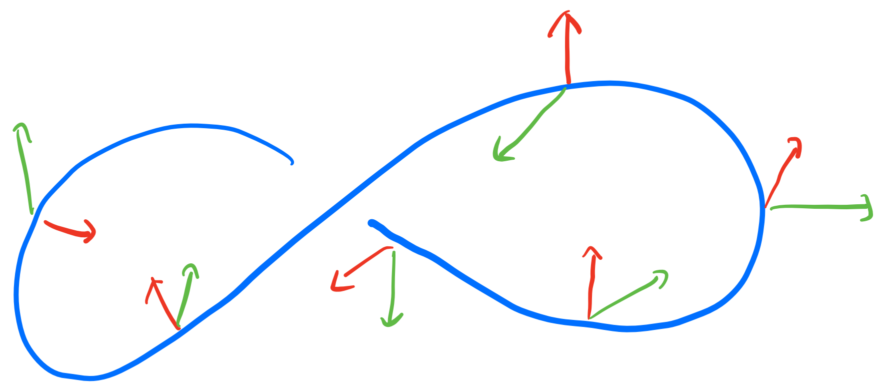
Non-linear proper Fredholm maps and the stable homotopy groups of spheres.

joint with Lauran Toussaint & Alberto Abbondandolo



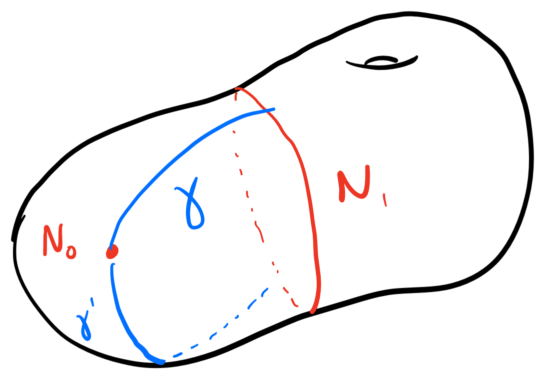
Thomas Rot
Vrije Universiteit Amsterdam

Prospects in geometry and global analysis
August 2023
Rauischolzhausen

Motivation: geometric ODE's / PDE's

(M, g) Riemannian mfd. N_0, N_1 submanifolds.

$$P = \{ \gamma : I \longrightarrow M \mid \gamma(0) \in N_0 \quad \gamma(1) \in N_1 \}$$ path space



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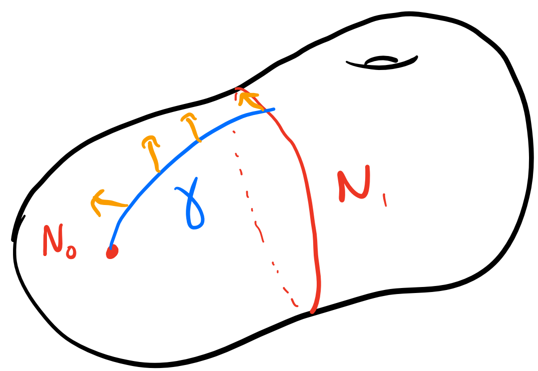

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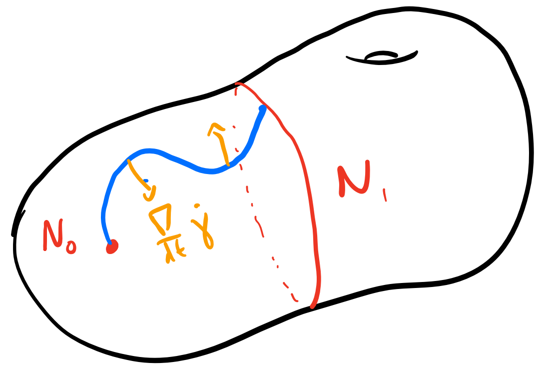

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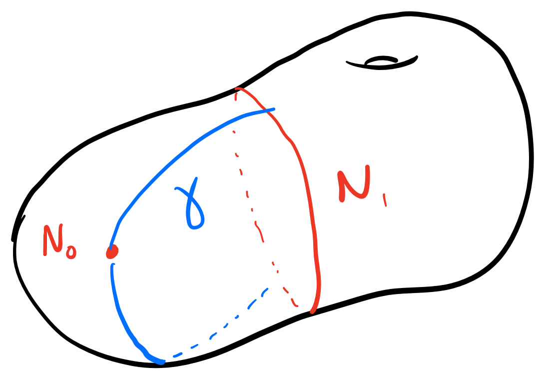

- $f^{-1}(0)$ is space of gedesics from N_0 to N_1
- f non-linear Fredholm of index $\dim N_0 + \dim N_1$
- bounded length $\Rightarrow$ f proper.

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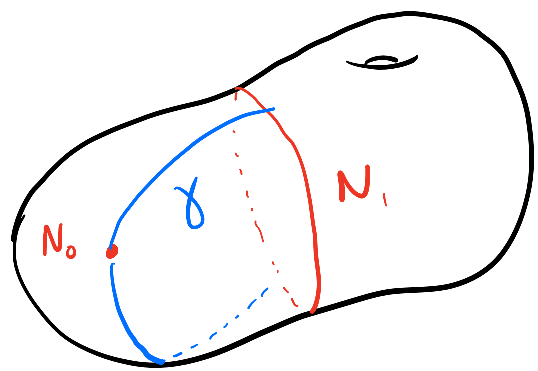

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- What behavior persists if we vary g?
- Always solutions?
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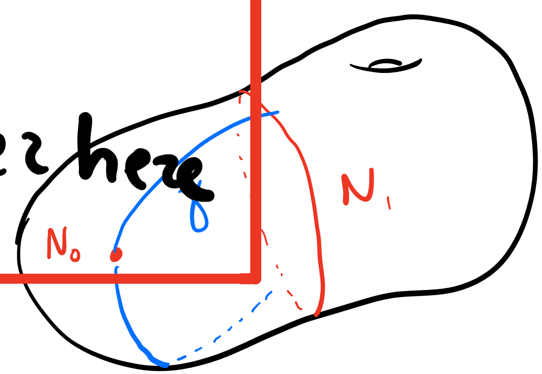
$T_\gamma P = \Gamma(\gamma^* TM)$ " vector fields along γ.

Warning!

Toy example

Morse theory is better here

$f : P \longrightarrow TP \qquad f(\gamma) = \frac{\nabla}{dt} \dot\gamma$



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Classify non linear proper Fredholm maps

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in terms of finite dimensional invariants

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understand the solution sets $f^{-1}(0)$.

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- $f : P \longrightarrow TP$
- $\tilde{f} : P \longrightarrow \mathbb{H}$

Notation:

$$\mathbb{H} = \ell^2(\mathbb{N}, \mathbb{R}) = \{ (x_1, x_2, \dots) \mid \sum_{i=1}^{\infty} x_i^2 < \infty \}.$$

canonical inclusions $\mathbb{R}^n \subset \mathbb{H}$

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$$\Phi(\mathbb{H}) = \left\{ L \in L(\mathbb{H}, \mathbb{H}) \;\middle|\; \begin{array}{l} \dim \ker L < \infty \\ \dim \operatorname{coker} L < \infty \end{array} \right\}$$

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index L = dim ker − dim coker.

