# Differential topology of symplectic toric manifolds

by

Harold Nieuwboer

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Supervisor: Dr. Thomas Rot

Second examiner: Dr. Federica Pasquotto

Department of Mathematics

Faculty of Science

VU UNIVERSITY AMSTERDAM

## Abstract

We study the topology of symplectic toric manifolds via properties of their associated moment map, whose image is a convex polytope, known as the moment polytope. The edges meeting at each corner of the moment polytope satisfy certain combinatorial properties, and there is a one-to-one correspondence between the polytopes satisfying these properties and symplectic toric manifolds up to equivalence. By applying Morse theory to the moment map, we recover the Betti numbers of the symplectic toric manifold. To determine the cohomological ring structure, we first compute the equivariant cohomology ring of the manifold, which also takes into account the given torus action, and then relate this to the ordinary cohomology ring. More precisely, we construct a collection of vector bundles, one for each corner of the polytope, whose equivariant Euler classes generate the equivariant cohomology, and use the combinatorial data on the edges to determine the ring structure.

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Author: Harold Nieuwboer,
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Department of Mathematics
VU University Amsterdam
de Boelelaan 1081, 1081 HV Amsterdam
`http://www.math.vu.nl/`

## Popular summary

Dynamical systems have been studied for a long time, and attract a lot of attention, given their widespread applications. A particularly nice class of dynamical systems is that of Hamiltonian dynamical systems, and symplectic toric manifolds are special examples of such Hamiltonian dynamical systems.

A natural example of a Hamiltonian system is a simple harmonic oscillator. Imagine a mass suspended from a spring; we are interested in the behaviour of the displacement of the mass from its equilibrium position, given a certain starting position. Hooke's law states that the force exerted by the spring is proportional to the displacement (for small displacements, at least), so by Newton's second law, the acceleration of the mass will be proportional to the displacement as well. We ignore friction in this situation, which is reasonable if the mass does not move too fast. A special property of this dynamical system is that its behaviour is *periodic*; if we start with a displacement of $x_0$ below the equilibrium position, then by the time the distance is $x_0$ *above* the equilibrium position, the mass will have come to rest again, and will start moving down again. One can show that if we consider at each time $t$ the pair $(x(t), v(t))$ consisting of the displacement $x(t)$ and the velocity $v(t)$ of the mass, this traces out an ellipse in the plane $\mathbb{R}^2$. These ellipses together form the entirety of $\mathbb{R}^2$, which is the *state space* for this dynamical system: each pair $(x, v)$ is a suitable state for our dynamical system (although our physical assumptions do not quite hold up for every state). A particularly interesting state is the rest state $(0, 0)$, which is the only state of our dynamical system which is stationary. Furthermore, each ellipse corresponds to a certain *energy level* (which in this case is the sum of the potential energy in the spring and the kinetic energy of the mass), and the rest state has minimal energy, namely 0.

Symplectic toric manifolds share many properties with the previous example: they are state spaces for multiple Hamiltonian dynamical systems, which are independent (think of two simple harmonic oscillators that do not interact with eachother), and also have periodic behaviour. These state spaces are interesting because they turn out to be completely described by their associated energy functions, also known as their *moment map*: the image of this moment map is a convex polytope in $\mathbb{R}^n$, where $n$ is the number of independent Hamiltonian systems under consideration. These polytopes turn out to have strong restrictions on the directions of the edges meeting at each corner, and polytopes satisfying these properties are actually in one-to-one correspondence with symplectic toric manifolds. A polytope is much easier to understand (and draw!) than a state space for a dynamical system. In this thesis, we relate properties of the polytope to properties of the state space. In particular, we compute the cohomology ring of the state space, which can roughly be considered to be the information determining the number of holes in the space, as well as the interactions between these holes. Generally, if one is given just a dynamical system, this cohomology ring is very difficult to compute, so there is a significant advantage to being able to do so from a concrete representation of the state space.

# 1. Introduction

Hamiltonian dynamical systems have been studied for a long time and arise naturally in many physical contexts, such as frictionless classical mechanical systems. Of particular importance is the phase space associated to such a dynamical system, which is a symplectic manifold. In this thesis, we investigate the topological properties of the phase spaces associated to completely integrable Hamiltonian systems with periodic orbits; that is, a collection of $n$ independent Hamiltonians on a (compact) $2n$-dimensional phase space, whose flows commute and are periodic. These phase spaces are known as symplectic toric manifolds. Natural examples of completely integrable systems include ($n$-dimensional) frictionless harmonic oscillators, but also the two-body problem. More complicated examples of symplectic toric manifolds are known to arise in the theory of mirror symmetry [10].

Symplectic toric manifolds are also special from a geometric viewpoint: they are symplectic manifolds with an effective torus action, almost all of whose orbits are Lagrangian tori, and thus symplectic manifolds with a high degree of homogeneity. Furthermore, they are classified by special combinatorial polytopes, and certain operations on symplectic manifolds, such as symplectically blowing up a point, correspond to a modification of the corresponding polytope. Therefore symplectic toric manifolds form a class of symplectic manifolds for which computing invariants is more tractable than for a general symplectic manifold.

As for prerequisites, we roughly assume knowledge of basic differential geometry and symplectic geometry, and familiarity with basic notions from algebraic topology, including characteristic classes of vector bundles and the Thom isomorphism theorem. Furthermore, we use Morse theory at various stages, but do explicitly give references whenever appropriate.

In Chapter 2, we first give a formal definition of symplectic toric manifolds, and give basic examples. The essential part of the definition concerns the so-called moment map, which serves as a collection of Hamiltonians for independent commuting dynamical systems. For a more extensive treatment of symplectic geometry and symplectic toric manifolds, we refer the reader to [6] and [4]. We then discuss the convexity theorem for the moment map, proven independently by Atiyah [1] and Guillemin and Sternberg [8], which states that the image of the moment map is a convex polytope, and is in fact the convex hull of the images of the fixed points of the torus action. This result was further strengthened by Delzant [7]: these convex polytopes satisfy certain combinatorial properties, and are in fact in one-to-one correspondence with symplectic toric manifolds (up to equivalence).

Given this classification, it is natural to ask whether there exist straightforward methods of computing topological invariants of the symplectic toric manifolds directly from their moment polytope. In Chapter 3, we compute the (co-)Betti numbers, by relating critical points of the Hamiltonian for a generic direction in the torus to the ranks of the cohomology groups via a standard Morse-theoretical argument.

In Chapter 4, we study the equivariant cohomology ring (with integer coefficients) of symplectic toric manifolds. Equivariant cohomology is a version of cohomology adapted to spaces endowed with a continuous group action. The primary difference between equivariant and ordinary cohomology is that the equivariant cohomology of a point is generally highly non-trivial, whereas the ordinary cohomology of a point is $\mathbb{Z}$. On the other hand, the equivariant cohomology of a single free orbit is trivial. Despite this difference, one can define characteristic classes for equivariant vector bundles; in particular, there exist non-trivial equivariant vector bundles over a point. We then proceed to use Morse theory, as in Chapter 3, to find a basis for the equivariant cohomology ring of symplectic toric manifolds, and we identify each basis element as the equivariant Euler class of an equivariant vector bundle over the symplectic toric manifold, with the main result being the following.

**Theorem** (4.2.5)**.** *Let $M$ be a symplectic toric manifold with moment map $\mu : M \to \mathfrak{t}^*$, and let $F$ be the set of fixed points of the action. Then for a choice of generic direction $X \in \mathfrak{t}$, write $A_v \subseteq M$ for the preimage under $\mu$ of the flow-up face of $v$, relative to $X$. Then for each $v \in F$ there exists an equivariant vector bundle $E_v \to M$ such that $E_v|_{A_v}$ is isomorphic to the normal bundle of $A_v$ in $M$. Furthermore, the inclusion-induced map $H_T^*(M) \to H_T^*(F)$ is injective, and the equivariant cohomology ring $H_T^*(M)$ is a free $H_T^*$-module on the equivariant Euler classes $e_T(E_v)$.*

The injectivity of $H_T^*(M) \to H_T^*(F)$ was already recognized by Atiyah and Bott [3], as was the fact that $H_T^*(M)$ is a free $H_T^*$-module. However, they do not provide an explicit basis for $H_T^*(M)$, although in [2] they show that in equivariant Morse theory, the Euler classes of negative normal bundles play a critical role in determining the rank of $H_T^*(M)$ in each degree; the negative normal bundles are in turn the restrictions of the normal bundles associated to the flow-up faces.

To explicitly construct the $E_v$, we use Delzant's construction of symplectic toric manifolds from their moment polytopes. This construction also allows us to compute the equivariant Euler classes of the bundles when restricted to the fixed point set of the torus action, as demonstrated by the following theorem.

**Theorem** (4.3.8)**.** *Let $v \in F$ be a fixed point and $E_v \to M$ be the extension of the normal bundle of preimage (under $\mu$) of the flow-up face $A_v \subseteq M$, as in Theorem 4.2.5. Then for any fixed point $v' \in F$, if we take $H_T^*(\{v'\}) = \mathbb{Z}[x_{vi} : i = 1, \ldots, n]$, we have*

$$e_T(E_v)|_{v'} = \begin{cases} 0 & \text{if } v' \notin A_v \\ \prod_{j=1}^{l} \sum_{i=1}^{n} (-u_{ji} x_{vi}) & \text{if } v' \in A_v \end{cases}$$

*where $-u_j \in \mathbb{Z}^n$, $j = 1, \ldots, l$ are the directions of the edges meeting at $\mu(v')$ that do not point into $A_v$, and $u_{ji}$ is the $i$-th coordinate.*

We now have generators for $H_T^*(M)$ as a $H_T^*$-module, and know their images along the injection $H_T^*(M) \to H_T^*(F)$. Moreover, in $H_T^*(F)$ it is easy to compute products, as it is a direct sum of polynomial rings; therefore we have now deduced the product structure of the equivariant cohomology ring. This may then be used to compute the product structure of the ordinary cohomology ring of the manifold, using the following theorem.

**Theorem** (4.2.7)**.** *Let $j : M \to M \times_T ET$ denote the inclusion. Then the map $j^*$ : $H_T^*(M) \to H^*(M)$ is surjective, and $H^*(M)$ is generated as an abelian group by the set of $e(E_v)$, $v \in F$. Furthermore, for any element $r \in H_T^*$ of degree at least 2, we have $j^*(re_T(E_v)) = 0$.*

This theorem effectively says that to go from the equivariant product structure to the product structure on ordinary cohomology, all we need to do is forget its module structure with respect to the equivariant coefficient ring. We have thus established that the cohomology ring of any symplectic toric manifold is (additively) generated by characteristic classes of a collection of vector bundles, one for each fixed point of the torus action. Although this result is not completely new (see Audin [4, Thm. VII.3.8]), our advantage is that we have identified an additive basis for $H^*(M)$ whose pairwise products are easy to compute, rather than providing a description of $H^*(M)$ as a quotient of a polynomial ring on degree-2 generators. Moreover, the proof given in Audin uses the de Rham model for equivariant cohomology, along with the Duistermaat–Heckman theorem [4, Thm. VI.2.3], whereas the proof we give relies more heavily on Morse theory.

# 2. Classification of symplectic toric manifolds

## 2.1. Introduction and definitions

In this section we give basic definitions related to symplectic toric manifolds, along with some examples.

