert mfds.

- $f : M \longrightarrow N$ is **Fredholm** if linearization

  $df_x : T_x M \longrightarrow T_{f(x)} N$ is **Fredholm**:

  $\dim \ker df_x < \infty$ & $\dim \operatorname{coker} df_x < \infty$. $= T_{f(x)}N / \operatorname{im} df_x$.

- $\operatorname{ind} f = \dim \ker df_x - \dim \operatorname{coker} df_x$ Fredholm index (indep of x)

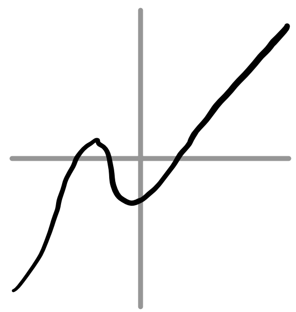
- In fin dim $f : X^m \longrightarrow Y^n$ always Fredholm

  $$\operatorname{ind} f = m - n$$

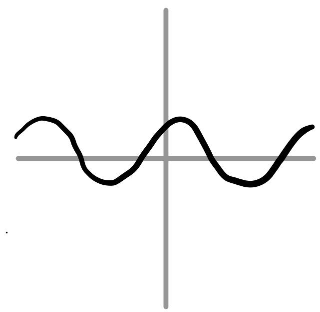
- Hilbert mfds & Fredholm maps arise naturally in PDE theory

Second good adjective

- f is proper if $f^{-1}(C)$ is compact for all C compact



proper map

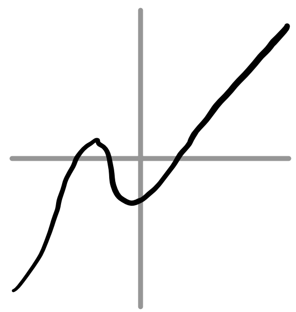


non proper map

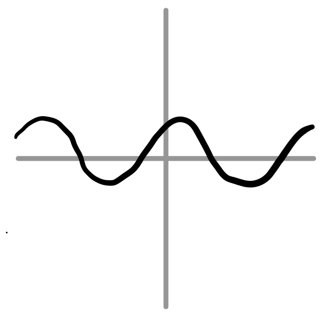
" ∞ goes to ∞ "

Second good adjective

- f is proper if $f^{-1}(C)$ is compact for all C compact



proper map



non proper map

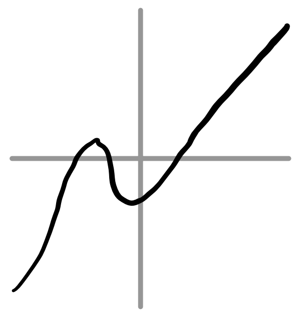
"∞ goes to ∞"

- Smale + IFT

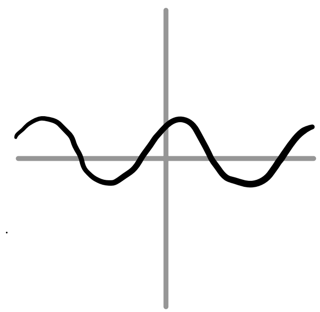
  - Regular values of proper Fredholm maps $f: M \to N$ are open and dense

Second good adjective

- f is proper if $f^{-1}(C)$ is compact for all C compact



proper map



non proper map

"∞ goes to ∞"

- Smale + IFT

  - Regular values of proper Fredholm maps $f: M \to N$ are open and dense

  - If $y \in N$ regular value, then $f^{-1}(y)$ is compact manifold of dimension $\dim f^{-1}(y) = \operatorname{ind} f$.

Fredholm solves genericity issues

Fredholms also solve lack of rigidity

	Fin dim	∞ dim
GL	Complicated	contractible.

Fredholms also solve lack of rigidity

	Fin dim	∞ dim
GL	complicated	contractible.
Fred	contractible	complicated $\simeq \mathbb{Z} \times BGL(\infty)$

Fredholms also solve lack of rigidity

	Fin dim	∞ dim
GL	complicated	contractible.
Fred	contractible	complicated $\simeq \mathbb{Z} \times BGL(\infty)$

In fin dim topology is in GL

In ∞ dim topology is in Fred.

An h-principle for Fredholm maps

- given $f: M \to N$ Fredholm map.

An h-principle for Fredholm maps

- given $f : M \to N$ Fredholm map.

$$df : M \to \Phi$$

linear Fredholm operators

An h-principle for Fredholm maps

- given $f: M \to N$ Fredholm map.

$$df : M \to \Phi$$

linear Fredholm operators

- h-principle:

f Fredholm homotopic to g

An h-principle for Fredholm maps.