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linear Fredholm operators.

"$\mathbb{H}/\operatorname{im} L$

Positive index = "high dimensional to low dimensional"

zero index = "same dimensions"

negative index = "low dimensional to high dimensional"

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$$\mathcal{F}(\mathbb{H}) = \left\{ f : \mathbb{H} \longrightarrow \mathbb{H} \;\middle|\; df : \mathbb{H} \longrightarrow \Phi(\mathbb{H}) \right\}$$

non-linear Fredholm mappings

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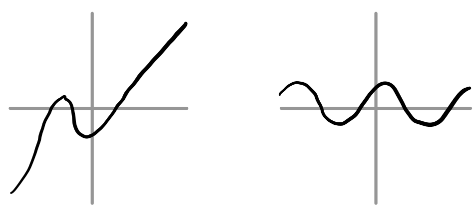
$$F^{prop}(\mathbb{H}) = \left\{ f : \mathbb{H} \longrightarrow \mathbb{H} \;\middle|\; \begin{array}{l} f \text{ proper} \\ df : \mathbb{H} \longrightarrow \Phi(\mathbb{H}) \end{array} \right\}$$

non-linear proper Fredholm mappings

f is proper if $f^{-1}(C)$ is compact for all C compact



proper map | non proper map

" ∞ goes to ∞ ".

Square brackets denote homotopy classes.

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$$g : \mathbb{R}^{n+k} \longrightarrow \mathbb{R}^k$$
$$Sg : \mathbb{R}^{n+k} \oplus \mathbb{H} \longrightarrow \mathbb{R}^k \oplus \mathbb{H}$$
$$(x, y) \longmapsto (g(x), y)$$

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$$C^\infty(\mathbb{R}^{n+k}, \mathbb{R}^k) \longrightarrow \mathcal{F}_n(\mathbb{H})$$

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what is $[\mathbb{R}^{n+k}, \mathbb{R}^k]^{prop}$?

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what is $[\mathbb{R}^{n+k}, \mathbb{R}^{k}]^{prop}$?

A proper map extends to one point compactification.

A diagram

$$[\mathbb{R}^{n+k}, \mathbb{R}^k]^{prop} \xrightarrow{S} \mathcal{F}_n^{prop}[\mathbb{H}]$$

A diagram

$$[S^{n+k-1}, S^{k-1}] \xrightarrow[\text{radial extension}]{\cong} [\mathbb{R}^{n+k}, \mathbb{R}^k]^{prop} \xrightarrow{S} \mathcal{F}_n^{prop}[\mathbb{H}]$$

A diagram

$$
\begin{tikzcd}
{[S^{n+k-1}, S^{k-1}]} \arrow[r, "\cong", "\text{radial extension}"'] \arrow[d, hook, "\text{Suspension}"'] & {[\mathbb{R}^{n+k}, \mathbb{R}^{k}]^{prop}} \arrow[r, "S"] \arrow[d, hook, "\times \mathbb{R}"] & \mathcal{F}_n^{prop}[\mathbb{H}] \\
{[S^{n+k}, S^{k}]} \arrow[r, "\cong"] & {[\mathbb{R}^{n+k+1}, \mathbb{R}^{k+1}]^{prop}} \arrow[ur, "S"'] &
\end{tikzcd}
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\pi_n^S \arrow[r, "\cong"] & {[\mathbb{R}^{n+k}, \mathbb{R}^{k}]^{prop}_S} \arrow[uuur, "S"'] &
\end{tikzcd}
$$

stable homotopy group of spheres . . .

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\begin{tikzcd}
{[S^{n+k-1}, S^{k-1}]} \arrow[r, "\cong"] \arrow[r, swap, "\text{radial extension}"] \arrow[d, hook, swap, "\text{Suspension}"] & {[\mathbb{R}^{n+k}, \mathbb{R}^{k}]^{prop}} \arrow[r, "S"] \arrow[d, hook, "\times \mathbb{R}"] & \mathcal{F}_n^{prop}[\mathbb{H}] \\
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\cdots \arrow[d, hook] & \cdots \arrow[d, hook] & \\
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stable homotopy group of spheres $\Rightarrow$ π_n^S

The Theorems:

<u>Thm</u> (Toussaint-R) $S : \Pi_n^S \longrightarrow \mathcal{F}_n^{prop}[\mathbb{H}]$

is surjective but <u>not</u> injective.

$$S(a) = S(b) \quad \text{iff} \quad a = \pm b.$$

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Previous work e.g. Švarc, Gęba, Berger, Bauer, Furuta, ... did not consider the fully nonlinear problem

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	FO_n^{prop}	$\# F^{prop}$
$k<0$	0	0
$k=0$	$\mathbb{Z}$	$\mathbb{N}$
$k=1$	$\mathbb{Z}/2$	2
$k=2$	$\mathbb{Z}/2$	2
$k=3$	$\mathbb{Z}/24$	13

Note

$$F^{prop}[H] = \bigcup_n F_n^{prop}[H] \cong \pi^S/\sim$$

does not have a natural group structure. ($FO^{prop}[H]$ does)

however the multiplicative structure remains.

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Thm (Toussaint-R) $n \neq 0$

Let $f \in F_n^{prop}(\mathbb{H})$. Then there excists a $k \in \mathbb{N}$ s.t.

$$f^k = \underbrace{f \circ \cdots \circ f}_{k \text{ times}}$$

is proper Fredholm homotopic to a non surjective proper Fredholm map.

Where does the identification come from?

In finite dimensions degree 1 map is not proper homotopic to degree -1 map.

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$f_i : \mathbb{R}^2 \to \mathbb{R}^2$

$$f_1 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} \qquad \text{degree } 1$$

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In ∞ dimensions $\int f_1 = \int f_{-1}$ ($GL(\mathbb{H})$ is contractible)

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$$f_{-1}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -x \\ y \end{pmatrix} \qquad \text{degree } -1.$$

In ∞ dimensions $Sf_1 = Sf_{-1}$ ($GL(\mathbb{H})$ is contractible)

$$\begin{pmatrix} \cos \pi t & \sin \pi t \\ -\sin \pi t & \cos \pi t \end{pmatrix} \qquad \underset{t=0}{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}} \qquad \underset{t=1}{\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}}$$

$$\begin{pmatrix} -1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & 1 & \\ & & & & & \ddots \end{pmatrix} \sim \begin{pmatrix} -1 & & & \\ & \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} & & \\ & & \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} & \\ & & & \ddots \end{pmatrix} \sim \begin{pmatrix} -1 & & & \\ & \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} & & \\ & & \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} & \\ & & & \ddots \end{pmatrix} \sim \begin{pmatrix} \begin{pmatrix} -1 & \\ & -1 \end{pmatrix} & & & \\ & \begin{pmatrix} -1 & \\ & -1 \end{pmatrix} & & \\ & & \begin{pmatrix} -1 & \\ & \end{pmatrix} & \\ & & & \ddots \end{pmatrix}$$

Where does the identification come from?