**Definition 2.1.1.** The 1-torus is the Lie group $S^1 = T^1$, consisting of complex numbers $z \in \mathbb{C}$ with $|z| = 1$, and whose multiplication is given by multiplication of complex numbers. Alternatively we may construct $T^1$ as the quotient group $\mathbb{R}/(2\pi\mathbb{Z})$, and we identify $\mathbb{R}/(2\pi\mathbb{Z})$ with $S^1$ by sending the equivalence class of $[x]$ to $e^{ix}$.

If $\partial/\partial x$ denotes the canonical basis element of the tangent space $T_0\mathbb{R}$ of $\mathbb{R}$ at 0, then its pushforward under the projection map $\mathbb{R} \to \mathbb{R}/(2\pi\mathbb{Z})$ is a nonzero vector in $\mathfrak{t} = T_eT^1$, and is also denoted by $\partial/\partial x$. Thus $\partial/\partial x$ is chosen such that the exponential map $\exp : \mathfrak{t} \to T^1$ satisfies

$$\exp\left(t\frac{\partial}{\partial x}\right) = e^{it}.$$

The dual basis element is then denoted by $dx \in \mathfrak{t}^*$, satisfying $dx(\partial/\partial x) = 1$. Similarly, for the $n$-torus $T^n = (T^1)^n$, its Lie algebra $\mathfrak{t}$ has a canonical basis $\partial/\partial x_1, \ldots, \partial/\partial x_n$, whose dual basis for $\mathfrak{t}^*$ is denoted by $dx_1, \ldots, dx_n$.

**Definition 2.1.2.** A closed symplectic manifold $(M^{2n}, \omega)$, endowed with an action of the $n$-torus $T^n$, is called a symplectic toric manifold if

- the torus action is effective (i.e., faithful), and
- the torus action is Hamiltonian. That is, there exists a $T^n$-invariant smooth map $\mu : M \to \mathfrak{t}^*$, satisfying
$$d\langle \mu, X\rangle = \omega(X^\#, \cdot)$$
for every $X \in \mathfrak{t}$. Here $\langle\cdot,\cdot\rangle$ is the natural pairing $\mathfrak{t}^* \times \mathfrak{t} \to \mathbb{R}$, and $X^\#$ is the vector field on $M$ defined as
$$X^\#(p) = \left.\frac{\partial}{\partial t}\right|_{t=0} (\exp(tX)\cdot p)$$
for any $p \in M$. The map $\mu$ is called a moment map, and serves as a collection of Hamiltonian functions for the vector fields $X^\#$.

We consider $T^n$ and $\mu$ to be part of the data, and so denote a symplectic toric manifold by a quadruple $(M, \omega, T^n, \mu)$.

**Example 2.1.3.** Our prototypical example for a torus action is as follows. We consider $\mathbb{C}^n$ for $n > 0$ with a $T^n$-action on it given by

$$(u_1, \dots, u_n) \cdot (z_1, \dots, z_n) = (u_1 z_1, \dots, u_n z_n)$$

with $(z_1, \dots, z_n) \in \mathbb{C}^n$ and $(u_1, \dots, u_n) \in T^n$. We endow $\mathbb{C}^n$ with its standard symplectic structure, defined as

$$\omega_0 = \frac{i}{2} \sum_{j=1}^{n} dz_j \wedge d\overline{z}_j = \sum_{j=1}^{n} dx_j \wedge dy_j$$

where $z_j = x_j + iy_j$. With respect to this symplectic form, the torus action is Hamiltonian, with a moment map given by

$$\mu : \mathbb{C}^n \to \mathfrak{t}^*, \quad \mu(z_1, \dots, z_n) = -\frac{1}{2}(|z_1|^2, \dots, |z_n|^2).$$

The latter term $(|z_1|^2, \dots, |z_n|^2)$ is with respect to our standard basis for the Lie coalgebra $\mathfrak{t}^*$ of the torus. We check that this is indeed satisfies our definition for a moment map. It is clearly $T^n$-invariant. Now let $X = (X_1, \dots, X_n) \in \mathfrak{t}$; then $\exp(tX) = (e^{itX_1}, \dots, e^{itX_n}) \in T^n$, so

$$\begin{aligned} X^{\#}(z_1, \dots, z_n) &= \frac{\partial}{\partial t}\bigg|_{t=0} (\exp(tX) \cdot (z_1, \dots, z_n)) \\ &= \frac{\partial}{\partial t}\bigg|_{t=0} (e^{itX_1} z_1, \dots, e^{itX_n} z_n). \end{aligned}$$

To compute these derivatives, note that

$$\begin{aligned} &\frac{\partial}{\partial t}\bigg|_{t=0} e^{itX_1}(x_1 + iy_1) \\ &= \frac{\partial}{\partial t}\bigg|_{t=0} ((\cos(tX_1)x_1 - \sin(tX_1)y_1) + i(\cos(tX_1)y_1 + \sin(tX_1)x_1)) \\ &= \left( \frac{\partial}{\partial t}\bigg|_{t=0} \cos(tX_1)x_1 - \sin(tX_1)y_1 \right) \frac{\partial}{\partial x_1} \\ &+ \left( \frac{\partial}{\partial t}\bigg|_{t=0} \cos(tX_1)y_1 + \sin(tX_1)x_1 \right) \frac{\partial}{\partial y_1} \\ &= -X_1 y_1 \frac{\partial}{\partial x_1} + X_1 x_1 \frac{\partial}{\partial y_1}. \end{aligned}$$

Therefore

$$\omega(X^{\#}, \cdot) = \sum_{j=1}^{n} X_j(-y_j dy_j - x_j dx_j).$$

On the other hand, we have

$$d\langle \mu, X\rangle = d\left(-\frac{1}{2}\sum_{j=1}^{n} X_j|z_j|^2\right) = -\frac{1}{2}d\left(\sum_{j=1}^{n} X_j(x_j^2+y_j^2)\right)$$
$$= -\sum_{j=1}^{n} X_j(x_j dx_j + y_j dy_j).$$

Therefore $d\langle \mu, X\rangle = \omega(X^{\#}, \cdot)$, and we conclude that the torus action is Hamiltonian. Note that $(\mathbb{C}^n, \omega_0)$ with this torus action does not form a symplectic toric manifold, because $\mathbb{C}^n$ is not closed.

**Example 2.1.4** (Complex projective space)**.** The prototypical example for a symplectic toric manifold is as follows. Let $\mathbb{CP}^1$ be complex projective space, with an $S^1 = T^1$-action given by

$$u \cdot [z_0 : z_1] = [z_0 : uz_1]$$

for $u \in S^1 \subset \mathbb{C}$. Clearly $T^1$ acts effectively on $\mathbb{CP}^1$, as every $u \in S^1$ acts on $[1 : 1]$ as $[1 : u]$, which is $[1 : 1]$ if and only if $u = 1$.

The standard symplectic form on $\mathbb{CP}^n$ is the Fubini–Study form $\omega_{FS}$, defined as follows. The Fubini–Study form $\widehat{\omega}_{FS}$ on $\mathbb{C}^{n+1} \setminus \{0\}$ is given by

$$\widehat{\omega}_{FS} = \frac{i}{2}\partial\overline{\partial}\log(|z|^2)$$

where $z = (z_0, \dots, z_n) \in \mathbb{C}^{n+1} \setminus \{0\}$. Here the operators $\partial$ and $\overline{\partial}$ refer to the Dolbeault operators. One can check that if $L_\lambda : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{C}^{n+1} \setminus \{0\}$ is defined by $z \mapsto \lambda z$ for $\lambda \in \mathbb{C}^*$, then $(L_\lambda)^*\widehat{\omega}_{FS} = \widehat{\omega}_{FS}$. This guarantees that $\widehat{\omega}_{FS}$ descends to a 2-form $\omega_{FS}$ on $\mathbb{CP}^n$, which one can check to be a symplectic form.

With respect to the Fubini–Study form, the torus action is also Hamiltonian, with moment map given by

$$\mu([z_0 : z_1]) = -\frac{1}{2} \cdot \frac{|z_1|^2}{|z_0|^2 + |z_1|^2}.$$

After identifying $\mathbb{CP}^1 \cong S^2$, this action can be viewed as rotating the sphere around its $z$-axis. The moment map under this identification becomes (up to scale) the height function on $S^2$, which is clearly invariant under the circle action.

This example easily generalizes to higher dimensions. Let $T^n$ act on $\mathbb{CP}^n$ by

$$(u_1, \dots, u_n) \cdot [z_0 : z_1 : \dots : z_n] = [z_0 : u_1 z_1 : \dots : u_n z_n].$$

This action is again effective, and Hamiltonian (with respect to the Fubini–Study form) with moment map given by

$$\mu([z_0 : z_1 : \dots : z_n]) = -\frac{1}{2}\left(\frac{|z_1|^2}{|z|^2}, \dots, \frac{|z_n|^2}{|z|^2}\right).$$

Observe now that the fixed points of this torus action are $[1 : 0 : \ldots : 0]$, $[0 : 1 : 0 : \ldots : 0]$, et cetera, and that the images of the fixed points under the moment map are $(0, \ldots, 0)$, $(-1/2, 0, \ldots, 0)$, $(0, \ldots, 0, -1/2)$ in $\mathfrak{t}^* \cong \mathbb{R}^n$. The image of $\mu$ is by inspection seen to be the convex subset of $\mathbb{R}^n$ spanned by the images of the fixed points of the action. Furthermore, each point in the image of $\mathbb{R}^n$ has its preimage consisting of exactly one orbit of the torus action. This turns out to be a general fact about symplectic toric manifolds.

**Theorem 2.1.5** (Atiyah[1], Guillemin–Sternberg[8] Convexity)**.** *Let $(M^{2n}, \omega)$ be a symplectic toric manifold with associated moment map $\mu : M \to \mathfrak{t}^*$, and let $F$ denote the set of fixed points of the $T^n$-action. Then $\mu(M)$ is the convex hull of $\mu(F)$, and each non-empty fiber $\mu^{-1}(p)$ for $p \in \mu(M)$ is connected.*

**Definition 2.1.6.** The image $\mu(M)$ is called the moment polytope of $(M, \omega, T, \mu)$.

**Example 2.1.7.** Any product of symplectic toric manifolds is again a symplectic toric manifold. More precisely, let $(M_i^{2n_i}, \omega_i, T_i^{n_i}, \mu_i)$, $i = 1, 2$ be symplectic toric manifolds. Define $M = M_1 \times M_2$, $\omega = p_1^*\omega_1 + p_2^*\omega_2$ where $p_i : M \to M_i$ is the projection, and let $T^{n_1+n_2} = T_1^{n_1} \times T_2^{n_2}$ act on $M$ via the product action. The associated moment map is then $\mu : M \to \mathfrak{t}^*$ defined by $\mu(p_1, p_2) = \mu_1(p_1) + \mu_2(p_2)$, where we identify $\mathfrak{t}^*$ with $\mathfrak{t}_1^* \oplus \mathfrak{t}_2^*$. The moment polytope of $M$ is then the product of the moment polytopes of $M_1$ and $M_2$.