- given $f : M \to N$ Fredholm map.

$$df : M \to \Phi$$ linear Fredholm operators

- h-principle:

f Fredholm homotopic to g

Iff.

f homotopic to g & df homotopic to dg.

as continuous maps

An h-principle for Fredholm maps

- given $f: M \to N$ Fredholm map.

$$df : M \to \Phi$$

linear Fredholm operators

- h-principle:

f Fredholm homotopic to g

Iff.

f homotopic to g & df homotopic to dg.

as continuous maps

Thm

$$\mathrm{Fred}[M,N] \cong [M,N] \times [M, \Phi].$$

Part II

homotopy theory of proper Fredholm maps.

S^∞ is not a proper Fredholm retract of B.

$$\begin{tikzcd} S^\infty \arrow[r, "i"] \arrow[dr, "\mathrm{id}"'] & B \arrow[d, "f"] \\ & S^\infty \end{tikzcd}$$

S^∞ is not a proper Fredholm retract of B.

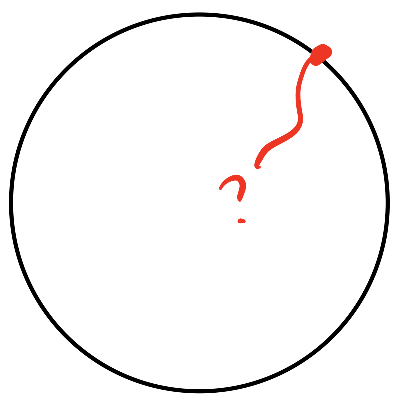
- $\operatorname{ind} f = 1$
- y regular value of f

$$\begin{tikzcd} S^\infty \arrow[r, "i"] \arrow[dr, "id"'] & B \arrow[d, "f"] \\ & S^\infty \end{tikzcd}$$

S^∞ is not a proper Fredholm retract of B.

$$\begin{tikzcd} S^\infty \arrow[r, "i"] \arrow[dr, "id"'] & B \arrow[d, "f"] \\ & S^\infty \end{tikzcd}$$

- $\operatorname{ind} f = 1$
- y regular value of f
- $f^{-1}(y)$ is compact 1 dim mfd with one boundary component.

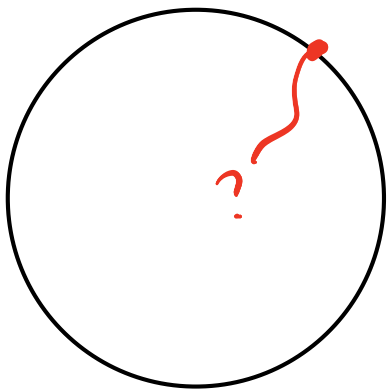


$\lightning$.

S^∞ is not a proper Fredholm retract of B.

$$\begin{tikzcd} S^\infty \arrow[r, "i"] \arrow[dr, "id"'] & B \arrow[d, "f"] \\ & S^\infty \end{tikzcd}$$

- ind $f = 1$
- y regular value of f
- $f^{-1}(y)$ is compact ~~th~~ 1 dim mfd with one boundary component.



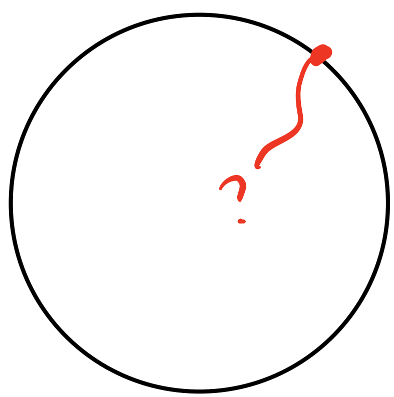
↯.

Exercise S^∞ is a Fredholm retract of B.

S^∞ is not a proper Fredholm retract of B.

- ind $f = 1$
- y regular value of f
- $f^{-1}(y)$ is compact 1 ~~th~~ dim mfd with one boundary component.

$$\begin{tikzcd} S^\infty \arrow[r, "i"] \arrow[dr, "id"'] & B \arrow[d, "f"] \\ & S^\infty \end{tikzcd}$$



$\lightning$.

<u>Exercise</u> S^∞ is a Fredholm retract of B.

Basic problem:

Classify proper Fredholm maps

$$f : \mathbb{H} \longrightarrow \mathbb{H}$$

up to proper Fredholm homotopy

$$\mathcal{F}_n^{prop}[\mathbb{H}]$$

Basic problem:

Classify proper Fredholm maps

$$f : \mathbb{H} \to \mathbb{H}$$

up to proper Fredholm homotopy

$$\mathcal{F}_n^{prop}[\mathbb{H}]$$

- Recall $\mathbb{H} \cong S^\infty$

Question : How to produce these maps from finite dimensional ones?

Question : How to produce these maps from finite dimensional ones?

Suspensions $\quad g: \mathbb{R}^{n+k} \longrightarrow \mathbb{R}^{n}$

Question: How to produce these maps from finite dimensional ones?

