Proper homotopy classes of proper Fredholm mappings

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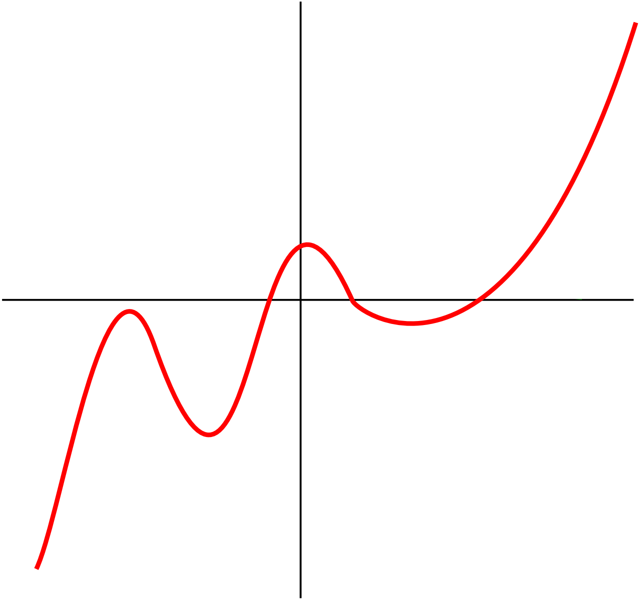
NDNS+ Twente

25-06-2018

How can we guarantee that an equation $f(x) = y$ has a solution?

Solving equations



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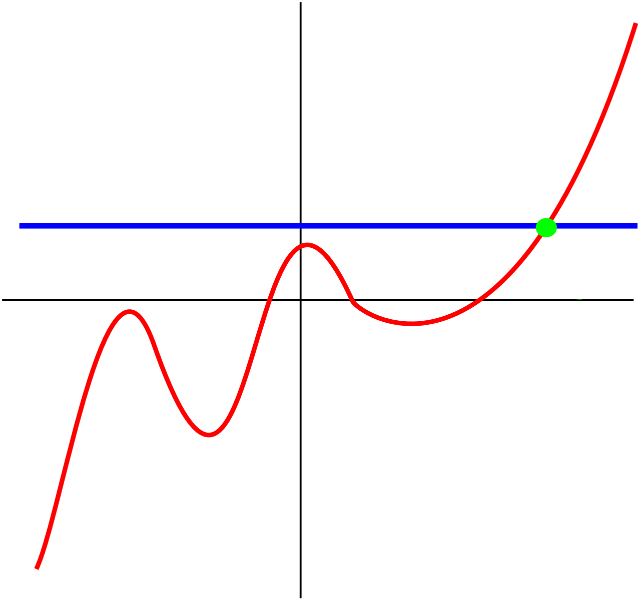

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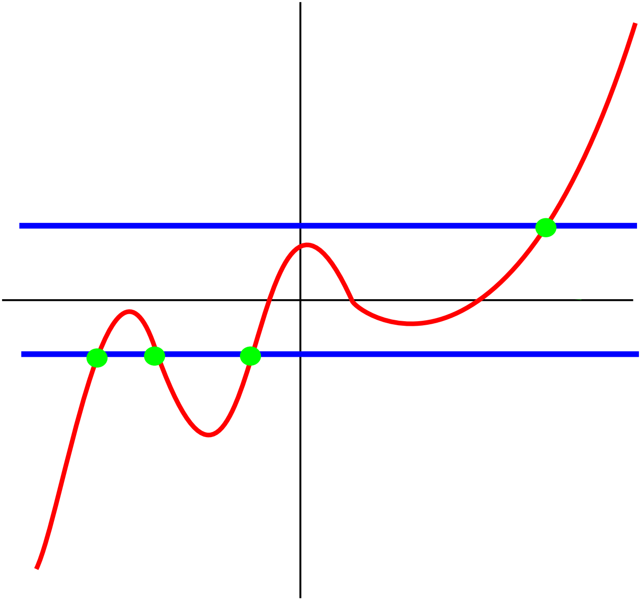

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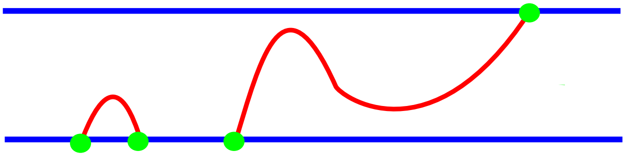