## 2.2. Equivariant Darboux charts and the Delzant classification

Given the Atiyah–Guillemin–Sternberg convexity theorem, it becomes a natural question to ask how much information about $M$ and its torus action can be recovered from its moment polytope. The first step towards answering this is to formulate a notion of equivalence of symplectic toric manifolds with their moment maps. Since translations of the moment map are moment maps, and invertible integral linear transformations of $\mathfrak{t}^* \cong \mathbb{R}^n$ correspond to Lie group automorphisms of the associated torus, we make the following definition of weak equivalence.

**Definition 2.2.1.** Two symplectic toric manifolds $(M_i^{2n}, \omega_i, T_i^n, \mu_i)$, $i = 1, 2$, are called weakly equivalent if there exists a symplectomorphism $f : M_1 \to M_2$ and a Lie group isomorphism $g : T_1^n \to T_2^n$ such that

$$f(up) = g(u)f(p)$$

for $u \in T_1^n$ and $p \in M_1$, and $(d_e g)^* \circ \mu_2 \circ f$ and $\mu_1$ differ by a constant. Here $(d_e g)^* : \mathfrak{t}_2^* \to \mathfrak{t}_1^*$ is the dual of $d_e g : \mathfrak{t}_1 \to \mathfrak{t}_2$.

**Example 2.2.2.** The symplectic toric manifolds $(\mathbb{CP}^1 \times \mathbb{CP}^1, p_1^*\omega_{FS}^1 + p_2^*\omega_{FS}^2, T^1 \times T^1, \mu_1 \times \mu_2)$ and $(\mathbb{CP}^2, \omega_{FS}, T^2, \mu)$ are not weakly equivalent, because there is no diffeomorphism $\mathbb{CP}^1 \times \mathbb{CP}^1 \to \mathbb{CP}^2$.

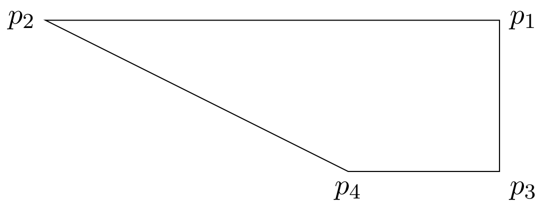


Figure 2.1.: The moment polytope for the $k$-th Hirzebruch surface $W_k$.

**Example 2.2.3** (Hirzebruch surface)**.** A more subtle example which incorporates the moment polytope is the following. Let $k > 0$ be an integer, and let $W_k$ be the subset of $\mathbb{CP}^1 \times \mathbb{CP}^2$ consisting of all points $([a : b], [x : y : z])$ such that $a^k y = b^k x$. This is a smooth complex hypersurface, as it is locally defined as a regular level set of a holomorphic function; for instance, on $\{a \neq 0, x \neq 0\}$, it is the zero set of $\frac{b^k}{a^k} - \frac{y}{x}$. Let $j : W \to \mathbb{CP}^1 \times \mathbb{CP}^2$ be the inclusion map, and set $\omega = j^* \omega_{FS}$ with $\omega_{FS}$ the Fubini–Study form on $\mathbb{CP}^1 \times \mathbb{CP}^2$. Note that $\omega$ is closed since $d\omega = j^*(d\omega_{FS}) = 0$. To see that $\omega$ is nondegenerate, let $g$ denote the Hermitian metric on $\mathbb{CP}^1 \times \mathbb{CP}^2$ associated to $\omega_{FS}$, and let $J$ be the almost complex structure on $\mathbb{CP}^1 \times \mathbb{CP}^2$. As $W_k$ is a complex submanifold, $J$ restricts to an almost complex structure on $W_k$. Now for $p \in W_k$ and $u \in T_p W_k$,

$$\omega(u, Ju) = \omega_{FS}(u, Ju) = g(Ju, Ju) > 0$$

so $\omega$ is nondegenerate on $W_k$.

Now $W_k$ with its projection onto the first coordinate is a $\mathbb{CP}^1$-bundle over $\mathbb{CP}^1$, and is known as a Hirzebruch surface[1]. Let $T^2$ act on $W_k$ by defining

$$(u, v) \cdot ([a : b], [x : y : z]) = ([ua : b], [u^k x : y : vz]).$$

This is a Hamiltonian action with moment map

$$\begin{aligned} \mu_k : W_k &\to \mathfrak{t}^* \cong \mathbb{R}^2 \\ ([a : b], [x : y : z]) &\mapsto -\frac{1}{2}\left( \frac{|a|^2}{|a|^2 + |b|^2} + k\frac{|x|^2}{|x|^2 + |y|^2 + |z|^2}, \frac{|z|^2}{|x|^2 + |y|^2 + |z|^2} \right). \end{aligned}$$

The fixed points of the action are $v_1 = ([0 : 1], [0 : 1 : 0])$, $v_2 = ([1 : 0], [1 : 0 : 0])$, $v_3 = ([0 : 1], [0 : 0 : 1])$ and $v_4 = ([1 : 0], [0 : 0 : 1])$, and their images under the moment map are $p_1 = (0, 0)$, $p_2 = (-(k+1)/2, 0)$, $p_3 = (0, -1/2)$ and $p_4 = (-1/2, -1/2)$, respectively. Hence by the convexity theorem, $\mu_k(W_k)$ is the convex hull of $p_1, \ldots, p_4$, as depicted in Figure 2.1.

We now show that for $k, k' > 0$ distinct integers, $W_k$ and $W_{k'}$ are not weakly equivalent as symplectic toric manifolds. Suppose for contradiction that we have a symplectomorphism $f : W_k \to W_{k'}$ and a Lie group isomorphism $g : T^2 \to T^2$ with $f(up) = g(u)f(p)$

[1] There are only two $S^2$-bundles over $S^2$: they correspond to homotopy classes of maps $S^1 \to \mathrm{Diff}(S^2)$ by the clutching construction, and the latter deformation retracts onto $O(3)$ [15]. Therefore $S^2$-bundles over $S^2$ are in one-to-one correspondence with elements of $\pi_1(O(3)) \cong \mathbb{Z}/2\mathbb{Z}$, where we ignore the issue of basepoints.

for all $u \in T^2$, $p \in W_k$, and such that $(d_e g)^* \circ \mu_{k'} \circ f - \mu_k$ is a constant, say $a \in \mathfrak{t}^*$. Since $f$ preserves fixed points, $(d_e g)^* - a$ sends the corners of $\mu_{k'}(W_{k'})$ to the corners of $\mu_k(W_k)$, and hence also sends edges to edges. Now assume without loss of generality that $k > k'$, and let $e_1, e_2 \in (\mathbb{R}^2)^*$ be the standard basis. Then the norm of $(d_e g)^* e_1$ is at least one, and hence the edge from $p_1 = (0,0)$ to $p_2 = (-(k+1)/2, 0)$ is mapped to an edge of length at least $(k+1)/2$. However, $\mu_{k'}(W_{k'})$ only has edges of length $1/2$, $\sqrt{(k'/2)^2 + (1/2)^2}$ and $(k'+1)/2$, each of which is less than $(k+1)/2$, so we have reached a contradiction.

The previous example shows that we can show that two symplectic toric manifolds are distinct by looking at "global" information, namely their moment polytope. In the other direction, it turns out that near fixed points, the symplectic toric manifolds are indistinguishable, up to a reparameterization of the torus.

**Definition 2.2.4.** Let $u \in \mathbb{Z}^n$ be an integral vector. Then $T^1$ is said to act on $\mathbb{C}^n$ with weight vector $u$ if for each $z \in T^1$, we have

$$z \cdot (z_1, \ldots, z_n) = (z^{u_1} z_1, \ldots, z^{u_n} z_n),$$

where $u_i$ denotes the $i$-th coordinate of $u$. Similarly, if $u_1, \ldots, u_k \in \mathbb{Z}^n$ are integral vectors, then $T^k$ is said to act on $\mathbb{C}^n$ with weight vectors $u_1, \ldots, u_k$ if for each $j = 1, \ldots, k$, the $j$-th factor of $T^k$ acts on $\mathbb{C}^n$ with weight $u_j$.

**Theorem 2.2.5.** *Let $(M^{2n}, \omega, T^n, \mu)$ be a symplectic toric manifold, and let $(\mathbb{C}^n, \omega_0)$ be linear complex space with its standard symplectic form. Let $v \in M$ be a fixed point of the torus action. Then there exist vectors $u_1, \ldots, u_n \in \mathbb{Z}^n$ and $T^n$-invariant neighbourhoods $U$ of $v \in M$ and $U_0$ of $0 \in \mathbb{C}^n$, and a $T^n$-equivariant symplectomorphism $f : (U, \omega) \to (U_0, \omega_0)$, where $T^n$ acts on $U_0$ with weights $u_1, \ldots, u_n$. Under this identification, $\mu$ takes the form*

$$\mu(z_1, \ldots, z_n) = \mu(v) - \frac{1}{2} \sum_{i=1}^{n} |z_i|^2 u_i$$

*where we now view the $u_i$ as vectors in $(\mathbb{R}^n)^* \cong \mathfrak{t}^*$.*

We do not prove this theorem in detail, although we do provide a sketch of its proof. A full proof can be found in [9, Section 32].

*Sketch of proof.* Since $T^n$ is compact, we can take a $T^n$-invariant Riemannian metric $g$ on $M$. The exponential map $\exp : T_v M \to M$ is then $T^n$-equivariant, where $T^n$ acts on $T_v M$ by differentiating the action on $M$ at $v$. Then there are $T^n$-invariant neighbourhoods $U_1 \ni v$ and $U_2 \ni 0$ such that $\exp : U_2 \to U_1$ is a diffeomorphism. The pullback $\omega_2 = \exp^*(\omega)$ is then a symplectic form on $U_2$, and we may define another symplectic form $\omega_{lin}$ on $T_v M$ by taking $\omega_{lin}|_0 = \omega_2|_0$, and asserting that $\omega_{lin}$ is translation-invariant. An equivariant version of Darboux's theorem can be proven with Moser's trick [9, Thm. 22.1], which then allows one to conclude that there exists a neighbourhood $U_3 \ni 0$ and a $T^n$-equivariant symplectomorphism $F : (U_3, \omega_{lin}) \to (U_2, \omega_2)$, shrinking $U_2$ if necessary. The final step is then to find an identification between $(U_3, \omega_{lin})$ and $(\mathbb{C}^n, \omega_0)$ and determining the vectors $u_1, \ldots, u_n$. Since $g$ is a $T^n$-invariant metric, $g_v$ is a $T^n$-invariant inner product on $T_vM$, which together with the symplectic form on $T_vM$ defines a complex structure and a $T^n$-invariant Hermitian form on $T_vM$, and hence we have a unitary representation $T^n \to U(T_vM)$. Since these linear operators commute, they can be simultaneously diagonalized, i.e., there exists a $\mathbb{C}$-linear basis $w_1, \ldots, w_n$ such that the image of each element of $T^n$ is a diagonal matrix. The weights $u_1, \ldots, u_n$ can then be recovered from this diagonal form. Since the complex structure is compatible with the symplectic structure, $w_1, iw_1, \ldots, w_n, iw_n$ is a symplectic basis for $T_vM$, so we get a symplectomorphism $(T_vM, \omega_{lin}) \to (\mathbb{C}^n, \omega_0)$, which is $T^n$-equivariant by definition of the action on $\mathbb{C}^n$. The local form for the moment map then follows from uniqueness. □

The local model is a key ingredient for the following complete classification, due to Delzant [7].