Suspensions

$$g : \mathbb{R}^{n+k} \longrightarrow \mathbb{R}^k$$

$$Sg : \underbrace{\mathbb{R}^{n+k} \oplus \mathbb{H}}_{\cong \mathbb{H}} \longrightarrow \underbrace{\mathbb{R}^k \oplus \mathbb{H}}_{\cong \mathbb{H}}$$

$$(x, y) \longmapsto (g(x), y)$$

Question: How to produce these maps from finite dimensional ones?

Suspensions

$$g : \mathbb{R}^{n+k} \longrightarrow \mathbb{R}^k$$

$$Sg : \underbrace{\mathbb{R}^{n+k} \oplus \mathbb{H}}_{\cong \mathbb{H}} \longrightarrow \underbrace{\mathbb{R}^k \oplus \mathbb{H}}_{\cong \mathbb{H}}$$

$$(x, y) \longmapsto (g(x), y)$$

$$C^\infty(\mathbb{R}^{n+k}, \mathbb{R}^k) \longrightarrow \mathcal{F}_n(\mathbb{H})$$

$$g \longmapsto Sg$$

Question: How to produce these maps from finite dimensional ones?

Suspensions

$$g : \mathbb{R}^{n+k} \longrightarrow \mathbb{R}^k$$

$$Sg : \underbrace{\mathbb{R}^{n+k} \oplus \mathbb{H}}_{\cong \mathbb{H}} \longrightarrow \underbrace{\mathbb{R}^k \oplus \mathbb{H}}_{\cong \mathbb{H}}$$

$$(x, y) \longmapsto (g(x), y)$$

$$C^\infty_{prop}(\mathbb{R}^{n+k}, \mathbb{R}^k) \longrightarrow \mathcal{F}^{prop}_n(\mathbb{H})$$

$$g \longmapsto Sg$$

Question: How to produce these maps from finite dimensional ones?

Suspensions

$$g : \mathbb{R}^{n+k} \longrightarrow \mathbb{R}^k$$

$$\begin{aligned} Sg : \underbrace{\mathbb{R}^{n+k} \oplus \mathbb{H}}_{\cong \mathbb{H}} &\longrightarrow \underbrace{\mathbb{R}^k \oplus \mathbb{H}}_{\cong \mathbb{H}} \\ (x, y) &\longmapsto (g(x), y) \end{aligned}$$

$$\begin{aligned} C^\infty_{prop}(\mathbb{R}^{n+k}, \mathbb{R}^k) &\longrightarrow \mathcal{F}_n^{prop}(\mathbb{H}) \\ g &\longmapsto Sg \end{aligned}$$

what is $[\mathbb{R}^{n+k}, \mathbb{R}^k]^{prop}$?

Question: How to produce these maps from finite dimensional ones?

Suspensions

$$g : \mathbb{R}^{n+k} \longrightarrow \mathbb{R}^k$$

$$Sg : \underset{\cong \, H}{\mathbb{R}^{n+k} \oplus H} \longrightarrow \underset{\cong \, H}{\mathbb{R}^k \oplus H}$$

$$(x, y) \longmapsto (g(x), y)$$

$$C^\infty_{prop}(\mathbb{R}^{n+k}, \mathbb{R}^k) \longrightarrow \mathcal{F}^{prop}_n(H)$$

$$g \longmapsto Sg$$

what is $[\mathbb{R}^{n+k}, \mathbb{R}^k]^{prop}$?

A proper map extends to one point compactification.
hence related to $[S^{n+k}, S^k]$

A diagram

$$[\mathbb{R}^{n+k}, \mathbb{R}^k]^{prop} \xrightarrow{S} \mathcal{F}_n^{prop}[\mathbb{H}]$$

A diagram

$$[S^{n+k-1}, S^{k-1}] \xrightarrow[\text{radial extension}]{\cong} [\mathbb{R}^{n+k}, \mathbb{R}^{k}]^{prop} \xrightarrow{S} \mathcal{F}_n^{prop}[\mathbb{H}]$$

A diagram

$$
\begin{tikzcd}
{[S^{n+k-1}, S^{k-1}]} \arrow[r, "\cong", "\text{radial extension}"'] \arrow[d, "\text{Suspension}"'] & {[\mathbb{R}^{n+k}, \mathbb{R}^{k}]^{prop}} \arrow[r, "S"] \arrow[d, "\times \mathbb{R}"] & \mathcal{F}_n^{prop}[\mathbb{H}] \\
{[S^{n+k}, S^{k}]} \arrow[r, "\cong"] & {[\mathbb{R}^{n+k+1}, \mathbb{R}^{k+1}]^{prop}} \arrow[ur, "S"'] &
\end{tikzcd}
$$

A diagram

$$
\begin{tikzcd}
{[S^{n+k-1}, S^{k-1}]} \arrow[r, "\cong", "\text{radial extension}"'] \arrow[d, hook, "\text{Suspension}"'] & {[\mathbb{R}^{n+k}, \mathbb{R}^{k}]^{prop}} \arrow[r, "S"] \arrow[d, hook, "\times \mathbb{R}"] & \mathcal{F}_n^{prop}[\mathbb{H}] \\
{[S^{n+k}, S^{k}]} \arrow[r, "\cong"] \arrow[d, hook] & {[\mathbb{R}^{n+k+1}, \mathbb{R}^{k+1}]^{prop}} \arrow[ur, "S"'] \arrow[d, hook] & \\
\cdots & \cdots &
\end{tikzcd}
$$