In finite dimensions degree 1 map is not proper homotopic to degree -1 map.

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$$\begin{pmatrix}-1&&&&\\&1&&&\\&&1&&\\&&&1&\\&&&&\ddots\end{pmatrix} \sim \begin{pmatrix}-1&&\\&\begin{pmatrix}1&\\&1\end{pmatrix}&&\\&&\begin{pmatrix}1&\\&1\end{pmatrix}&\\&&&\ddots\end{pmatrix} \sim \begin{pmatrix}-1&&\\&\begin{pmatrix}-1&\\&-1\end{pmatrix}&&\\&&\begin{pmatrix}-1&\\&-1\end{pmatrix}&\\&&&\ddots\end{pmatrix} \sim \begin{pmatrix}\begin{pmatrix}-1&\\&-1\end{pmatrix}&&\\&\begin{pmatrix}-1&\\&-1\end{pmatrix}&&\\&&\begin{pmatrix}-1&\\&\cdot\end{pmatrix}&\\&&&\ddots\end{pmatrix} \sim \begin{pmatrix}1&&&&\\&1&&&\\&&1&&\\&&&1&\\&&&&\ddots\end{pmatrix}$$

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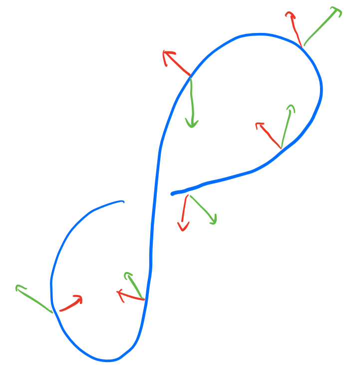
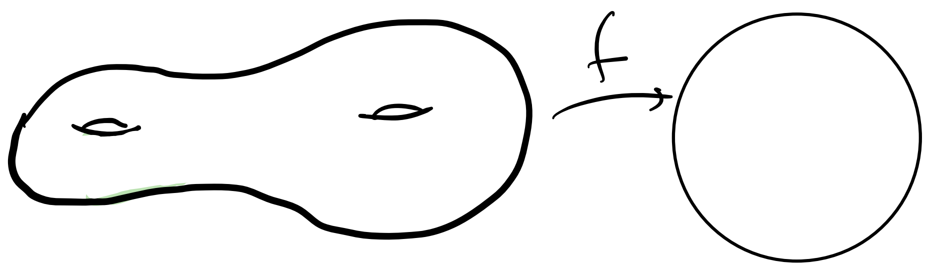
The inverse of $f: S^{n+k} \longrightarrow S^k$ is $T\circ f$ (T coordinate flip)

$$S(T\circ f) = S(T)\,S(f) = S(f)$$

One more ingredient: Framed cobordism.

$f : \mathbb{R}^{n+k} \longrightarrow \mathbb{R}^k.$ proper





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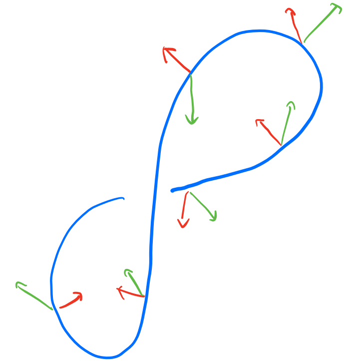
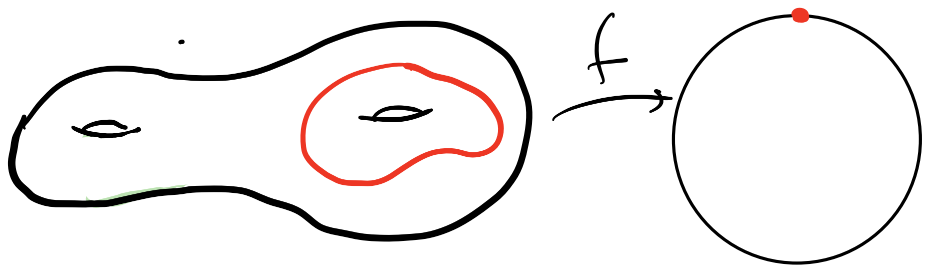
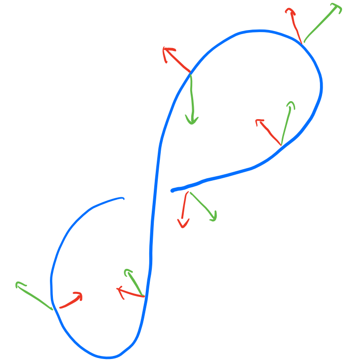


$f: \mathbb{R}^{n+k} \longrightarrow \mathbb{R}^k$. proper

y regular value : $X = f^{-1}(y)$ closed n-dim mfd.





One more ingredient: Framed cobordism.

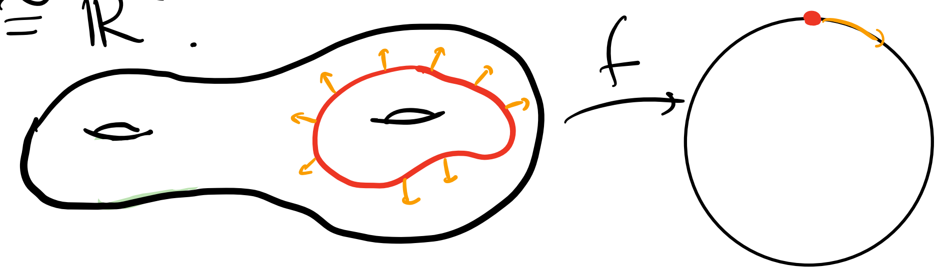
$$f : \mathbb{R}^{n+k} \longrightarrow \mathbb{R}^k. \qquad \text{proper}$$

y regular value : $X = f^{-1}(y)$ closed n-dim mfd. Framed by df:

$$0 \to T_x X \longrightarrow \mathbb{R}^{n+k} \xrightarrow{df} \mathbb{R}^k \to 0 \qquad \forall x \in X.$$

df induces iso $N_x X \cong \mathbb{R}^k$.