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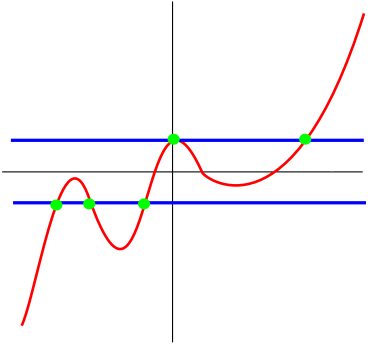

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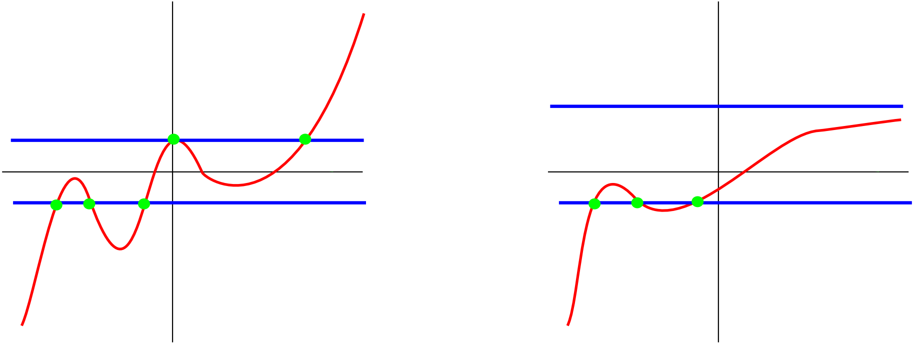
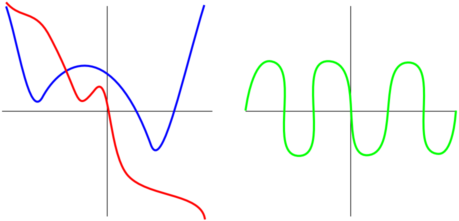


Definition

A map $f : M \to N$ is proper if $f^{-1}(C)$ is compact for every C.



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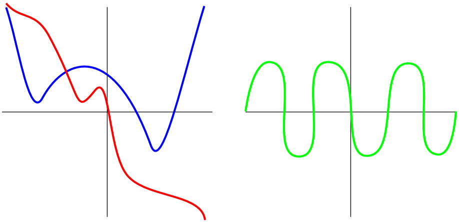

Graphs of proper and non-proper maps

Theorem

$$[\mathbb{R}^n, \mathbb{R}^m]_{\mathrm{prop}} \cong [S^{n-1}, S^{m-1}] \cong \pi_{n-1}(S^{m-1})$$

(Last isomorphism holds if $n > 1$)

Theorem

Let $f : M^m \to N^n$ a proper smooth map. Then for $y \in N$ regular value $X = f^{-1}(y)$ is a closed $m - n$ dimensional manifold. The cobordism class of X does not depend on y or the proper homotopy class of f.

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Let M^m be an m-dimensional closed manifold. There is a bijection

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- Fredholm maps makes differential topology available (Smale-Sard)
- Such maps arise in the study of non-linear elliptic PDE's

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In finite dimensions the index is the difference in dimensions of domain and codomain. Examples of such maps are easy to find. Let $g : \mathbb{R}^m \to \mathbb{R}^n$ be proper. Then define $f : \mathbb{R}^m \times \mathbb{H} \to \mathbb{R}^n \times \mathbb{H}$ by

$$f(z, x) = (g(z), x)$$

This is a proper Fredholm map of index $k = m - n$.

Some funny facts in infinite dimensions.

- Any infinite dimensional Hilbert manifold can be embedded as an open set in $\mathbb{H}$ (Palais, Kuiper-Burghelea, Moulis, Eells-Elworthy,...).
- Thus its tangent bundle is trivial!

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The differential of a Fredholm map can be viewed as a map

$$df : M \to \Phi_k(\mathbb{H})$$

where $\Phi_k(\mathbb{H})$ is the space of (linear) Fredholm operators of index k.

Homotopy of Fredholm maps

A necessary requirement for two Fredholm maps to be Fredholm homotopic:

- $f, g : M \to N$ are homotopic as continuous maps
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Proof.