A diagram

$$
\begin{tikzcd}
{[S^{n+k-1}, S^{k-1}]} \arrow[r, "\cong", "\text{radial extension}"'] \arrow[d, hook, "\text{Suspension}"'] & {[\mathbb{R}^{n+k}, \mathbb{R}^{k}]^{prop}} \arrow[r, "S"] \arrow[d, hook, "\times \mathbb{R}"] & \mathcal{F}_n^{prop}[\mathbb{H}] \\
{[S^{n+k}, S^{k}]} \arrow[r, "\cong"] \arrow[d, hook] & {[\mathbb{R}^{n+k+1}, \mathbb{R}^{k+1}]^{prop}} \arrow[ur, "S"'] \arrow[d, hook] & \\
\cdots \arrow[d, hook] & \cdots \arrow[d, hook] & \\
\pi_n^S \arrow[r, "\cong"] & {[\mathbb{R}^{n+k}, \mathbb{R}^{k}]^{prop}_S} \arrow[uuur, "S"'] &
\end{tikzcd}
$$

A diagram

$$
\begin{tikzcd}
{[S^{n+k-1}, S^{k-1}]} \arrow[r, "\cong", "\text{radial extension}"'] \arrow[d, hook, "\text{Suspension}"'] & {[\mathbb{R}^{n+k}, \mathbb{R}^k]^{prop}} \arrow[r, "S"] \arrow[d, hook, "\times \mathbb{R}"] & \mathcal{F}_n^{prop}[\mathbb{H}] \\
{[S^{n+k}, S^k]} \arrow[r, "\cong"] \arrow[d, hook] & {[\mathbb{R}^{n+k+1}, \mathbb{R}^{k+1}]^{prop}} \arrow[ur, "S"'] \arrow[d, hook] & \\
\cdots \arrow[d, hook] & \cdots \arrow[d, hook] & \\
\pi_n^S \arrow[r, "\cong"] \arrow[uuurr, bend right=60, "S"'] & {[\mathbb{R}^{n+k}, \mathbb{R}^k]^{prop}_S} \arrow[uuur, "S"'] &
\end{tikzcd}
$$

Stable homotopy groups of spheres

The Theorems (with Toussaint)

Thm $S : \Pi_n^S \longrightarrow \mathcal{F}_n^{prop}[\mathbb{H}]$ is surjective but not injective

The Theorems (with Toussaint)

<u>Thm</u> $S : \pi_n^S \longrightarrow F_n^{prop}[H]$ is surjective but not injective

$$S(a) = S(b) \quad \text{iff} \quad a = \pm b.$$

The Theorems (with Toussaint)

Thm $S : \Pi_n^S \longrightarrow F_n^{prop}[\mathbb{H}]$ is surjective but not injective

$$S(a) = S(b) \quad \text{iff} \quad a = \pm b.$$

Thm Let $FO_n^{prop}[\mathbb{H}]$ oriented maps $S : \Pi_n^S \longrightarrow FO_n^{prop}[\mathbb{H}]$ is a bijection.

The Theorems (with Toussaint)

Thm $S : \pi_n^S \longrightarrow F_n^{prop}[\mathbb{H}]$ is surjective but not injective

$$S(a) = S(b) \quad \text{iff} \quad a = \pm b.$$

Thm Let $FO_n^{prop}[\mathbb{H}]$ oriented maps $S : \Pi_n^S \longrightarrow FO_n^{prop}[\mathbb{H}]$ is a bijection.

Thm $n \neq 0$ let $f \in F_n^{prop}(\mathbb{H})$. Then there exists a $k \in \mathbb{N}$ s.t.

$$f^k = \underbrace{f \circ \cdots \circ f}_{k \text{ times}}$$

is proper Fredholm homotopic to a non surjective proper Fredholm map

Where does the identification come from?

In finite dimensions degree 1 map is not proper homotopic to degree -1 map.

Where does the identification come from?

In finite dimensions degree 1 map is not proper homotopic to degree -1 map.

- $$I\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \qquad T\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} -1 & & \\ & 1 & \\ & & \ddots & \\ & & & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} -x_1 \\ \vdots \\ x_n \end{pmatrix}$$

Where does the identification come from?

In finite dimensions degree 1 map is not proper homotopic to degree -1 map.

- $I\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ $\quad T\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} -1 & & \\ & 1 & \\ & & \ddots & \\ & & & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} -x_1 \\ \vdots \\ x_n \end{pmatrix}$

- SI proper Fredholm homotopic to ST ($GL(\mathbb{H})$ is contractible)

Where does the identification come from?