$\ker df|_X = TX$



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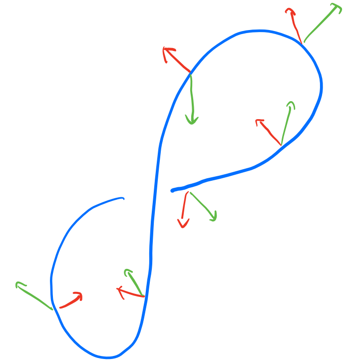

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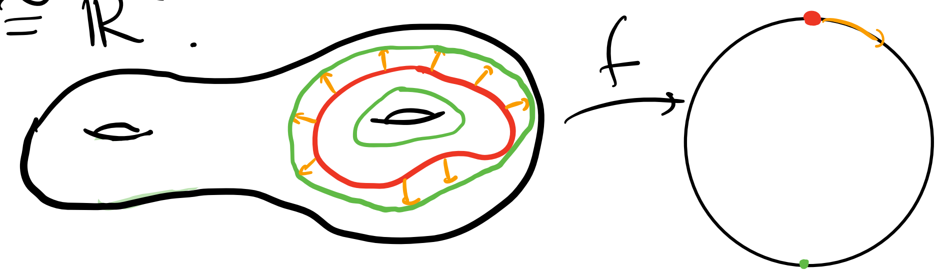

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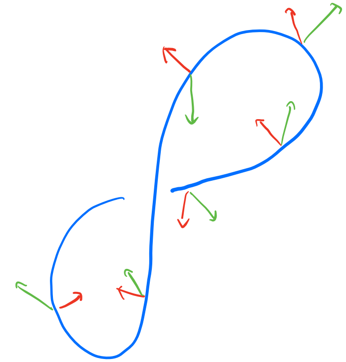

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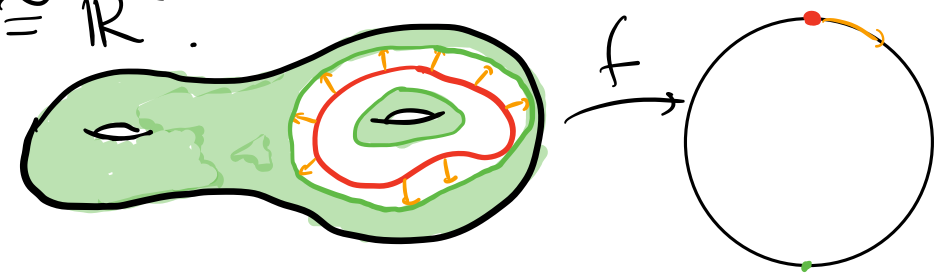

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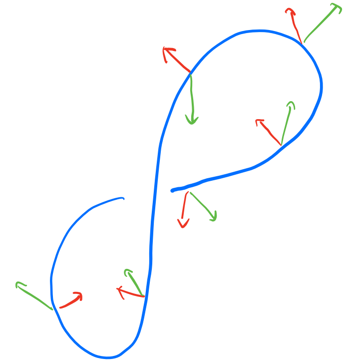

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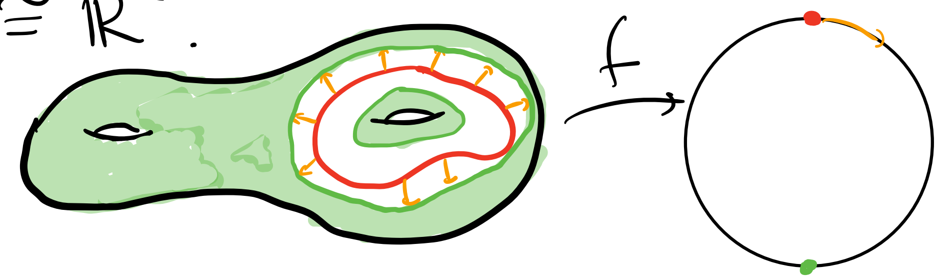
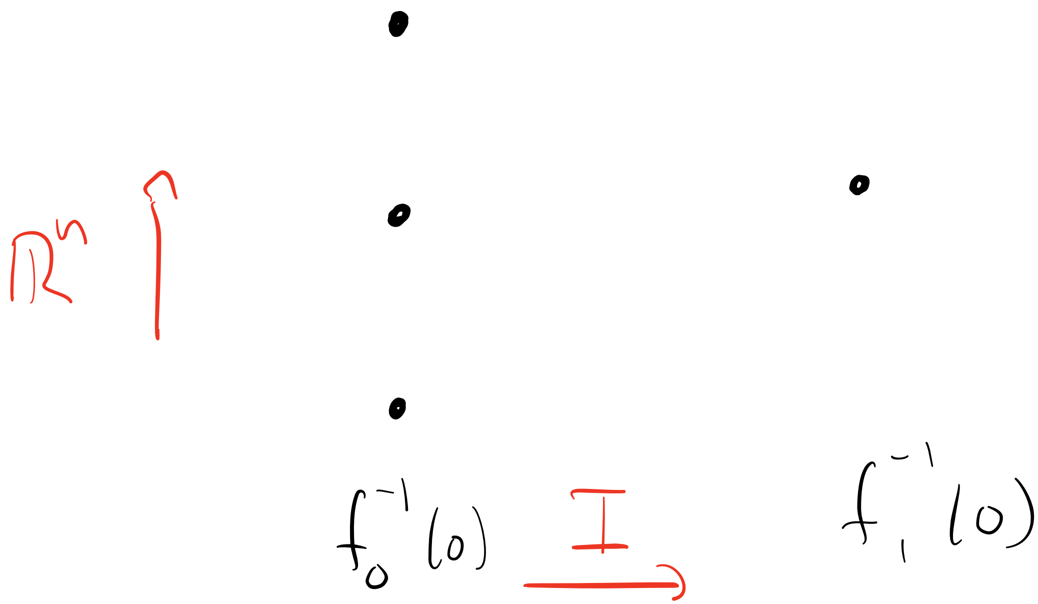


Thm (Csepai): for k sufficiently large

$$[\mathbb{R}^{n+k}, \mathbb{R}^k]^{prop} \cong \Omega_n^{fr}(\mathbb{R}^{n+k}) = \left\{ \begin{matrix} X^n \subset \mathbb{R}^{n+k} \\ NX \xrightarrow{\cong} \underline{\mathbb{R}^k} \end{matrix} \right\} \Big/ \text{cobordism}$$

Basic example $\mathbb{R}^n \to \mathbb{R}^n$



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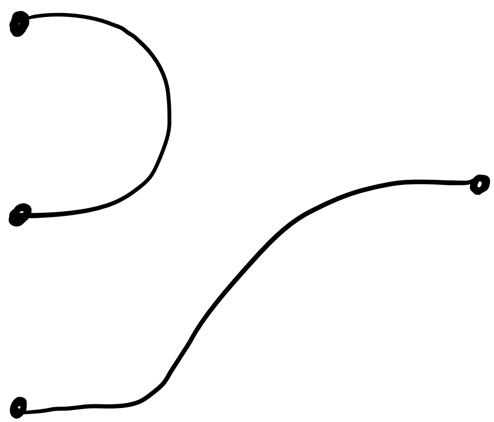