**Theorem 2.2.6** (Delzant classification). *Symplectic toric manifolds are classified by their moment polytopes: let $(M_i^{2n}, \omega_i, T^n, \mu_i)$, $i = 1, 2$ be two symplectic toric manifolds, such that* $\operatorname{im}\mu_1 = \operatorname{im}\mu_2 \in \mathfrak{t}^*$. *Then there exists a $T^n$-equivariant symplectomorphism $\varphi : M_1 \to M_2$ such that $\mu_2 = \mu_1 \circ \varphi$.*

*Furthermore, given a symplectic toric manifold $(M^{2n}, \omega, T^n, \mu)$, for each fixed point $v \in M$:*

1. *there are exactly $n$ edges meeting at $\mu(v)$,*
2. *each edge is of the form $\mu(v) - tu_i$ for $t \geq 0$, $u_i \in \mathbb{Z}^n$, and*
3. *$u_1, \ldots, u_n$ form a $\mathbb{Z}$-basis for $\mathbb{Z}^n$.*

*Any convex polytope satisfying the above conditions at each corner is called a Delzant polytope. Furthermore, for each Delzant polytope, one can construct a symplectic toric manifold with precisely this polytope as image.*

**Example 2.2.7.** Let $T^2$ act on $\mathbb{CP}^2$ as in Example 2.1.4, and let $\mu : \mathbb{CP}^2 \to \mathfrak{t}^*$ be the moment map. The fixed points $v_0 = [1 : 0 : 0]$, $v_1 = [0 : 1 : 0]$ and $v_2 = [0 : 0 : 1]$ are mapped to $p_0 = (0, 0)$, $p_1 = (-1/2, 0)$ and $p_2 = (0, -1/2)$ respectively, so the moment polytope is the convex hull of $p_0$, $p_1$ and $p_2$, as depicted in Figure 2.2. At $p_0$, the edges of the moment polytope are of the form $p_0 - t(1, 0)$ and $p_0 - t(0, 1)$ for $0 \leq t \leq 1/2$, and $(1, 0)$ and $(0, 1)$ form a $\mathbb{Z}$-basis for $\mathbb{Z}^n$. Similarly, at $p_1$, the edges of the moment polytope are of the form $p_1 - t(-1, 0)$ and $p_1 - t(-1, 1)$, and $(-1, 0)$ and $(-1, 1)$ also form a $\mathbb{Z}$-basis for $\mathbb{Z}^n$.

**Example 2.2.8.** Let $k > 0$ be an integer and let $W_k$ be the $k$-th Hirzebruch surface as in Example 2.2.3. The moment polytope is the convex hull of the points $p_1 = (0, 0)$, $p_2 = (-(k + 1)/2, 0)$, $p_3 = (0, -1/2)$ and $p_4 = (-1/2, -1/2)$, as depicted in Figure 2.1. At $p_2$, the edges are of the form $p_2 - t(-1, 0)$ and $p_2 - t(-k, 1)$, and $(-1, 0)$ and $(-k, 1)$ form a $\mathbb{Z}$-basis for $\mathbb{Z}$.

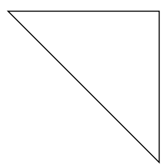


Figure 2.2.: The moment polytope for the complex projective plane $\mathbb{CP}^2$.

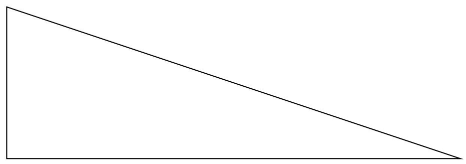


Figure 2.3.: An example of a polytope in $(\mathbb{R}^2)^*$ that is not a Delzant polytope. Here, $k > 1$ is an integer. The polytope is not a Delzant polytope because the directions of the edges at $(0, 1)$ and $(k, 0)$ do not form a $\mathbb{Z}$-basis for $\mathbb{Z}^n$.

**Example 2.2.9.** An example of a non-Delzant polytope is the convex hull of $(0, 0)$, $(0, 1)$ and $(k, 0)$ for integral $k > 1$, as shown in Figure 2.3: at $(0, 1)$, the edges will be of the form $(0, 1) - t(-k, 1)$ and $(0, 1) - t(0, 1)$. However, $(-k, 1)$ and $(0, 1)$ do not form a $\mathbb{Z}$-basis for $\mathbb{Z}^n$.

**Example 2.2.10.** An example of a polytope in $(\mathbb{R}^3)^*$ which is not Delzant is a pyramid with a square base, that is, the convex hull of $(0, 0, 0)$, $(1, 0, 0)$, $(0, 1, 0)$, $(1, 1, 0)$ and $(1/2, 1/2, 1/2)$. Every edge in this polytope has integral direction, but there are 4 edges meeting at $(1/2, 1/2, 1/2)$.

In Section 4.3, we will use the explicit construction of the symplectic toric manifolds to build vector bundles with special properties. Note that the Delzant classification the above classification gives also gives a one-to-one correspondence between $2n$-dimensional symplectic toric manifolds, up to weak equivalence, and Delzant polytopes in $(\mathbb{R}^n)^*$, up to invertible integral linear transformation and translation.

# 3. Morse theory on symplectic toric manifolds

We recall some of the basic definitions and properties of Morse theory that will be relevant later on. For proofs and more detailed statements, we refer the reader to [13]. Let $M^n$ be a smooth manifold of dimension $n$.

**Definition 3.1** ([13, §2])**.** Let $f : M \to \mathbb{R}$ be a smooth function. A critical point $x \in M$ of $f$ is called nondegenerate if in local coordinates $x_1, \ldots, x_n$ around $x$, the Hessian

$$H_x(f) = \left( \frac{\partial^2 f}{\partial x_i \partial x_j} \right)_{1 \le i,j \le n}$$

is invertible. If $x$ is nondegenerate, its index $\lambda$ is defined as the number of negative eigenvalues of $H_x(f)$, which does not depend on the choice of coordinates. If each critical point of $f$ is nondegenerate, then $f$ is called a Morse function.

**Lemma 3.2** (Morse Lemma, [13, §2.2])**.** *Let $f : M \to \mathbb{R}$ be a smooth function and $x \in M$ a nondegenerate critical point of $f$ with index $\lambda$. Then there exist local coordinates $x_1, \ldots, x_n$ centered at $x$ (so $(0, \ldots, 0) = x$) such that*

$$f(x_1, \ldots, x_n) = f(x) - x_1^2 - \cdots - x_\lambda^2 + x_{\lambda+1}^2 + \cdots + x_n^2.$$

The following two theorems then allow one to determine the homotopy type of a (compact) smooth manifold.

**Theorem 3.3** ([13, §3.1])**.** *Let $f : M \to \mathbb{R}$ be a smooth function, and for $c \in \mathbb{R}$ define the sublevel set $\overline{M}^c = f^{-1}((-\infty, c])$. Assume we have $a, b \in \mathbb{R}$, $a < b$, such that $f^{-1}([a, b])$ is compact and does not contain any critical points of $f$. Then the sublevel sets $\overline{M}^a$ and $\overline{M}^b$ are diffeomorphic, and $\overline{M}^a$ is a deformation retract of $\overline{M}^b$.*

**Theorem 3.4** ([13, §3.2, §3.4])**.** *Let $f : M \to \mathbb{R}$ be a smooth function, $x \in M$ a nondegenerate critical point of $f$ with index $\lambda$. Set $f(p) = c$, let $\epsilon > 0$ and assume $f^{-1}[c - \epsilon, c + \epsilon]$ is compact and does not contain any other critical points of $f$. Then for all $\epsilon$ sufficiently small, $M^{c+\epsilon}$ has the homotopy type of $M^{c-\epsilon}$ with a $\lambda$-cell attached. Furthermore, $M^c$ is a deformation retract of $M^{c+\epsilon}$.*

From here onwards, let $(M^{2n}, \omega, T^n, \mu)$ be a fixed symplectic toric manifold. Let $X \in \mathfrak{t}$ be a generic direction so that $\{\exp(tX) : t \in \mathbb{R}\}$ is a dense subgroup of $T^n$. Define $f : M \to \mathbb{R}$ by $f(p) = \langle \mu(p), X \rangle$, where we recall that $\langle \cdot, \cdot \rangle$ denotes the natural pairing $\mathfrak{t}^* \times \mathfrak{t} \to \mathbb{R}$.

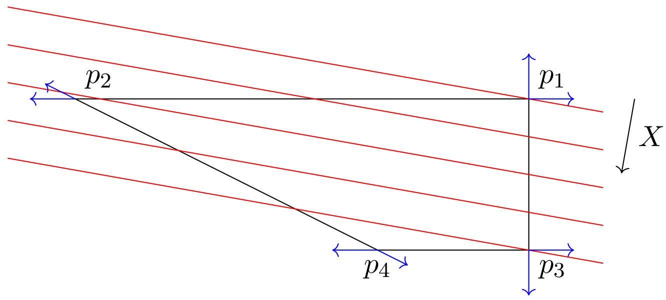


Figure 3.1.: The moment polytope corresponding to the $k$-th Hirzebruch surface $W_k$. The red lines indicate the level sets of the pairing $\langle \cdot, X\rangle : (\mathbb{R}^2)^* \to \mathbb{R}$, where $X$ is the vector as depicted. The blue arrows at each corner $p_i$ indicate the direction of the vectors $u \in \mathbb{Z}^2$ such that the edges at $p_i$ are of the form $p_i - tu$, with $t \geq 0$.

**Theorem 3.5.** *The function $f = \langle \mu, X\rangle$ is a Morse function, whose critical points are precisely the fixed points of the torus action. Furthermore, each critical point $v \in M$ has even index: if the edges meeting at $\mu(v)$ of the form $\mu(v) - tu_i$ for $t \geq 0$ and $u_i \in \mathbb{Z}^n$ for $i = 1, \dots, n$, then the index of $v$ is determined as twice the number of $u_i$ with $\langle u_i, X\rangle > 0$.*

**Definition 3.6.** Let $u_i \in \mathbb{Z}^n$ be as in Theorem 3.5. Then the edge with direction $u_i$ is called a negative edge, relative to $X$, if $\langle u_i, X\rangle > 0$. Similarly, if $\langle u_i, X\rangle < 0$, then the edge with direction $u_i$ is referred to as a positive edge, relative to $X$.

**Example 3.7** (Hirzebruch surfaces)**.** Let $k > 0$ be an integer and let $W_k$ be the $k$-th Hirzebruch surface, cf. Example 2.2.3. The moment polytope for $W_k$ is the convex hull of $p_1 = (0, 0)$, $p_2 = (-(k + 1)/2, 0)$, $p_3 = (0, -1/2)$ and $p_4 = (-1/2, -1/2)$, as shown in Figure 3.1. Let $X$ be a small clockwise rotation of $(-1, 0) \in \mathbb{R}^2$; then the red lines indicate the level sets of $\langle \cdot, X\rangle : (\mathbb{R}^2)^* \to \mathbb{R}$. At each corner $p_i$, the blue arrows indicate (the directions of) the $u_1, u_2$ associated to that corner; for instance, at $p_4$, the arrows point in the direction of $(k, -1)$ and $(-1, 0)$. Theorem 3.5 says that the index at the critical point corresponding to each $p_i$ is twice the number of blue arrows along which the evaluation with $X$ would increase; so for $p_1, \dots, p_4$, the indices are 0, 2, 2 and 4, respectively.