In finite dimensions degree 1 map is not proper homotopic to degree -1 map.

- $I\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ $\quad T\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} -1 & & \\ & 1 & \\ & & \ddots & \\ & & & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} -x_1 \\ \vdots \\ x_n \end{pmatrix}$

- SI proper Fredholm homotopic to ST
  ($GL(\mathbb{H})$ is contractible)

- Inverse of $f : S^{n+k} \longrightarrow S^k$ is $f^{-1} = T \circ f$

Where does the identification come from?

In finite dimensions degree 1 map is not proper homotopic to degree -1 map.

- $I\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ $\quad T\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} -1 & & \\ & \ddots & \\ & & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} -x_1 \\ \vdots \\ x_n \end{pmatrix}$

- SI proper Fredholm homotopic to ST
  ($GL(H)$ is contractible)

- Inverse of $f : S^{n+k} \to S^k$ is $f^{-1} = T \circ f$

$$S(f^{-1}) = S(T \circ f) = S(T)\,S(f) = S(f)$$

Where does the identification come from?

In finite dimensions degree 1 map is not proper homotopic to degree -1 map.

- $I\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ $\quad T\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} -1 & & \\ & \ddots & \\ & & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} -x_1 \\ \vdots \\ x_n \end{pmatrix}$

- SI proper Fredholm homotopic to ST
  ($GL(H)$ is contractible)

- Inverse of $f: S^{n+k} \longrightarrow S^k$ is $f^{-1} = T \circ f$

  $$S(f^{-1}) = S(T \circ f) = S(T)S(f) = S(f)$$

- No other identifications!

Generalizations

Generalizations

with Toussaint

- M is of finite type $\qquad M \cong X \times \mathbb{H}$

fin dim mfd.

describe $\mathcal{F}_n^{prop}[M, \mathbb{H}]$ in terms of suitable quotient of twisted stable cohomotopy of X.

Generalizations

with Toussaint

fin dim mfd.

- M is of finite type $\quad M \cong X \times \mathbb{H}$

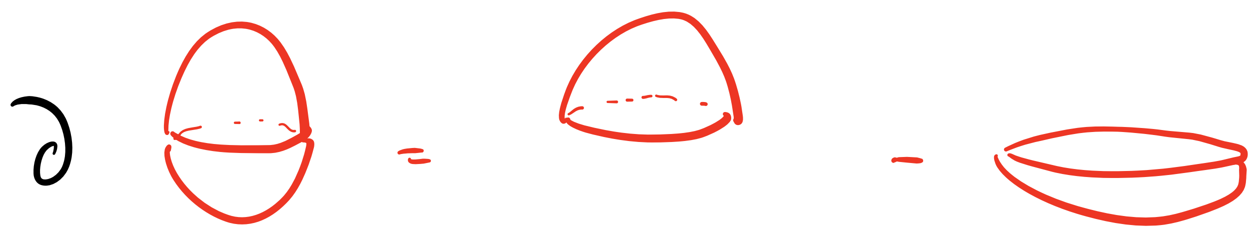
  describe $\mathcal{F}^{prop}_n[M, \mathbb{H}]$ in terms of suitable quotient of twisted stable cohomotopy of X.

Jung & Toussaint

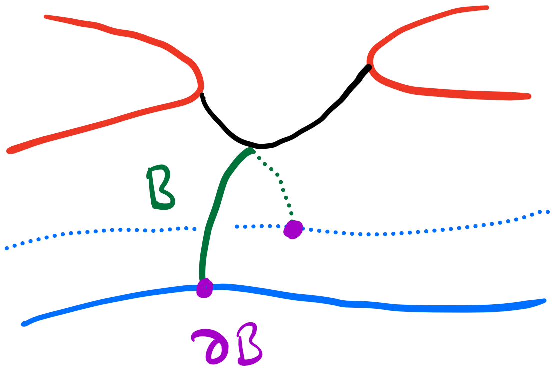
- describe $\mathcal{F}^{prop}_n[\mathbb{H}, Y]$ $\quad Y$ universal bundle in terms of bordism groups with structure.

Part III

homology theory of proper Fredholm maps.