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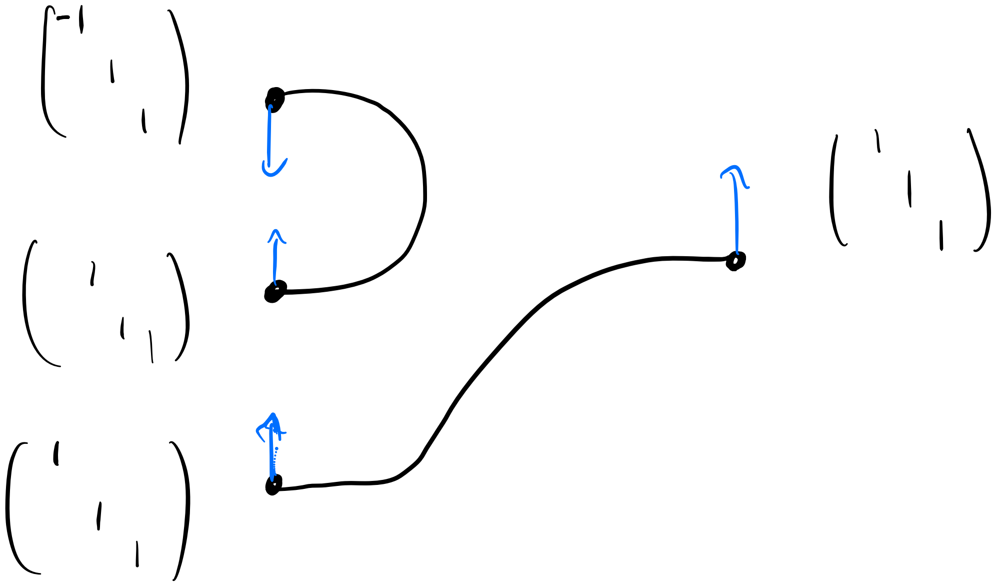

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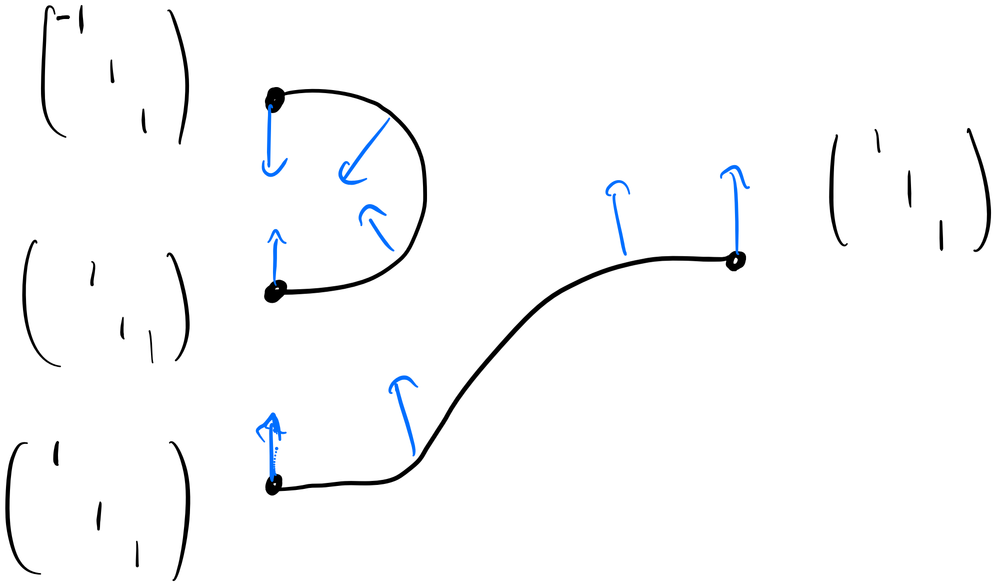

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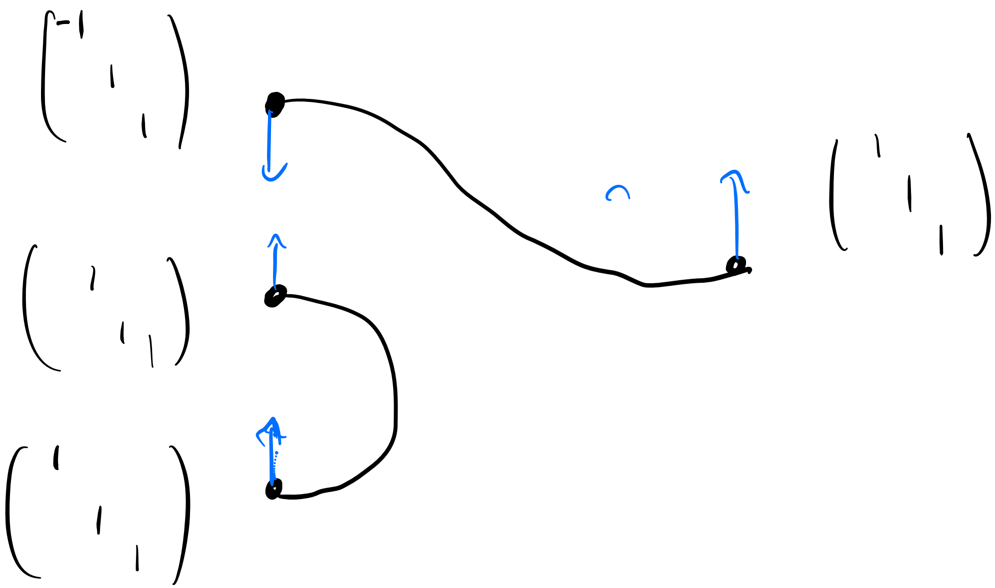

This does <u>not</u> work.

$$[\mathbb{R}^{n+k}, \mathbb{R}^{k}]^{prop} \xrightarrow{\cong} \Omega_n^{fr}(\mathbb{R}^{n+k}) = \left\{ \begin{array}{l} X \subset \mathbb{R}^{n+k} \\ A : X \longrightarrow \mathrm{Hom}(\mathbb{R}^{n+k}, \mathbb{R}^{k}) \\ \ker A_x = TX \end{array} \right. \Big/ \text{cobordism}$$

$$\begin{tikzcd}
{[\mathbb{R}^{n+k}, \mathbb{R}^k]^{prop}} \arrow[r, "\cong"] \arrow[d, "S"'] & \Omega_n^{fr}(\mathbb{R}^{n+k}) = \left\{ \begin{array}{l} X \subset \mathbb{R}^{n+k} \\ A : X \longrightarrow \mathrm{Hom}(\mathbb{R}^{n+k}, \mathbb{R}^k) \\ \ker A_x = T_x X \end{array} \right. \Big/ \text{cobordism} \\
F_n^{prop}[\mathbb{H}] &
\end{tikzcd}$$

$$
\begin{tikzcd}
{[\mathbb{R}^{n+k}, \mathbb{R}^k]^{prop}} \arrow[r, "\cong"] \arrow[d, "S"'] & \Omega^{fr}_n(\mathbb{R}^{n+k}) \arrow[d, "S"] \\
F^{prop}_n[\mathbb{H}] \arrow[r, "\cong"', "\text{Abbondandolo-R.}"'] & \Omega^{fr}_n(\mathbb{H})
\end{tikzcd}
$$

$$
\Omega^{fr}_n(\mathbb{R}^{n+k}) = \left\{ \begin{array}{l} X \subset \mathbb{R}^{n+k} \\ A : X \longrightarrow \mathrm{Hom}(\mathbb{R}^{n+k}, \mathbb{R}^k) \\ \mathrm{Ker}\, A_x = T_x X \end{array} \right. \Big/ \text{cobordism}
$$