*Proof of Theorem 3.5.* First observe that $\omega(X^\#, \cdot) = d\langle \mu, X\rangle = df$ by the definition of the moment map. Therefore, for each $p \in M$, $df_p$ is zero only when $\omega(X^\#(p), \cdot)$ is zero, which is true if and only if $X^\#(p)$ is zero, by nondegeneracy of the symplectic form.

Now assume $v$ is a critical point of $f$. We show that $v$ must be a fixed point of the action. Note that since $X^\#(v) = 0$, for each $s \in \mathbb{R}$ we also have

$$\begin{aligned} X^\#(\exp(sX)v) &= \frac{\partial}{\partial t}(\exp(tX)\exp(sX)v) = \frac{\partial}{\partial t}(\exp(sX)\exp(tX)v) \\ &= d_v(\exp(sX))X^\#(v) = 0, \end{aligned}$$

where we view $\exp(sX)$ as a map $M \to M$ by left-multiplication. Therefore the map $\mathbb{R} \to M$, $s \mapsto \exp(sX)v$ has zero derivative everywhere and must be constant. In particular, as $\exp(0X)v = v$, $\exp(tX)v = v$ for all $t \in \mathbb{R}$. By density of the subgroup generated by $X$ in $T^n$, we conclude that every $g \in T^n$ has $gv = v$. The other direction is simpler: if $v$ is a fixed point of the torus action, then $\exp(tX)v = v$ is the constant curve, and so $X^{\#}(v) = 0$. Now it remains to check that each critical point is nondegenerate, and of even degree. Fix a critical point $v \in M$ of $f$, and let the edges meeting at $\mu(v)$ be of the form $\mu(v) - tu_i$, $t \geq 0$, for $u_1, \ldots, u_n \in \mathbb{Z}^n$. By Theorem 2.2.5, there exist local coordinates $x_1, \ldots, x_n, y_1, \ldots, y_n$ centered at $v$ (i.e., $v$ corresponds to 0) such that $\omega = \sum_i dx_i \wedge dy_i$, with $z_j = x_j + iy_j$ such that

$$\mu(z_1, \ldots, z_n) = \mu(v) - \frac{1}{2} \sum_{j=1}^{n} |z_j|^2 u_j.$$

In this local model, we see that

$$f(z_1, \ldots, z_n) = \langle \mu(v), X \rangle - \frac{1}{2} \sum_{j=1}^{n} |z_j|^2 \langle u_j, X \rangle$$

Therefore $f$ clearly has even index at 0, determined as twice the number of $i$ with $\langle u_i, X \rangle > 0$. □

**Corollary 3.8.** *Let $H^*(-) = H^*(-;\mathbb{Z})$ denote the (singular) integral cohomology functor. The cohomology ring $H^*(M)$ is zero in odd degree, and is torsion-free. Furthermore, $H^k(M)$ is nonzero whenever $0 \leq k \leq 2n$ is even, with rank equal to the number of critical points of $f$ of index $k$.*

*Proof.* This is Morse's lacunary principle: the indices of the critical points of $f$ do not contain any consecutive integers, and hence the rank of $H^k(M)$ is equal to the number of critical points of $f$ with index $k$. To see this, assume we have chosen $X$ such that $f = \langle \mu, X \rangle$ has different critical values for each fixed point $v$ of the torus action. Label the critical values $c_1 < \cdots < c_l$, and the corresponding indices by $\lambda_1, \ldots, \lambda_l$. We now proceed by induction on the critical values, with $\epsilon > 0$ small enough: $\overline{M}^{c_1-\epsilon} = \emptyset$, and so $H^*(\overline{M}^{c_1-\epsilon}) = 0$, which is torsion-free and zero in odd degree. Now for each $1 \leq j \leq l$, the long exact sequence for the pair $(\overline{M}^{c_j+\epsilon}, \overline{M}^{c_j-\epsilon})$ is

$$\cdots \to H^*(\overline{M}^{c_j+\epsilon}, \overline{M}^{c_j-\epsilon}) \to H^*(\overline{M}^{c_j+\epsilon}) \to H^*(\overline{M}^{c_j-\epsilon}) \to H^{*+1}(\overline{M}^{c_j+\epsilon}, \overline{M}^{c_j-\epsilon}) \to \cdots$$

By Theorem 3.4, $H^k(\overline{M}^{c_j+\epsilon}, \overline{M}^{c_j-\epsilon})$ is zero if $k \neq \lambda_j$, and otherwise it is $\mathbb{Z}$. Furthermore, by the induction assumption, $H^k(\overline{M}^{c_j-\epsilon})$ is zero for $k = \lambda_j - 1, \lambda_j + 1$. Therefore the long exact sequence splits into short exact sequences, and $H^k(\overline{M}^{c_j+\epsilon}) \cong H^k(\overline{M}^{c_j-\epsilon})$ if $k \neq \lambda_j$. On the other hand, if $k = \lambda_j$, then we have a short exact sequence

$$0 \longrightarrow H^{\lambda_j}(\overline{M}^{c_j+\epsilon}, \overline{M}^{c_j-\epsilon}) \longrightarrow H^{\lambda_j}(\overline{M}^{c_j+\epsilon}) \longrightarrow H^{\lambda_j}(\overline{M}^{c_j-\epsilon}) \longrightarrow 0.$$

Since $c_j$ is a critical value of index $\lambda_j$, $H^{\lambda_j}(\overline{M}^{c_j+\epsilon}, \overline{M}^{c_j-\epsilon}) \cong \mathbb{Z}$, which is torsion-free. Furthermore, by the induction hypothesis, $H^{\lambda_j}(\overline{M}^{c_j-\epsilon})$ is torsion-free, and so $H^{\lambda_j}(\overline{M}^{c_j+\epsilon})$ is torsion-free as well[1]. Moreover, the rank of $H^{\lambda_j}(\overline{M}^{c_j+\epsilon})$ is one plus the rank of $H^{\lambda_j}(\overline{M}^{c_j-\epsilon})$. Theorem 3.3 gives $H^k(\overline{M}^{c_j+\epsilon}) \cong H^k(\overline{M}^{c_{j+1}-\epsilon})$. Induction now gives the claimed result for $H^k(\overline{M}^{c_l+\epsilon})$, and $\overline{M}^{c_l+\epsilon} = M$.

The fact that $H^k(M)$ is nonzero whenever $0 \leq k \leq 2n$ is even then follows from either considerations of the structure of the moment polytope of $M$, or one may appeal to the fact that $M$ is symplectic; the latter fact gives that the $k$-th exterior power of the symplectic form is cohomologically nontrivial for every $0 \leq k \leq n$ (see e.g. [6]). □

[1] Given a short exact sequence $0 \to A \to B \to C \to 0$ of abelian groups, if $A, C$ are torsion-free, then $B$ is as well. Indeed, suppose $x \in B$ and $nx = 0$ for $n \neq 0$; then the image of $nx$ is zero in $C$, so the image of $x$ is zero in $C$ (since $C$ is torsion-free). Therefore $x$ is in the image of $A$, say $x$ is the image of $y \in A$. Then $ny$ is mapped to $0 \in B$, but $A \to B$ is injective, so $ny$ must be zero. Since $A$ is torsion-free, $y = 0$, and $x = 0$ as well.

# 4. Equivariant cohomology

## 4.1. Introduction

In the previous chapter, we determined the rank of degree of the cohomology ring of a given symplectic toric manifold $(M^{2n}, \omega, T^n, \mu)$. However, determining the ring structure (i.e., the cup product of any two generators) of $H^*(M)$ is a more delicate matter. In this chapter, we compute the equivariant cohomology[3][2], a version of cohomology which remembers the torus action (or more generally any group action), and then we relate the ring structure on equivariant cohomology to that of the original cohomology ring.

**Definition 4.1.1.** Let $G$ be a compact Lie group. A locally trivial principal $G$-bundle over a paracompact Hausdorff space $X$ consists of a fiber bundle $\pi : P \to X$ with a continuous right $G$-action, such that there is an open covering $\{U_\alpha\}_{\alpha \in A}$ of $X$ and homeomorphisms $\phi_\alpha : U_\alpha \times G \to \pi^{-1}(U_\alpha)$ with

1. $\phi_\alpha(p, g) = \phi_\alpha(p, e)g$ for $g \in U_\alpha$, $g \in G$, and
2. $\pi\phi_\alpha = p_{U_\alpha}$ where $p_{U_\alpha}$ is the projection $U_\alpha \times G \to U_\alpha$.

A universal $G$-bundle is then a principal $G$-bundle $EG \to BG$ with $EG$ (weakly) contractible, that is, all homotopy groups of $EG$ vanish. The space $BG$ is known as a classifying space for $G$. Furthermore, any two classifying spaces are (weakly) homotopy equivalent. For more information, see for instance [11].

**Definition 4.1.2.** Let $M$ be a topological space endowed with a continuous right $G$-action. Then $G$ acts freely on $M \times EG$ with the diagonal action, given by

$$(p, x)g = (pg, xg).$$

We denote the quotient space $(M \times EG)/G$ by $M \times_G EG$, and define the equivariant cohomology $H^*_G(M)$ as $H^*(M \times_G EG)$.[1] If $\{p\}$ is a single-point space with the trivial $G$-action, then $\{p\} \times_G EG \cong EG/G \cong BG$, and so $H^*_G(\{p\}) \cong H^*(BG)$. Given any space $M$ with a continuous $G$-action, the unique map $M \to \{p\}$ is then $G$-equivariant, and hence defines a map $H^*_G(\{p\}) \to H^*_G(M)$ (of rings). Therefore $H^*_G(M)$ is a module over $H^*_G(\{p\})$, and we shall abbreviate the latter by $H^*_G$.

**Example 4.1.3.** Set $G = T^1$. Let $\mathbb{C}^\infty$ be the complex vector space of all tuples $(z_0, z_1, \dots)$ where only finitely many $z_i$ are nonzero. Then

$$S^\infty = \{(z_0, z_1, \dots) \in \mathbb{C}^\infty : |z|^2 = 1\},$$

[1]Note that some authors define $M \times_G EG$ to be the quotient of $EG \times M$ by the equivalence relation $(eg, x) \sim (e, g^{-1}x)$, where $M$ is endowed with a left-action rather than a right-action.

is contractible[2], and admits a $T^1$-action via

$$u \cdot (z_0, z_1, \dots) = (uz_0, uz_1, \dots)$$

where $u \in T^1$. The quotient $S^\infty/T^1$ is the complex projective space $\mathbb{CP}^\infty$ consisting of lines in $\mathbb{C}^\infty$, and the quotient map $\pi : S^\infty \to \mathbb{CP}^\infty$ has the structure of a principal $T^1$-bundle. Therefore $\pi : S^\infty \to \mathbb{CP}^\infty$ is a universal $T^1$-bundle, and we will henceforth write $ET^1 = S^\infty$, $BT^1 = \mathbb{CP}^\infty$. If we let $T^1$ act trivially on a single point $\{p\}$, then

$$H^*_{T^1}(\{p\}) = H^*(\{p\} \times_{T^1} ET^1) \cong H^*(BT^1) = H^*(\mathbb{CP}^\infty) = \mathbb{Z}[c_1(\tau)],$$

where $\tau \to \mathbb{CP}^\infty$ is the tautological vector bundle, and $c_1(\tau)$ is the first Chern class of $\tau$.