Morse theory primer. $f : M \longrightarrow \mathbb{R}$ Morse function

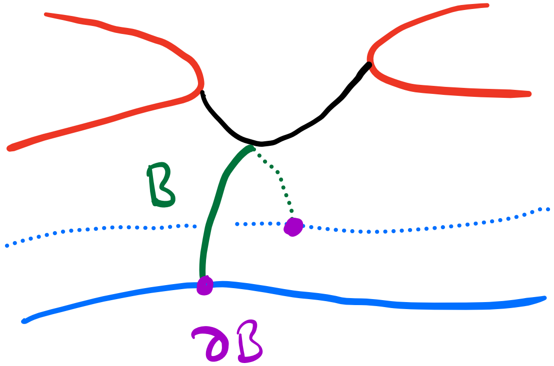


$f^{\leq c+\varepsilon} := f^{-1}(-\infty, c+\varepsilon]$

$f^{\leq c-\varepsilon} := f^{-1}(-\infty, c-\varepsilon]$

$(f^{\leq c+\varepsilon}, f^{\leq c-\varepsilon})$

Morse theory primer. $f: M \longrightarrow \mathbb{R}$ Morse function

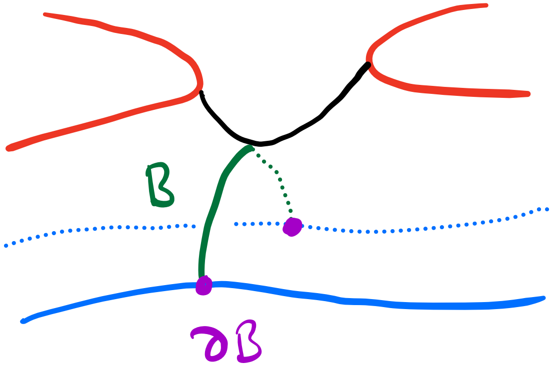


$f^{\leq c+\varepsilon} := f^{-1}(-\infty, c+\varepsilon]$

$f^{\leq c-\varepsilon} := f^{-1}(-\infty, c-\varepsilon]$

$$\left(f^{\leq c+\varepsilon}, f^{\leq c-\varepsilon}\right) \simeq \left(f^{\leq c-\varepsilon} \cup_{\partial B} B, f^{\leq c-\varepsilon}\right)$$

Morse theory primer. $f: M \longrightarrow \mathbb{R}$ Morse function



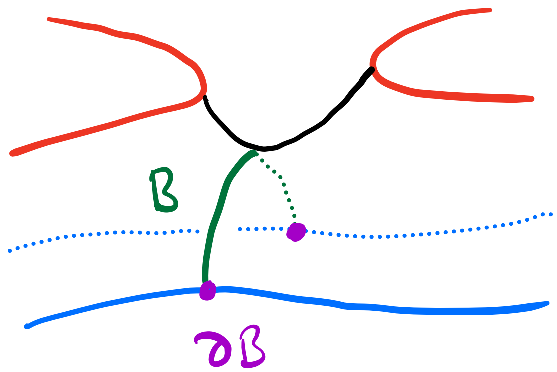
$f^{\leq c+\varepsilon} := f^{-1}(-\infty, c+\varepsilon]$

$f^{\leq c-\varepsilon} := f^{-1}(-\infty, c-\varepsilon]$

$$\left(f^{\leq c+\varepsilon}, f^{\leq c-\varepsilon}\right) \simeq \left(f^{\leq c-\varepsilon} \cup_{\partial B} B, f^{\leq c-\varepsilon}\right) \simeq (B, \partial B)$$

$K = \dim B =$ Morse index of critical point

Morse theory primer. $f: M \longrightarrow \mathbb{R}$ Morse function



$f^{\leq c+\varepsilon} := f^{-1}(-\infty, c+\varepsilon]$

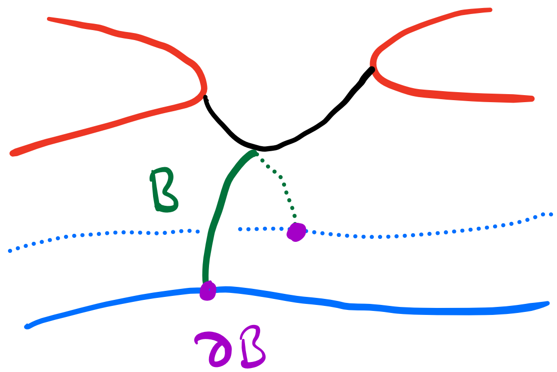
$f^{\leq c-\varepsilon} := f^{-1}(-\infty, c-\varepsilon]$

$$(f^{\leq c+\varepsilon}, f^{\leq c-\varepsilon}) \simeq (f^{\leq c-\varepsilon} \cup_{\partial B} B, f^{\leq c-\varepsilon}) \simeq (B, \partial B)$$

$k = \dim B =$ Morse index of critical point

$$H_*(f^{\leq c+\varepsilon}, f^{\leq c-\varepsilon}) = \begin{cases} \mathbb{Z} & * = k \\ 0 & \text{otherwise} \end{cases} \quad \text{if } k \text{ finite.}$$

Morse theory primer. $\quad f: M \longrightarrow \mathbb{R}$ Morse function



$$f^{\leq c+\varepsilon} := f^{-1}(-\infty, c+\varepsilon]$$

$$f^{\leq c-\varepsilon} := f^{-1}(-\infty, c-\varepsilon]$$

$$\left(f^{\leq c+\varepsilon}, f^{\leq c-\varepsilon}\right) \simeq \left(f^{\leq c-\varepsilon} \cup_{\partial B} B,\ f^{\leq c-\varepsilon}\right) \simeq (B, \partial B)$$