Extention theorems. □

Invariants of proper Fredholm maps

- Given a regular value y of a proper Fredholm map $f : M \to N$ we get a closed manifold $X = f^{-1}(y) \subset M$, the *Pontryagin manifold*.
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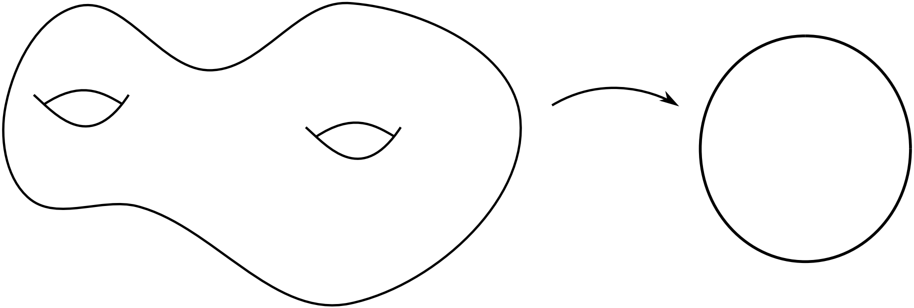
Theorem (Abbondandolo-R.)

This is a poor invariant! If $M = \mathbb{H}$ and $k \neq 0$ the image is 0

Upgrading the Pontryagin manifold: Framings in finite dimensions



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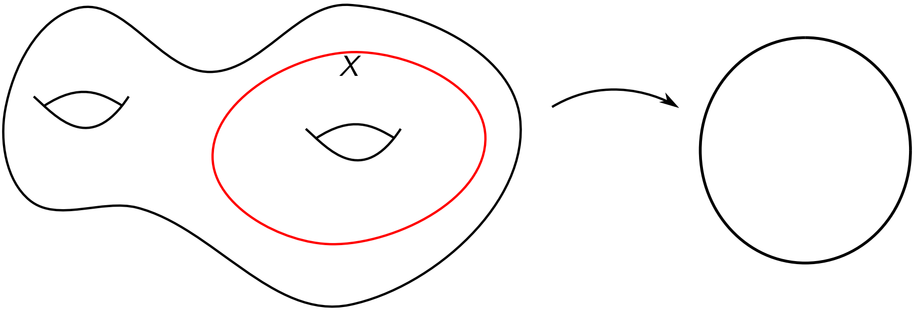

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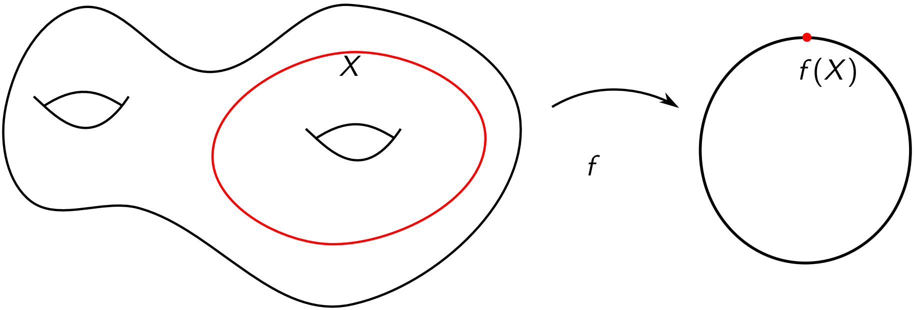

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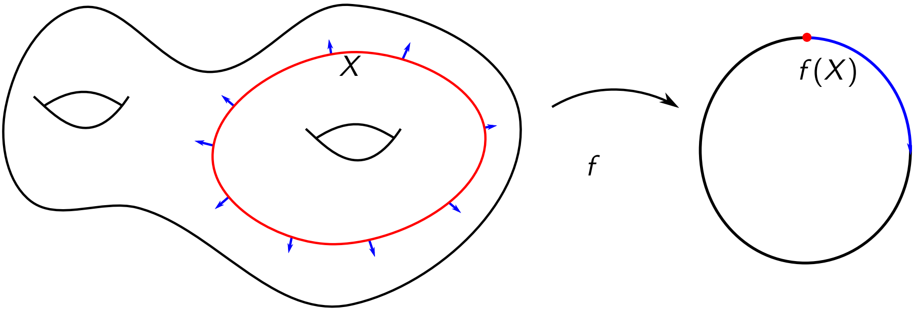

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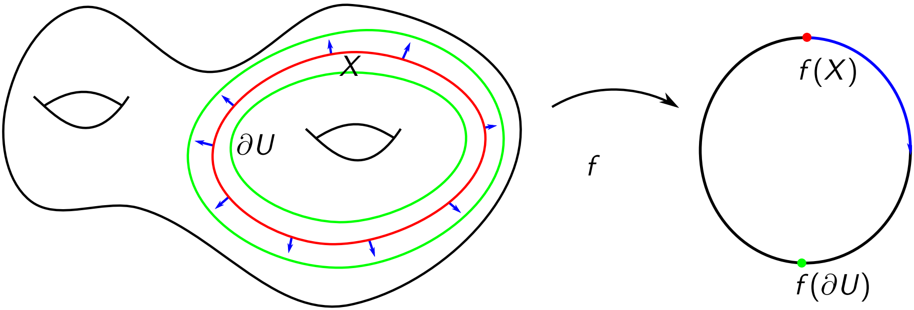