**Remark 4.1.4.** For any topological group $G$, there exists a universal $G$-bundle $EG \to BG$ [11, Ch. 4]. We will only be interested in $ET^n \to BT^n$ for $n \geq 0$, and we have constructed $ET^1 \to BT^1$ above. For $n > 1$, we then set $ET^n = (ET^1)^n$ and $BT^n = (BT^1)^n$.

**Example 4.1.5.** Let $T^n$ act trivially on a point $p$. Then, if $\pi_i : ET^n = (ET^1)^n \to ET^1$ denotes the projection on the $i$-th coordinate, we obtain a map

$$\pi_i : \{p\} \times_{T^n} ET^n \to \{p\} \times_{T^1} ET^1.$$

Then $H^*_{T^n}(\{p\}) = H^*((\mathbb{CP}^\infty)^n)$, and we identify

$$H^*((\mathbb{CP}^\infty)^n) = \mathbb{Z}[x_1, \dots, x_n]$$

where $x_i = \pi_i^*(c_1(\tau)) = c_1(\pi_i^*(\tau))$ is the pullback of our earlier choice of generator for $H^*(\mathbb{CP}^\infty)$.

**Remark 4.1.6.** Many basic properties of cohomology still remain valid for equivariant cohomology, as long as all the data is compatible with the $G$-action. Some examples include:

- If $X$, $Y$ are spaces with continuous $G$-actions and $f : X \to Y$ is a $G$-equivariant continuous map, then we obtain a ring morphism $f^* : H^*_G(Y) \to H^*_G(X)$. Furthermore, if $f' : X \to Y$ is another $G$-equivariant map and $H : X \times [0,1] \to Y$ is a homotopy between $f$ and $f'$ with $H(xg, \cdot) = H(x, \cdot)g$ for all $g \in G$, then $f^* = (f')^*$ as maps $H^*_G(Y) \to H^*_G(X)$.

- If $X$ is a space with a continuous $G$-action and $A$ a $G$-invariant subspace, then one can define the relative equivariant cohomology $H^*_G(X, A) = H^*(X \times_G EG, A \times_G EG)$. This gives rise to a long exact sequence in equivariant cohomology.

[2] Write $r : S^\infty \to S^\infty$ for the right-shift map given by $r(z_0, z_1, \dots) = (0, z_0, z_1, \dots)$, and let $c : S^\infty \to S^\infty$ be the constant map $c(z) = (1, 0, \dots)$. Then the identity on $S^\infty$ is homotopic to $r$, and $r$ is homotopic to $c$, with homotopies between these maps given by taking convex combinations and dividing by their norm.

- The excision theorem still holds: if $X$ is a space with a continuous $G$-action, $A$ a $G$-invariant subspace and $U \subset X$ a $G$-invariant subspace such that the closure of $U$ is in the interior of $A$, then the inclusion map $(X \setminus U, A \setminus U) \to (X, A)$ induces an isomorphism in equivariant cohomology.

We now move on to equivariant (complex) vector bundles and their equivariant Euler classes.

**Definition 4.1.7.** Let $\pi : E \to X$ be a rank $k$ complex vector bundle over a (paracompact) Hausdorff space $X$, such that

- we have a (left) $T^n$-action on both $E$ and $X$,
- $\pi$ is $T^n$-equivariant, and
- for $x \in X$ and $u \in T^n$, the map $E_x \to E_{ux}$ given by left-multiplication by $u$ is linear.

Then $\pi : E \to X$ is called a ($T^n$-)equivariant (complex) vector bundle over $X$.

Given such an equivariant vector bundle $\pi : E \to X$ the natural map $\pi_T : E \times_T ET \to X \times_T ET$ gives $E \times_T ET$ the structure of a rank $k$ complex vector bundle, by Lemma A.2. Since $X \times_T ET$ is paracompact, there exists a Thom class

$$u \in H^k(E \times_T ET, E \times_T ET - s_0(X) \times_T ET) = H^k_T(E, E - s_0(X))$$

where $s_0 : X \to E$ is the zero section of $E$ [14, Thm. 9.1, Lem. 14.1]. If $i : (E, \emptyset) \to (E, E - s_0(X))$ denotes the inclusion, the equivariant Euler class $e_T(E) \in H^k_T(X)$ is then defined as

$$e_T(E) = s_0^*(i^* u).$$

**Definition 4.1.8.** Let $X$ be a (paracompact) Hausdorff space with a left $T^n$-action, and $k \geq 0$ an integer. An equivariant complex vector bundle $\pi : E \to X$ of rank $k$ is called trivial if there exists an equivariant isomorphism of vector bundles $E \to X \times \mathbb{C}^k$, where the $T^n$-action on $X \times \mathbb{C}^k$ is given by acting on the first component.

**Lemma 4.1.9.** *The equivariant Euler class has the following properties:*

1. *Let $X, Y$ be two spaces with $T^n$ acting on both, and let $f : X \to Y$ be an equivariant continuous map. If $E \to Y$ is an equivariant rank $k$ complex vector bundle over $Y$, then $f^*E$ is an equivariant complex vector bundle over $X$, and*

$$f^* e_T(E) = e_T(f^* E) \in H^k_T(X).$$

2. *If $j : X \to X \times_T ET$ denotes the inclusion of $X$ as the "base fiber" in $X \times_T ET$, then $j^*(e_T(E)) = e(E)$, where $e(E)$ is the Euler class of $E$.*

*Proof.* The first part follows from the fact that the pullback of a Thom class is again a Thom class for the pullback of the original bundle. The second part is then a consequence of the first part, along with the observation that $E \cong j^*(E \times_T ET)$ as vector bundles over $X$. □

**Example 4.1.10.** As a first example of a $T^1$-equivariant vector bundle, we consider the standard $T^1$-action on $\mathbb{C}$. Let $\pi : \mathbb{C} \to \{p\}$ be the complex line bundle over a point, and let $T^1 = S^1$ act linearly on $\mathbb{C}$ by multiplication. Note that although this bundle is trivial as a topological vector bundle, it is not trivial as an equivariant bundle. The map $\pi_T : \mathbb{C} \times_T ET \to \{p\} \times_T ET$ given by

$$\pi_T[(v, x)] = [(p, x)].$$

is well-defined, because $u \cdot (v, x) = (uv, ux)$ is sent to $(p, ux)$ under the projection, which is equivalent to $(p, x)$. The fiber over $[(p, x)]$ admits a canonical vector space structure because $T^1$ acts linearly on $\mathbb{C}$. Therefore $\pi_T$ is a vector bundle, and is actually isomorphic to the dual tautological bundle $\tau^* \to \mathbb{CP}^\infty$. To see this, define a map $\phi : \mathbb{C} \times_T ET \to \tau^*$ by $\phi([(v, x)]) = (f, [x])$, where $f : [x] \to \mathbb{C}$ is $\mathbb{C}$-linear such that $f(x) = v$. Then $f(ux) = uf(x) = uv$ for all $u \in T^1$, and $f$ is independent of the choice of representative for $[(v, x)]$, so $\phi$ is well-defined. It is easy to check $\phi$ is an isomorphism.

**Lemma 4.1.11.** *Let $k \in \mathbb{Z}$ and define a $T^1$-action on the vector bundle $E_k = \mathbb{C} \to \{p\}$ by*

$$u \cdot v = u^k v, \quad u \in T^1, v \in \mathbb{C}.$$

*Then the vector bundle $E_k \times_T ET \to \{p\} \times_T ET$ is isomorphic to the $(-k)$-th tensor power of the tautological bundle $\tau \to \mathbb{CP}^\infty$. Here we view the $(-1)$-th tensor power of $\tau$ as the dual tautological bundle $\tau^*$.*

*Proof.* First assume $k \geq 0$. Then define $\phi : E_k \times_T ET \to (\tau^*)^{\otimes k}$ by sending the equivalence class $[(v, x)]$ to the unique $\mathbb{C}$-linear map $f : [x]^{\otimes k} \to \mathbb{C}$ with $f(x \otimes \cdots \otimes x) = v$. This is well-defined and an isomorphism.

On the other hand, if $k < 0$, define $\phi : E_k \times_T ET \to \tau^{\otimes(-k)}$ by sending the equivalence class $[(v, x)] \in E_k \times_T ET$ to $v(x \otimes \cdots \otimes x) \in [x]^{\otimes(-k)}$. This is well-defined: if $u \in T^1$, then the representative $(u^k v, ux)$ of $[(v, x)]$ is sent to

$$u^k v(ux \otimes \cdots \otimes ux) = u^k u^{-k} v(x \otimes \cdots \otimes x) = v(x \otimes \cdots \otimes x). \qquad \square$$

**Corollary 4.1.12.** *Let $k \in \mathbb{Z}$ and let $T^1$ act on the equivariant vector bundle $E_k = \mathbb{C} \to \{p\}$ by $u \cdot v = u^k v$ for $u \in T^1$ and $v \in \mathbb{C}$. Then the equivariant Euler class $e_T(E_k)$ is*

$$e_T(E_k) = -kx \in \mathbb{Z}[x] = H_T^*(\{p\}).$$

*Proof.* Observe that $E_k$ is a line bundle, so that $E_k \times_T ET \to \{p\} \times_T ET$ is a line bundle as well. Then

$$e_T(E_k) = e(E_k \times_T ET) = c_1(E_k \times_T ET) = c_1((\tau)^{\otimes(-k)}) = -kc_1(\tau) = -kx,$$

noting that we have defined $x \in H_T^*(\{p\})$ to be the first Chern class of the tautological bundle. $\square$

**Corollary 4.1.13.** *Let $\lambda_1, \dots, \lambda_n \in \mathbb{Z}$ and let $T^n$ act on $E = \mathbb{C} \to \{p\}$ by*

$$(u_1, \dots, u_n) \cdot v = \left( \prod_{i=1}^{n} u_i^{\lambda_i} \right) v.$$

*Then the equivariant Euler class $e_T(E) \in H_T^*(\{p\}) = \mathbb{Z}[x_1, \dots, x_n]$ is given by*

$$e_T(E) = \sum_{i=1}^{n} (-\lambda_i) x_i.$$

*Proof.* Let $E_i = \mathbb{C} \to \{p\}$ be as before with $T^1$-action given by $u \cdot v = u^{\lambda_i} v$, and let $\pi_i : ET^n = (ET^1)^n \to ET^1$ be the projection on the $i$-th coordinate. Then

$$\begin{aligned} e_{T^n}(E) &= e(E \times_{T^n} ET^n) = e(\otimes_{i=1}^{n} \pi_i^*(E_i \times_{T^1} ET^1)) \\ &= \sum_{i=1}^{n} \pi_i^* e(E_i \times_{T^1} ET^1) = \sum_{i=1}^{n} \pi_i^*(-\lambda_i x) = \sum_{i=1}^{n} -\lambda_i x_i. \end{aligned}$$ □