$$
\Omega^{fr}_n(\mathbb{H}) = \left\{ \begin{array}{l} X \subset \mathbb{H} \\ A : \mathbb{H} \longrightarrow \Phi_n(\mathbb{H}) \\ \mathrm{Ker}\, A_x = T_x X \end{array} \right. \Big/ \text{cobordism.}
$$

$$
\begin{tikzcd}
{[\mathbb{R}^{n+k}, \mathbb{R}^k]^{prop}} \arrow[r, "\cong"] \arrow[d, "S"'] & \Omega_n^{fr}(\mathbb{R}^{n+k}) \arrow[d, "S"] \\
F_n^{prop}[\mathbb{H}] \arrow[r, "\cong"', "\text{Abbondandolo-R.}"' yshift=-12pt] & \Omega_n^{fr}(\mathbb{H})
\end{tikzcd}
$$

$$
\Omega_n^{fr}(\mathbb{R}^{n+k}) = \left\{ \begin{array}{l} X \subset \mathbb{R}^{n+k} \\ A : X \longrightarrow \mathrm{Hom}(\mathbb{R}^{n+k}, \mathbb{R}^k) \\ \mathrm{Ker}\, A_x = T_x X \end{array} \right. \Big/ \text{cobordism}
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$$

- $\mathrm{Hom}(\mathbb{R}^{n+k}, \mathbb{R}^k)$ contractible.
- $GL(\mathbb{R}^k)$ non-trivial topology

$$
\begin{tikzcd}
{[\mathbb{R}^{n+k}, \mathbb{R}^k]^{prop}} \arrow[r, "\cong"] \arrow[d, hook, "S"'] & \Omega_n^{fr}(\mathbb{R}^{n+k}) \arrow[d, hook, "S"] \\
F_n^{prop}[\mathbb{H}] \arrow[r, "\cong"'] & \Omega_n^{fr}(\mathbb{H})
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$$

Abbondandolo-R.

$$\Omega_n^{fr}(\mathbb{R}^{n+k}) = \left\{ \begin{array}{l} X \subset \mathbb{R}^{n+k} \\ A : X \longrightarrow \mathrm{Hom}(\mathbb{R}^{n+k}, \mathbb{R}^k) \\ \mathrm{Ker}\, A_x = T_x X \end{array} \right. \Big/ \text{cobordism}$$

$$\Omega_n^{fr}(\mathbb{H}) = \left\{ \begin{array}{l} X \subset \mathbb{H} \\ A : \mathbb{H} \longrightarrow \Phi_n(\mathbb{H}) \\ \mathrm{Ker}\, A_x = T_x X \end{array} \right. \Big/ \text{cobordism.}$$

- $\mathrm{Hom}(\mathbb{R}^{n+k}, \mathbb{R}^k)$ contractible.
- $GL(\mathbb{R}^k)$ non-trivial topology
- $\Phi_n(\mathbb{H}) \simeq BO$ non trivial topology
- $GL(\mathbb{H})$ contractible
- need framing defined on $\mathbb{H}$.

$$
\begin{tikzcd}
{[\mathbb{R}^{n+k}, \mathbb{R}^k]^{prop}} \arrow[r, "\cong"] \arrow[d, "S"'] & \Omega^{fr}_n(\mathbb{R}^{n+k}) \arrow[d, "S"] \\
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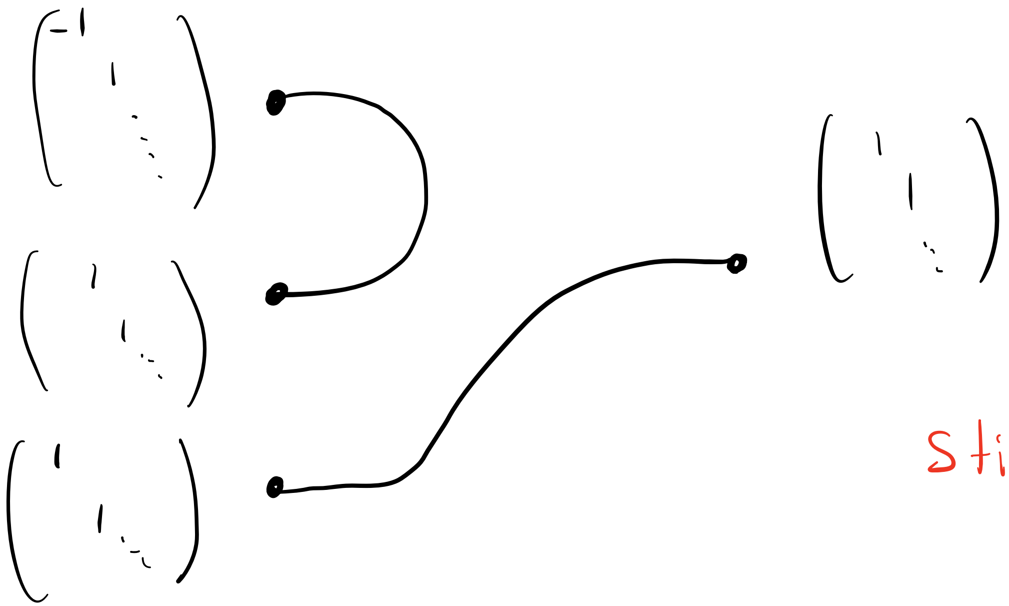
Equivalent definition

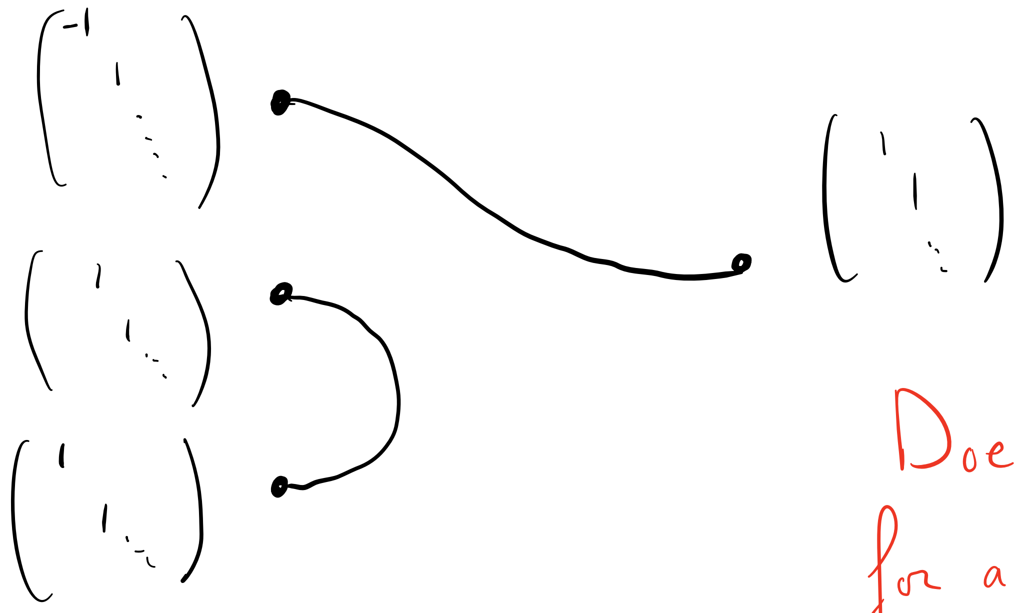
$$
\Omega^{fr}_n(\mathbb{R}^{n+k}) = \left\{ \begin{array}{l} X \subset \mathbb{R}^{n+k} \\ A : \mathbb{R}^{n+k} \longrightarrow \mathrm{Hom}(\mathbb{R}^{n+k}, \mathbb{R}^k) \\ \operatorname{Ker} A_x = T_x X \end{array} \right. \Big/ \text{cobordism}
$$