$k = \dim B =$ Morse index of critical point

$$H_*\left(f^{\leq c+\varepsilon}, f^{\leq c-\varepsilon}\right) = \begin{cases} \mathbb{Z} & * = k \\ 0 & \text{otherwise} \end{cases} \quad \text{if } k \text{ finite.}$$

$$H_*\left(f^{\leq c+\varepsilon}, f^{\leq c-\varepsilon}\right) = 0 \quad \text{if } k \text{ infinite.}$$

Question Is there a homology theory

- sensitive to proper Fredholm maps,

<u>Question</u> Is there a homology theory

- sensitive to proper Fredholm maps,
- $H_*^{?}(f^{\le c+\varepsilon}, f^{\le c-\varepsilon}) \neq 0$?

<u>Question</u> Is there a homology theory

- Sensitive to proper Fredholm maps,

- $H_*^?(f^{\leq c+\varepsilon}, f^{\leq c-\varepsilon}) \neq 0$?

- For experts:
  Will surpsess discussion of polarization.
  importand for Floer type problems.

Ingredients

nice subset of M

Hilbert manifold with polarization

- Measure the topology of "$X \subset M$"

Ingredients

- Measure the topology of "$X \subset M$"

  X: nice subset of M

  M: Hilbert manifold with polarization

- For every k, ℓ "Cimplices", proper Fredholm maps

$$\sigma : I^\ell \times B \longrightarrow M \qquad \operatorname{ind} \sigma = k$$

with image in X.

ℓ independent of k!

Ingredients : nice subset of M

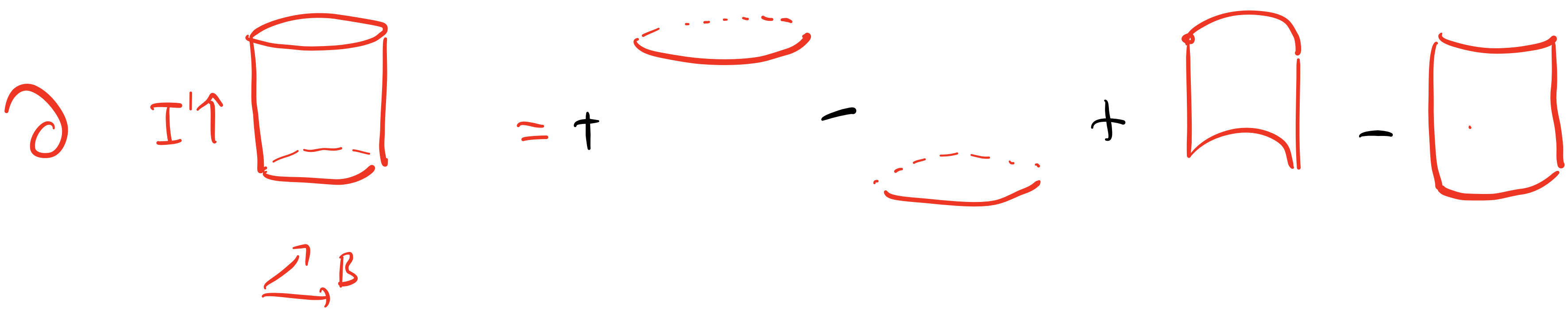
- Measure the topology of "$X \subset M$" ← Hilbert manifold with polarization

- For every k, ℓ "Cimplices", proper Fredholm maps

$$\sigma : I^\ell \times B \longrightarrow M \qquad \operatorname{ind} \sigma = k.$$

with image in X. ℓ independent of k!

- $\partial \sigma$ restriction of σ to faces with signs



- Q_k generated by Cimplices of index k. for all l.

- Q_k generated by Cimplices of index k. for all ℓ.
- D_k degenerate simplices $I^\ell \times B \longrightarrow M$

$(x_1, \dots x_\ell, y_1, y_2, \dots)$

independent of $x_1, \dots x_\ell$ or y_1

- Q_k generated by Cimplices of index k. for all ℓ.
- D_k degenerate simplices $I^\ell \times B \longrightarrow M$

  $(x_1, \dots x_\ell, y_1, y_2, \dots)$

  independent of $x_1, \dots x_\ell$ or y_1

  issue also arises in fin dim with cubical homology
- $C_k = Q_k / D_k$.

- Q_k generated by Cimplices of index k. for all l.
- D_k degenerate simplices $I^l \times B \longrightarrow M$

  $(x_1, \dots x_l, y_1, y_2, \dots)$

  independent of $x_1, \dots x_l$ or y_1

  issue also arises in fin dim with cubical homology

- $C_k = Q_k / D_k$.
- $H^?_k(X) := H_k(C_k, \partial_k)$.

What do we know?

- Functoriality, exact sequence of pair, ...

What do we know?

- Functoriality, exact sequence of pair, ...
- Dimension axiom (This is hard!)

$$H_*^{?}(\mathbb{H}) = \begin{cases} 0 & * < 0 \\ & \\ & \end{cases}$$

What do we know?