## 4.2. Equivariant cohomology of toric manifolds

**Example 4.2.1.** Let $(\mathbb{CP}^1, \omega_{FS}, T^1, \mu)$ be complex projective space with its standard $T^1$-action and moment map. We write $M = \mathbb{CP}^1$, $T = T^1$, and compute the equivariant cohomology ring $H_T^*(M)$ as follows. Let $X \in \mathfrak{t}$ be nonzero and define $f : M \to \mathbb{R}$ by $f(p) = \langle \mu(p), X \rangle$. Then $f$ is a Morse function by Theorem 3.5, and its critical points are the fixed points $v_0 = [1 : 0]$ and $v_1 = [0 : 1]$. Assume $a = f(v_0) < f(v_1) = b$; otherwise replace $X$ by $-X$. If $J$ is the canonical almost-complex structure on $M = \mathbb{CP}^1$, then the Riemannian metric $g(u, v) = \omega(u, Jv)$ is $T$-invariant. Recall that for $c \in \mathbb{R}$, we write $\overline{M}^c = f^{-1}((-\infty, c])$. Then since $g$ and $f$ are both $T$-invariant, the gradient $\nabla f$ is $T$-invariant as well. Therefore a standard Morse-theoretical argument (as for Theorem 3.3) shows that if $a < c < d < b$, then $H_T^*(\overline{M}^c) \cong H_T^*(\overline{M}^d)$, because $M^c$ is an equivariant deformation retract of $M^d$. Now, for instance by using Theorem 2.2.5, $\{v_0\}$ can be seen to be an equivariant deformation retract of $\overline{M}^c$ for $a < c < b$ with $c - a$ small enough, so $H_T^*(\overline{M}^c) \cong H_T^*(\{v_0\}) = \mathbb{Z}[x_0]$.

We consider the long exact sequence in equivariant cohomology for the pair $(\overline{M}^b, \overline{M}^{b-\epsilon})$, given by

$$\cdots \to H_T^{k-1}(\overline{M}^{b-\epsilon}) \to H_T^k(\overline{M}^b, \overline{M}^{b-\epsilon}) \to H_T^k(\overline{M}^b) \to H_T^k(\overline{M}^{b-\epsilon}) \to \cdots$$

for $\epsilon > 0$ small enough. Let $\nu$ denote the negative normal bundle at $v_1$; then $\nu_{v_1} = T_{v_1}\mathbb{CP}^1$, and we have a linear $T$-action on $\nu_{v_1}$, by differentiating the action on $\mathbb{CP}^1$. By equivariant excision and an equivariant form of the tubular neighbourhood theorem, we have an isomorphism

$$H_T^*(\overline{M}^b, \overline{M}^{b-\epsilon}) \cong H_T^*(\nu_\epsilon, \partial\nu_\epsilon),$$

where $\nu_\epsilon$ denotes the normal vectors of length at most $\epsilon$ (with respect to the metric $g$). With equivariant excision and equivariant homotopy invariance we obtain an isomorphism

$$H_T^*(\nu_\epsilon, \partial\nu_\epsilon) \cong H_T^*(\nu, \nu - \{0\})$$

where $\{0\}$ denotes the zero section of $\nu$. After identifying $\nu_{v_1}$ with $\mathbb{C}$, the $T$-action is given by multiplication with the conjugate. Therefore, its equivariant Euler class $e_T(\nu) = x_1$, where $H_T^*(\{v_1\}) = \mathbb{Z}[x_1]$. We conclude that the image of the natural map $H_T^*(\overline{M}^b, \overline{M}^{b-\epsilon}) \to H_T^*(\{v_1\})$ is the free $H_T^*$-submodule generated by $x_1$, and in particular that this map is an injection.

We now return to the long exact sequence. Above we have determined both $H_T^*(\overline{M}^{b-\epsilon})$ and $H_T^*(\overline{M}^b, \overline{M}^{b-\epsilon})$, and both only live in even degrees (as $x_0, x_1$ are both degree 2). Since we have a $T$-equivariant inclusion $(\{v_1, v_0\}, \{v_0\}) \to (\overline{M}^b, \overline{M}^{b-\epsilon})$, we obtain a morphism on the respective long exact sequences, and hence for each $k \geq 0$ a morphism of short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H_T^k(\overline{M}^b, \overline{M}^{b-\epsilon}) & \longrightarrow & H_T^k(\overline{M}^b) & \longrightarrow & H_T^k(\overline{M}^{b-\epsilon}) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & H_T^k(\{v_1, v_0\}, \{v_0\}) & \longrightarrow & H_T^k(\{v_1, v_0\}) & \longrightarrow & H_T^k(\{v_0\}) & \longrightarrow & 0 \end{array}$$

Since the left and right vertical maps are injective, the middle vertical map is injective as well. To determine $H_T^*(\overline{M}^b)$ (as a ring) it therefore suffices to determine its image in $H_T^k(\{v_1, v_0\})$. We may identify $H_T^k(\{v_1, v_0\}, \{v_0\})$ with $H_T^k(\{v_1\})$ via excision, so the image of the left vertical map is the ideal $(x_1)$. Furthermore, the right vertical map is an isomorphism.

We now have a short exact sequence of $H_T^*$-modules

$$0 \longrightarrow H_T^*\langle e_T(\nu)\rangle \longrightarrow H_T^*(\overline{M}^b) \longrightarrow H_T^*(\overline{M}^{b-\epsilon}) \longrightarrow 0$$

where $H_T^*\langle e_T(\nu)\rangle \subset H_T^*(\{v_1\})$ and $H_T^*(\overline{M}^{b-\epsilon})$ are both free modules, with generators $e_T(\nu)$ and 1, respectively. Therefore, $H_T^*(\overline{M}^b)$ is a free module, with two generators: the image $\alpha$ of $e_T(\nu)$ under the inclusion, and a lift of $1 \in H_T^*(\overline{M}^{b-\epsilon})$ along the surjection. But $1 \in H_T^*(\overline{M}^b)$ is already a lift of 1, since the last map is a morphism of rings as well. Therefore, the image of the injection $H_T^*(\overline{M}^b) \to H_T^*(\{v_1, v_0\})$ is the free $H_T^*$-submodule generated by $1 = (1, 1)$ and $\alpha|_{\{v_1, v_0\}} = x_1$.

Note that $H_T^*(\{v_1, v_0\})$ is the direct sum of the rings $\mathbb{Z}[x_0] \oplus \mathbb{Z}[x_1]$. Therefore the image of $1 \in H_T^*(\overline{M}^b)$ is the pair $(1, 1)$; in particular, if $x \in H_T^* = \mathbb{Z}[x]$, then $x \cdot 1 \in H_T^*(\overline{M}^b)$ is sent to $(x_0, x_1) = x_0 + x_1 \in H_T^*(\{v_1, v_0\})$. As a ring, $H_T^*(\overline{M}^b)$ is isomorphic to the polynomial ring $\mathbb{Z}[y_0, y_1]/(y_0 y_1)$: define a map $\mathbb{Z}[y_0, y_1] \to H_T^*(\overline{M}^b)$ by sending $y_1 \mapsto \alpha$ and $y_0 \mapsto x \cdot 1 - \alpha$.

**Example 4.2.2.** Consider $\mathbb{CP}^2$ with its standard $T^2$-action, and let $v_0 = [1 : 0 : 0]$, $v_1 = [0 : 1 : 0]$, $v_2 = [0 : 0 : 1]$ be the fixed points of the action. We write $H_T^*(\{v_i\}) = \mathbb{Z}[x_i, y_i]$ for the canonical choices of generators $x_i, y_i$. Choose $X \in \mathfrak{t}$ such that $v_0$ is the first critical point, $v_1$ is the second, and $v_2$ is the last, and write $f(v_i) = c_i$. Let $\nu$ be the negative normal bundle at $v_1$. Then we have a $T^2$-representation on $\nu = \mathbb{C} \to v_1$, where the first factor acts nontrivially by multiplication (with exponent 1) and the second factor acts trivially. Therefore $\nu \times_T ET$ has equivariant Euler class $x_1$ in $\mathbb{Z}[x_1, y_1]$. As in the previous example, we see

$$H_T^*(\overline{M}^{c_1}) = H_T^*\langle 1, x_1\rangle \subseteq H_T^*(\{v_1, v_0\}),$$

with the additional note that $H_T^* = \mathbb{Z}[x, y]$. However, the same argument does not give $H_T^*(\overline{M}^{c_2})$, as it is not clear what the image under $H_T^*(\overline{M}^{c_2}) \to H_T^*(\{v_2, v_1, v_0\})$ of any lift of $x_1$ should be. In the computation for $\mathbb{CP}^1$, this was not an issue, as we only had to determine the image of any lift of 1.

**Definition 4.2.3.** Let $X \in \mathfrak{t}$ be such that $\{\exp(tX) : t \in \mathbb{R}\}$ is a dense subgroup of $T^n$, and set $f = \langle \mu, X\rangle$. Let $v \in M$ be a fixed point of the $T^n$-action, and let $u_1, \dots, u_n \in \mathbb{Z}^n$ be such that the edges meeting at $\mu(v)$ are of the form $\mu(v) - tu_i$ for small $t \geq 0$ (see Theorem 2.2.6). Then *flow-up face* of $v$ relative to $X$ is defined to be the smallest face in the polytope containing each edge $\mu(v) - tu_i$ which is positive relative to $X$, that is, $\langle u_i, X\rangle < 0$.

**Example 4.2.4** (Hirzebruch surfaces)**.** Let $k > 0$ and let $W_k$ be the $k$-th Hirzebruch surface (Example 2.2.3), with its standard $T^2$-action and moment map $\mu$. The moment polytope is the convex hull of the points $p_1 = (0, 0)$, $p_2 = (-(k+1)/2, 0)$, $p_3 = (0, -1/2)$ and $p_4 = (-1/2, -1/2)$ in $(\mathbb{R}^2)^*$, as shown in Figure 4.1. Let $X \in \mathfrak{t}$ be a small clockwise rotation of the vector $(0, -1)$ as in Example 3.7, so that $\{\exp(tX) : t \in \mathbb{R}\}$ is a dense subgroup of $T^2$. Then the edges meeting at $(0, 0)$ are of the form $(0, 0) - tu_1$ and $(0, 0) - tu_2$, with $u_1 = (1, 0)$ and $u_2 = (0, 1)$. For these edges, we have

$$\langle u_1, X\rangle < 0, \quad \langle u_2, X\rangle < 0.$$

Therefore both edges at $(0, 0)$ are positive relative to $X$, and the flow-up face of $(0, 0)$ relative to $X$ is the entire polytope. On the other hand, the edges meeting at $(0, -1/2)$ are of the form $(0, -1/2) - tu_i'$ with

$$\begin{aligned} u_1' &= (1, 0) \quad \langle u_1', X\rangle < 0 \\ u_2' &= (0, -1) \quad \langle u_2', X\rangle > 0 \end{aligned}$$

and hence only $u_1'$ is positive relative to $X$. Therefore, the flow-up face of $(0, -1/2)$ consists only of the edge from $(0, -1/2)$ to $(-1/2, -1/2)$.