The suspended version $Sf : \mathbb{H} \to \mathbb{H}$



Still works

The suspended version $Sf : \mathbb{H} \longrightarrow \mathbb{H}$

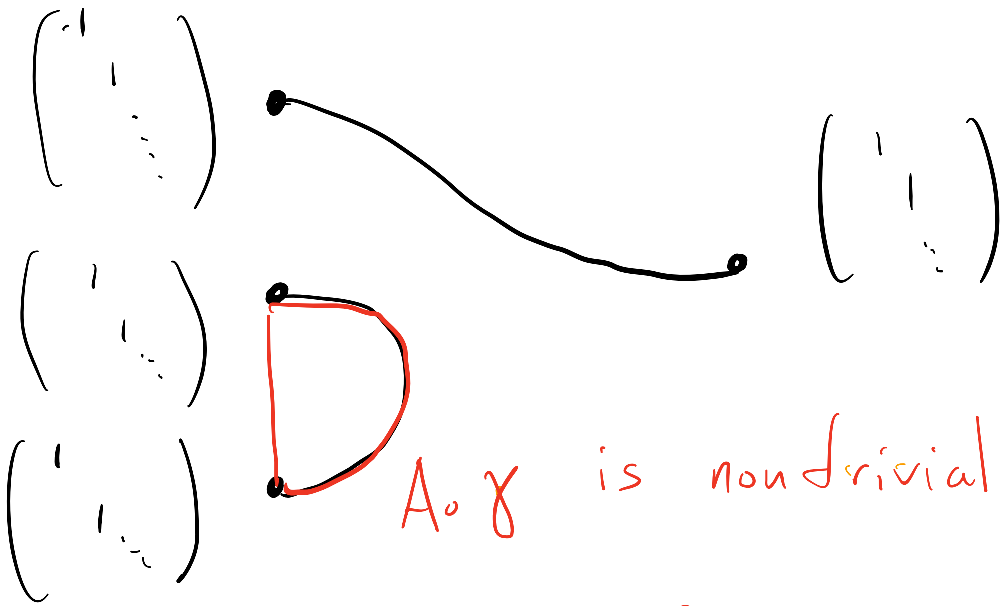


Does not work for a different reason.

as

$$\begin{pmatrix} -1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & \end{pmatrix} \sim \begin{pmatrix} 1 & & \\ & 1 & \\ & & \ddots \end{pmatrix}$$

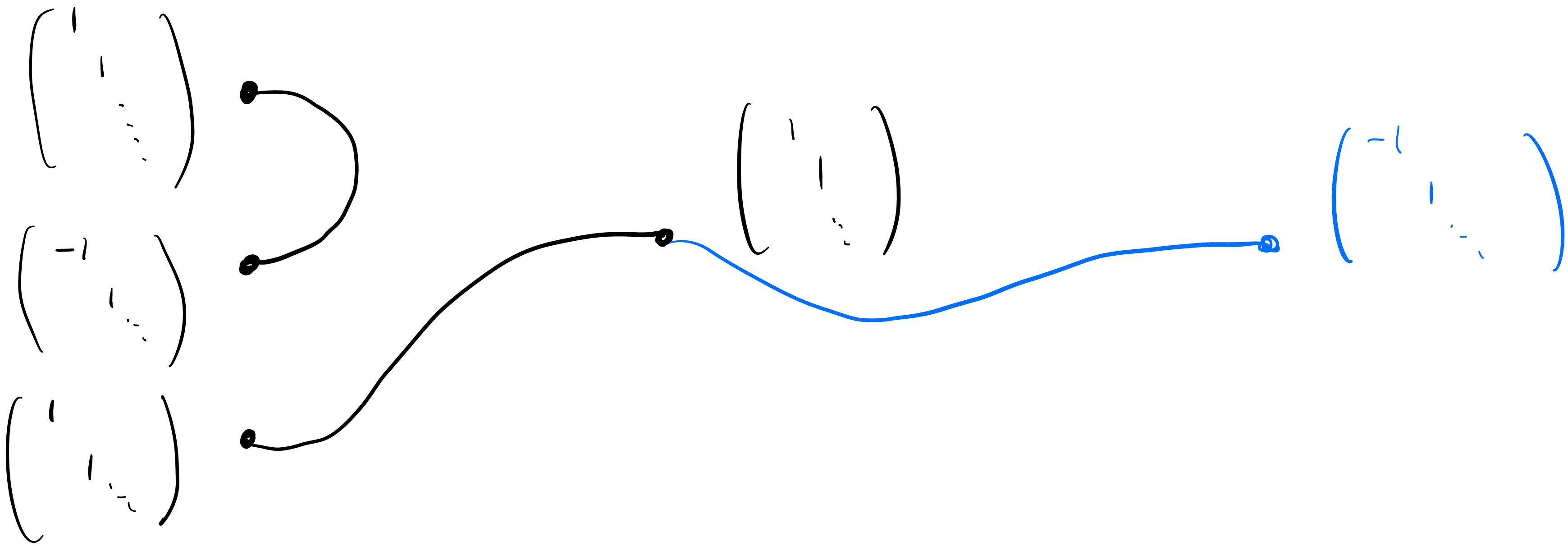
The suspended version $Sf : \mathbb{H} \longrightarrow \mathbb{H}$



$A_0 \gamma$ is nontrivial loop in $\pi_1(\Phi(\mathbb{H}))$

can't fill in.

The suspended version $Sf : \mathbb{H} \longrightarrow \mathbb{H}$



Thus degree k map equals degree $-k$ map

$$F_0^{prop}[\mathbb{H}] = \mathbb{Z}/_{x \sim -x} \cong \mathbb{N}_0$$

The specialstructure of a suspended framed submanifold.

The special structure of a suspended framed submanifold.

- $X \subset \mathbb{R}^{n+k} \subset \mathbb{H}$ (first $n+k$ coordinates) embedded in

The special structure of a suspended framed submanifold.