- Functoriality, exact sequence of pair, ...
- Dimension axiom (This is hard!)

$$H^{?}_{*}(\mathbb{H}) = \begin{cases} 0 & * < 0 \\ \mathbb{Z} \text{ or } \mathbb{Z}_2 & * = 0 \\ & \end{cases}$$

What do we know? 

- Functoriality, exact sequence of pair, ...
- Dimension axiom (This is hard!)

$$H^{?}_{*}(\mathbb{H}) = \begin{cases} 0 & * < 0 \\ \mathbb{Z} \text{ or } \mathbb{Z}_2 & * = 0 \\ ? & * > 0 \end{cases}$$

What do we know?

- Functoriality, exact sequence of pair, ...
- Dimension axiom (This is hard!)

$$H^?_*(\mathbb{H}) = \begin{cases} 0 & * < 0 \\ \mathbb{Z} \text{ or } \mathbb{Z}_2 & * = 0 \\ ? & * > 0 \end{cases}$$

- Let $S^{\infty-k} = \{(x_1, \dots) \in S^\infty \mid x_1, \dots, x_k = 0\}$

codim $k+1$ sphere

Then

$$H^?_*(\mathbb{H}) \cong H^!_{*-k-1}(S^{\infty-k})$$

What do we know?

- Functoriality, exact sequence of pair, ...
- Dimension axiom (This is hard!)

$$H^{?}_{*}(\mathbb{H}) = \begin{cases} 0 & * < 0 \\ \mathbb{Z} \text{ or } \mathbb{Z}_2 & * = 0 \\ ? & * > 0 \end{cases}$$

- let $S^{\infty-k} = \{(x_1, \dots) \in S^{\infty} \mid x_1, \dots, x_k = 0\}$ codim $k+1$ sphere

  Then

$$H^{?}_{*}(\mathbb{H}) \cong H^{?}_{*-k-1}(S^{\infty-k})$$

- let $\mathring{B}$ be the open unit ball

$$H^{?}_{*}(\mathring{B}) = 0$$

- We know explicit generator of $H_0^{?}(\mathbb{H})$

- We know explicit generator of $H_0^?(\mathbb{H})$

$$B \cong \mathbb{H} - \mathring{B}$$

"flipping ball inside out".

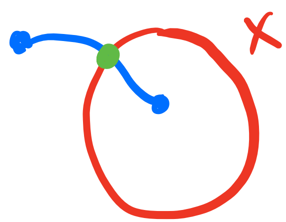
- We know explicit generator of $H_0^?(\mathbb{H})$

$$B \cong \mathbb{H} - \mathring{B}$$

"flipping ball inside out".

- There is a non-trivial intersection pairing for $X \subsetneq \mathbb{H}$

$$\bullet : H_k^?(X) \times \underset{\substack{\cong \\ H_{\ell-1}(\mathbb{H}-X)}}{H_\ell(\mathbb{H}, \mathbb{H}-X; \mathbb{Z}_2)} \longrightarrow H_{k-\ell}(X; \mathbb{Z}_2)$$



What do we hope / expect / plan to do?

- $$H_*^{?}(\mathbb{H}) = \begin{cases} \mathbb{Z}_2 & * = 0 \\ 0 & * \neq 0 \end{cases}$$

What do we hope / expect / plan to do?

- $$H^{?}_{*}(\mathbb{H}) = \begin{cases} \mathbb{Z}_2 & * = 0 \\ 0 & * \neq 0 \end{cases}$$

- Intersection pairing induces Alexander type duality

$$H^{?}_{-k}(X) \cong H_{k-1}(\mathbb{H} - X ; \mathbb{Z}_2) \qquad \star$$

What do we hope / expect / plan to do?

- $H^{?}_{*}(\mathbb{H}) = \begin{cases} \mathbb{Z}_2 & * = 0 \\ 0 & * \neq 0 \end{cases}$

- Intersection pairing induces Alexander type duality

$$H^{?}_{-k}(X) \cong H_{k-1}(\mathbb{H} - X ; \mathbb{Z}_2) \qquad \star$$

- Can define <u>oriented</u> $H^{?,\mathrm{or}}_{*}$ for which $\star$ is an iso with $\mathbb{Z}$ coefficients.

What do we hope / expect / plan to do?

- $H^{?}_{*}(\mathbb{H}) = \begin{cases} \mathbb{Z}_2 & * = 0 \\ 0 & * \neq 0 \end{cases}$

- Intersection pairing induces Alexander type duality

$$H^{?}_{-k}(X) \cong H_{k-1}(\mathbb{H} - X ; \mathbb{Z}_2) \qquad \star$$

- Can define oriented $H^{?,or}_{*}$ for which $\star$ is an iso with $\mathbb{Z}$ coefficients.

- Can do Floer theory via sublevel sets.

Thank you for your attention.