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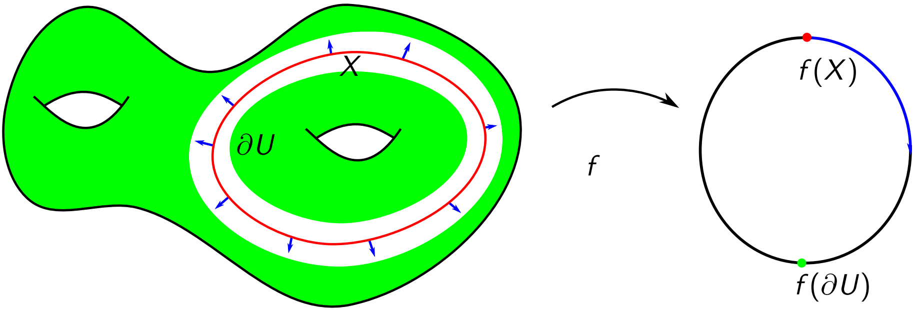

- A framing of X is a (smooth) choice of basis of each normal space.
- A choice of framing allows us to construct a map $f : M \to S^n$ with X as a Pontryagin manifold.
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Infinite dimensional framings contain no information

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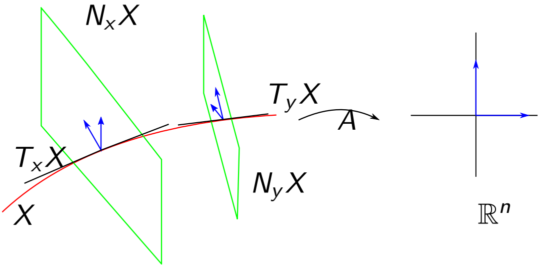
This notion of framing is *not* useful in infinite dimensions!

Rethinking finite dimensional framings.



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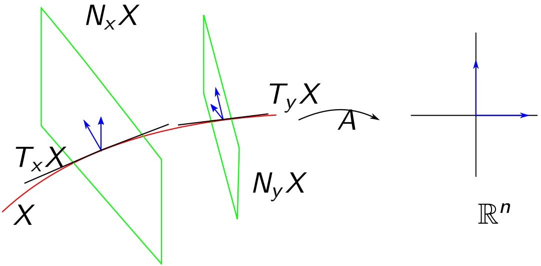

A (finite dimensional) framing is a section of $\mathrm{Iso}(NX, \mathbb{R}^n)$ this defines a section of $\mathrm{Hom}(TM, \mathbb{R}^n)$ by mapping the tangent space to X to zero. We see the framing as a map $A : X \to \mathrm{Hom}\left(TM\big|_X, \mathbb{R}^n\right)$ such that

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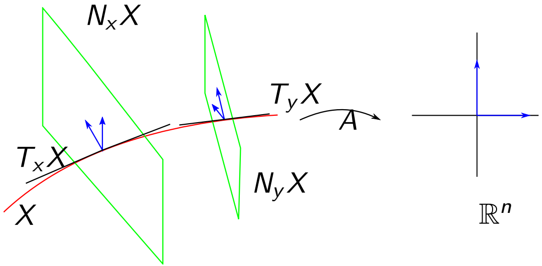

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In finite dimensions this map always extends to the whole of M, i.e. a map

$$A : M \to \mathrm{Hom}\left(TM, \mathbb{R}^n\right)$$

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Definition

Let $X \subset M$ by a finite dimensional submanifold of a Hilbert manifold M. A framing is a map $A : M \to \Phi_k(\mathbb{H})$ such that

$$\ker A(x) = T_x X \qquad \text{for all} \qquad x \in X$$

There is a similar notion of framed cobordism. This gives rise to framed cobordism sets $\Omega_k^{\mathrm{fr}}(M)$.

The main theorem

Theorem (Abbondandolo-R.)

A proper Fredholm map $f : M \to N$ gives rise to a framed cobordism class via the map $f \mapsto [(f^{-1}(y), df)]$.

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Proof.

Can't collapse as in finite dimensions. Proper Fredholm extention theorem. □

Computation of $\Omega_k^{\mathrm{fr}}(M)$ when $k < 0$

Proposition

For $k < 0$ there is a bijection

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Theorem (Atiyah-Jänich)

If the topology of M is mild (e.g. when M deformation retracts to a finite dimensional closed manifold) we have

$$[M, \Phi_k(\mathbb{H})] \cong \tilde{K}(M)$$

as $\Phi_k(\mathbb{H}) \simeq \lim_{n\to\infty} Gr(n, \mathbb{H})$.