- $X \subset \mathbb{R}^{n+k} \subset \mathbb{H}$

- framing $A : \mathbb{H} \longrightarrow \Phi_n(\mathbb{H})$ has the form

$$A = \begin{array}{c c} & \begin{array}{cc} \mathbb{R}^{n+k} & (\mathbb{R}^{n+k})^{\perp} \end{array} \\ \begin{array}{c} \mathbb{R}^{k} \\ \\ (\mathbb{R}^{k})^{\perp} \end{array} & \begin{pmatrix} \tilde{A} & 0 \\ & \\ 0 & S_n \end{pmatrix} \end{array}$$

The special structure of a suspended framed submanifold.

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$$

$\tilde{A} \in \mathrm{Hom}(\mathbb{R}^{n+k}, \mathbb{R}^{k})$

$S_n(x_1, x_2, \dots) = (x_n, x_{n+1}, \dots)$

The special structure of a suspended framed submanifold.

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$$

only depends on first nth coordinates

$\Downarrow$

$$\tilde{A} \in \mathrm{Hom}(\mathbb{R}^{n+k}, \mathbb{R}^{k})$$

$$S_n(x_1, x_2, \dots) = (x_n, x_{n+1}, \dots)$$

left n-shift

The special structure of a suspended framed submanifold.

- $X \subset \mathbb{R}^{n+k} \subset \mathbb{H}$
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$\tilde{A} \in \mathrm{Hom}(\mathbb{R}^{n+k}, \mathbb{R}^{k})$

$S_n(x_1, x_2, \dots) = (x_n, x_{n+1}, \dots)$

Define Ω_s^{fr} as the set of these up to cobordism

The surjectivity proof

Thm (Toussaint-R)

$$S : \pi_n^S \longrightarrow \mathcal{F}_n^{prop}[\mathbb{H}]$$

is surjective

The surjectivity proof

Thm (Toussaint-R)

$$S : \Pi_n^S \longrightarrow F_n^{prop}[\mathbb{H}] \iff S : \Omega_S^{fr} \longrightarrow \Omega^{fr}(\mathbb{H})$$

is surjective $\qquad\qquad\qquad$ is surjective

The surjectivity proof

Thm (Toussaint-R)

$$S : \pi_n^S \longrightarrow F_n^{prop}[\mathbb{H}] \iff S : \Omega_S^{fr} \longrightarrow \Omega^{fr}(\mathbb{H})$$

is surjective $\qquad\qquad\qquad$ is surjective

Step 1 manifolds up to cobordism in $\mathbb{R}^\ell$ $\iff$ manifolds up to cobordism in $\mathbb{H}$.
ℓ suf

The surjectivity proof

Thm (Toussaint-R)

$$S : \pi_n^S \longrightarrow F_n^{prop}[\mathbb{H}] \iff S : \Omega^{fr}_S \longrightarrow \Omega^{f.}(\mathbb{H})$$

is surjective $\qquad\qquad$ is surjective

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$W = I \times X$

$B = A_t$

while preserving $\operatorname{Ker} A|_X = TX$

Linear algebra

Let $A \in \Phi(\mathbb{H})$

$\exists$ $(n+k)$, $V = (\mathbb{R}^{n+k})^{\perp}$ s.t.

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$GL(\mathbb{H})$ is connected so there exists path τ_t from $\mathrm{id}_{\mathbb{H}}$ to τ.

hence $A \sim \tau A$ while preserving $\operatorname{Ker} A$

<u>Then</u>

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$GL(V,(\mathbb{R}^k)^\perp)$ is connected (contractible)

hence there is a path D_t

$D \sim S_n$

$$\begin{pmatrix} B & 0 \\ (1-t)C & D_t \end{pmatrix} \longrightarrow \quad TA \sim \begin{pmatrix} B & 0 \\ 0 & S_n \end{pmatrix}$$

Crucial property: if $\ker A \subset \mathbb{R}^{n+k}$ then

$$\ker A_t = \ker A \quad \forall t$$

Pointwise we can solve the problem.

Now: parametrically

K-theory to the rescue!

Let $A : M \longrightarrow \Phi(\mathbb{H})$ M reasonable.

$\exists$ $(n+k)$, $V = (\mathbb{R}^{n+k})^\perp$ s.t.

- $\ker A \cap V = 0$
- $\operatorname{coker}_V A := (A(V))^\perp$ is a k dim vector bundle.

$\left(\text{Atiyah-Jänich: } [\mathbb{R}^{n+k}] - [\operatorname{coker}_V A] \in K(M) \text{ is well defined} \right.$
$\left. \Phi(\mathbb{H}) \text{ is a classifying space for } K \right)$

This thus works parametrically

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This proves surjectivity.

Failure of injectivity

let (X_0, A_0), (X_1, A_1) finite dimensionally framed submflds

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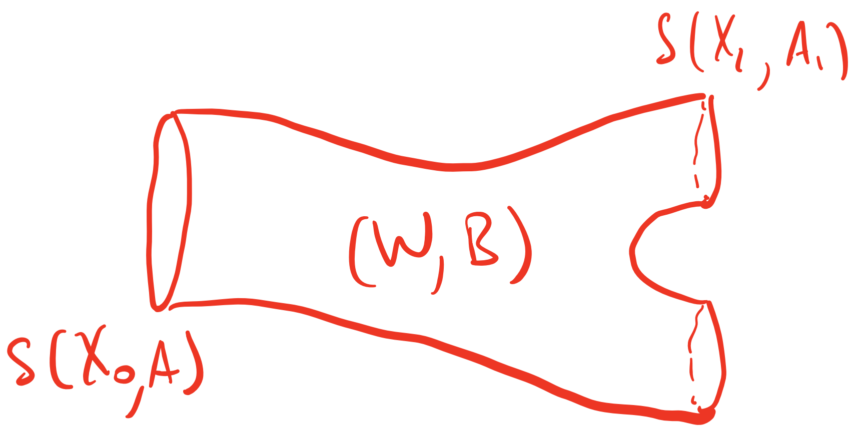

in $\mathbb{H}$

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$$\rightsquigarrow \quad \begin{array}{c} \widetilde{\operatorname{coker}_V B} \\ \downarrow \\ \mathbb{H}\times S^1 \end{array} = \operatorname{coker}_V B \Big/ (x,0)\sim(x,1)$$

$$\mathrm{Vect}_{\mathbb{R}}(\mathbb{H} \times S^1) \cong \mathrm{Vect}_{\mathbb{R}}(S^1) \cong \mathbb{Z}/_{2\mathbb{Z}}$$

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This describes failure of injectivity

Work in progress (speculative)

fin dim manifold.

Goal: Classify $\mathcal{F}_n^{prop}[M, \mathbb{H}]$ $M \cong N \times \mathbb{H}$

What is the suspension operation now?

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Then $\qquad S_g : E \oplus \mathbb{H} \longrightarrow \mathbb{R}^k \oplus \mathbb{H}$

$$S_g(x, y) = (g(x), y)$$

is proper Fredholm

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quotient of twisted stable cohomotopy of N

" Vector bundles F over $N \times S^1$ st $i_t : N \to N \times S^1$ $i_t^* F \cong E$.

Thank you for your attention

Arxiv: "Nonlinear proper Fredholm maps
and the stable homotopy groups of spheres"