(We understand the topology of this space better than the topology of the two sphere S^2)

The case of index 0: Orientable framings.

Definition

A map $A : M \to \Phi_0(\mathbb{H})$ is called *orientable* if

$$A_* : \pi_1(M) \to \pi_1(\Phi_0(\mathbb{H})) \cong \mathbb{Z}/2\mathbb{Z}$$

is trivial. Otherwise it is *non-orientable*.

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The set $[M, \Phi_0(\mathbb{H})]$ decomposes as

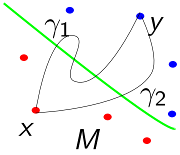
$$[M, \Phi_0(\mathbb{H})] = [M, \Phi_0(\mathbb{H})]_{\mathrm{or}} \cup [M, \Phi_0(\mathbb{H})]_{\mathrm{no}}$$

Degree theory

Given an oriented framed submanifold (X, A) we define an equivalence relation on elements of X.

$$x \sim y \Leftrightarrow [A \circ \gamma] = 0 \in \pi_1(\Phi_0(\mathbb{H}), GL(\mathbb{H})) \cong \mathbb{Z}/2\mathbb{Z}$$

where γ is a path in M from x to y.



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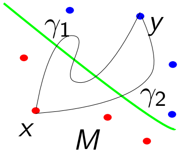

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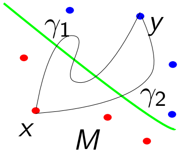

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If (X, A) is not orientable we define the Cacciopoli-Smale mod 2 degree by

$$\deg_2(X, A) = \#X \pmod 2.$$

Theorem (Abbondandolo-R.)

We have a bijection

$$\Omega_0^{\mathrm{fr}}(M) = \Omega_0^{\mathrm{fr}}(M)_{\mathrm{or}} \oplus \Omega_0^{\mathrm{fr}}(M)_{\mathrm{no}}$$

where

$$\Omega_0^{\mathrm{fr}}(M)_{\mathrm{or}} \cong [M, \Phi_0(\mathbb{H})]_{\mathrm{or}} \times \mathbb{N}$$

and

$$\Omega_0^{\mathrm{fr}}(M)_{\mathrm{no}} \cong [M, \Phi_0(\mathbb{H})]_{\mathrm{no}} \times \mathbb{Z}/2\mathbb{Z}$$

The isomorphisms are given by the degrees.

Explicit representatives

$$\mathcal{F}_0^{\mathrm{prop}}[\mathbb{H},\mathbb{H}] \cong [\mathbb{H},\Phi_0(\mathbb{H})]_{\mathrm{or}} \times \mathbb{N} \cong \mathbb{N}$$

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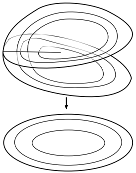
Write $\mathbb{H} \cong \mathbb{C} \times \mathbb{H}$. Then we have

$$f_n(z,x) = \begin{cases} (|z|^2, x) & n = 0 \\ (z^n, x) & n > 1 \end{cases}$$

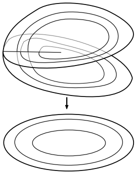


Explicit representatives

$$\mathcal{F}_0^{\mathrm{prop}}[\mathbb{H}, \mathbb{H}] \cong [\mathbb{H}, \Phi_0(\mathbb{H})]_{\mathrm{or}} \times \mathbb{N} \cong \mathbb{N}$$

Write $\mathbb{H} \cong \mathbb{C} \times \mathbb{H}$. Then we have

$$f_n(z, x) = \begin{cases} (|z|^2, x) & n = 0 \\ (z^n, x) & n > 1 \end{cases}$$

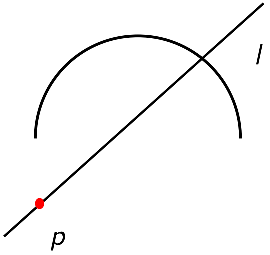


The theorem implies that f_n and $\overline{f_n}$ are proper Fredholm homotopic! This does not hold in finite dimensions!

Explicit representative of degree 1 non-orientable map

Consider the tautological line $\gamma \to \mathbb{RP}^1$. Define the map: $f : \gamma \times \mathbb{H} \to \mathbb{R}^2 \times \mathbb{H}$ by

$$f(l, p, x) = (p, x).$$



Thank you for your attention.

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