**Theorem 4.2.5.** *Let $(M^{2n}, \omega, T^n, \mu)$ be a symplectic toric manifold, and let $F$ be the set of fixed points of the action. Let $X \in \mathfrak{t}$ be such that $\{\exp(tX) : t \in \mathbb{R}\}$ is a dense subgroup of $T = T^n$. For each $v \in F$, if $A_v$ is the flow-up face of $v$ relative to $X$,*

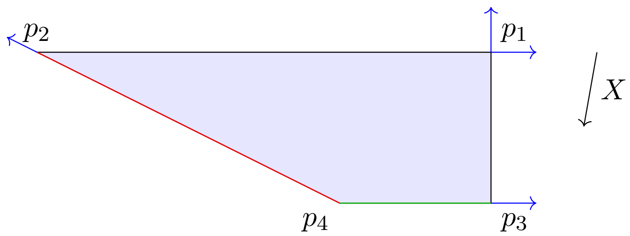


Figure 4.1.: The moment polytope of the $k$-th Hirzebruch surface $W_k$. The flow-up face for $p_1$ relative to $X$ is the entire polytope, the flow-up face for $p_i$, $i = 2, 3$, is the edge from $p_i$ to $p_4$, and the flow-up face for $p_4$ consists only of $p_4$. The blue arrows at each corner indicate the directions of the edges which are positive relative to $X$.

*we write $\nu^v$ for the normal bundle of $\mu^{-1}(A_v)$ in $M$. Then the inclusion-induced map $H_T^*(M) \to H_T^*(F)$ is injective, and its image is the free $H_T^*$-submodule of $H_T^*(F)$ with generators*

$$\{e_T(\nu^v)|_{F\cap A_v} : v \in F\}.$$

As stated in the introduction, the freeness of $H_T^*(M)$ and injectivity of $H_T^*(M) \to H_T^*(F)$ is already shown in [3] (for the case of cohomology with real coefficients), but their proof depends on a general localization theorem. Our proof is based on the following theorem, which we prove in the next section.

**Theorem** (4.3.8)**.** *Let $N$ be the preimage of a face of $\mu(M)$, and let $\nu$ be its normal bundle. Then there exists a $T$-equivariant complex vector bundle $E \to M$ such that $E|_N \cong \nu$ (as equivariant vector bundles), and such that $e_T(E)|_v = 0$ whenever $v \in F$ is not in $N$.*

The significance of the theorem is that it solves the problem we encounter in Example 4.2.2 when trying to build up $M$ inductively with Morse theory: it provides lifts of the equivariant Euler classes of the negative normal bundles at each critical point.

*Proof of Theorem 4.2.5.* Let $X \in \mathfrak{t}$ be such that $\{\exp(tX) : t \in \mathbb{R}\}$ is a dense subgroup of $T$, and let $f : M \to \mathbb{R}$ be given by $f(p) = \langle \mu(p), X\rangle$. We may assume without loss of generality that each $v \in F$ has distinct image under $f$, by slightly perturbing $X$. By Theorem 3.5, $f$ is a Morse function, whose set of critical points is $F$. Label the fixed points $v_1, \ldots, v_r$, ordered such that the critical values $f(v_i) = c_i$ form an increasing sequence. Recall that the sublevel sets of $f$ are defined as $\overline{M}^c = f^{-1}((-\infty, c])$, which are $T$-invariant, since $f$ is $T$-invariant. We write $F = \{v_1, \ldots, v_r\}$ for the set of fixed points, set $F^c = \overline{M}^c \cap F$ for $c \in \mathbb{R}$, and let $j_s : F^c \to \overline{M}^c$ be the inclusion map for $s = 1, \ldots, r$. We now prove by induction, for each $s = 1, \ldots, r$, and for $\epsilon > 0$ small enough (with $c_{s+1} - c_s > 2\epsilon$ for $s = 1, \ldots, r-1$):

1. for odd $k$, $H_T^k(\overline{M}^{c_s+\epsilon})$ is zero,

2. the inclusion-induced map $j_s^* : H_T^*(\overline{M}^{c_s+\epsilon}) \to H_T^*(F^{c_s})$ is injective, and

3. the image of $j_s^*$ is generated, as $H_T^*$-module, by

$$\{e_T(E_v)|_{F^{c_s}} : v \in F^{c_s}\}$$

where $E_v \to M$ is the extension of the normal bundle of $\mu^{-1}(A_v)$, the preimage of the flow-up face of $v$, obtained from Theorem 4.3.8.

Note that the choice of $\epsilon$ guarantees that each interval $[c_s - \epsilon, c_s + \epsilon]$ contains only one critical value of $f$, and they are pairwise disjoint, so that $F^{c_s+\epsilon} = F^{c_s}$. Furthermore, for $s = r$, the above three statements give the statement of the theorem.

The base case is $s = 1$. Since $v_1$ is a fixed point, by Theorem 2.2.5, there exists a local chart $(z_1, \ldots, z_n)$ centered at $v_1$ such that

$$f(z_1, \ldots, z_n) = c_1 - \frac{1}{2}\sum_{i=1}^{n} \langle u_i, X\rangle |z_i|^2$$

where the $u_i \in \mathbb{Z}^n$ are the edges meeting at $\mu(v_1)$. Since $\overline{M}^{c_1} = \{v_1\}$ ($c_1$ is the unique minimum of $f$), we may assume $\epsilon$ is small enough such that the chart contains all of $\overline{M}^{c_1+\epsilon}$. From the local description, scaling to the origin gives an equivariant deformation retraction of $\overline{M}^{c_1+\epsilon}$ to $\{v_1\}$. Therefore the inclusion of $v_1$ in $\overline{M}^{c_1+\epsilon}$ induces an isomorphism $H_T^*(\overline{M}^{c_1+\epsilon}) \cong H_T^*(\{v_1\})$, and $H_T^k(\overline{M}^{c_1+\epsilon})$ is zero for $k$ odd, and $j_1^* : H_T^*(\overline{M}^{c_1+\epsilon}) \to H_T^*(F^{c_1})$ is an isomorphism, hence injective. For the last part, it suffices to check that the unit $1 \in H_T^*(\{v_1\})$ is the equivariant Euler class of the normal bundle of the flow-up face of $v_1$. But the flow-up face of $v_1$ is $\mu(M)$, since there are only "positive edges" at $v_1$; therefore the normal bundle has rank 0, and $e_T(0) = 1 \in H_T^*(\{v_1\})$ is the equivariant Euler class of the trivial bundle.

Now assume that the above three statements hold for $s$. First, since $f$ is $T$-invariant and $[c_s + \epsilon, c_{s+1} - \epsilon]$ does not contain any critical values of $f$, the inclusion $\overline{M}^{c_s+\epsilon} \to \overline{M}^{c_{s+1}-\epsilon}$ induces an isomorphism in equivariant cohomology. The equivariant long exact sequence for the pair $(\overline{M}^{c_{s+1}+\epsilon}, \overline{M}^{c_{s+1}-\epsilon})$ is

$$\cdots \longrightarrow H_T^{k-1}(\overline{M}^{c_{s+1}-\epsilon}) \longrightarrow$$

$$\longrightarrow H_T^k(\overline{M}^{c_{s+1}+\epsilon}, \overline{M}^{c_{s+1}-\epsilon}) \longrightarrow H_T^k(\overline{M}^{c_{s+1}+\epsilon}) \longrightarrow H_T^k(\overline{M}^{c_{s+1}-\epsilon}) \longrightarrow \cdots$$

and by assumption, $H_T^k(\overline{M}^{c_{s+1}-\epsilon}) \cong H_T^k(\overline{M}^{c_s+\epsilon})$ is zero whenever $k$ is odd. Therefore, determining $H_T^*(\overline{M}^{c_{s+1}+\epsilon}, \overline{M}^{c_{s+1}-\epsilon})$ is now the key to the induction step.

**Lemma 4.2.6.** *$H_T^k(\overline{M}^{c_{s+1}+\epsilon}, \overline{M}^{c_{s+1}-\epsilon})$ is zero whenever $k$ is odd. Moreover, if*

$$i : (F^{c_{s+1}}, F^{c_s}) \to (\overline{M}^{c_{s+1}+\epsilon}, \overline{M}^{c_{s+1}-\epsilon})$$

*denotes the inclusion, and $\nu$ is a maximal negative-definite subspace of the Hessian of $f$ at $v_{s+1}$, then the composition*

$$H_T^*(\{v_{s+1}\}) \xleftarrow[\cong]{} H_T^*(F^{c_{s+1}}, F^{c_s}) \xleftarrow[i^*]{} H_T^*(\overline{M}^{c_{s+1}+\epsilon}, \overline{M}^{c_{s+1}-\epsilon})$$

*of $i^*$ and the excision isomorphism is injective, and has image the free $H_T^*$-module on $e_T(\nu) \neq 0$.*

*Proof.* Recall that we have a $T$-invariant metric $g$ on $M$. Let $\epsilon' > 0$ be small enough such that on $\nu_{\epsilon'} = \{w \in \nu : \|w\|_g \leq \epsilon'\}$, the exponential map $\exp : \nu_{\epsilon'} \to M$ is injective. Then for $\epsilon > 0$ sufficiently small, $\exp(\partial \nu_{\epsilon'}) \subset \overline{M}^{c_{s+1}-\epsilon}$. Consider the diagram

$$\begin{array}{ccccc}
H_T^*(\nu, \nu - s_0(\{v_{s+1}\})) & \xrightarrow{\cong} & H_T^*(\nu_\epsilon, \partial\nu_\epsilon) & \xleftarrow[(\exp)^*]{} & H_T^*(\overline{M}^{c_{s+1}+\epsilon}, \overline{M}^{c_{s+1}-\epsilon}) \\
\downarrow j^* & & & & \downarrow \\
H_T^*(\nu) & \xrightarrow{s_0^*} & H_T^*(\{v_{s+1}\}) & \xleftarrow[\cong]{} & H_T^*(F^{c_{s+1}}, F^{c_s})
\end{array}$$

where $s_0 : \{v_{s+1}\} \to \nu$ denotes the zero section, $j_\epsilon : \nu_\epsilon \to (\nu_\epsilon, \partial\nu_\epsilon)$ the inclusion. The map $(\exp)^*$ is an isomorphism; this is a direct consequence of the standard argument for Theorem 3.4, along with the equivariant local model from Theorem 2.2.5 to ensure that all the deformation retractions are equivariant. Since $\nu$ is an equivariant complex vector bundle, $\nu \times_T ET$ is a complex vector bundle over $\{v_{s+1}\} \times_T ET$, and hence has a Thom class $t \in H_T^{2\lambda}(\nu, \nu - s_0(\{v_{s+1}\}))$, with $\lambda$ the rank of $\nu$ as a complex vector bundle. Then if $\pi : \nu \to \{v_{s+1}\}$ denotes the projection, by the Thom isomorphism theorem [14, Thm. 9.1], the map $\psi : H_T^*(\{v_{s+1}\}) \to H_T^{*+2\lambda}(\nu, \nu - s_0(\{v_{s+1}\}))$ given by

$$\psi(y) = \pi^*(y) \cup t$$

is a bijection. But since $\{v_{s+1}\}$ is a single-point space, $H_T^*(\{v_{s+1}\}) \cong H_T^*$, and the Thom isomorphism theorem gives that $H_T^*(\nu, \nu - s_0(\{v_{s+1}\}))$ is a free $H_T^*$-module on a single generator $t$, that is,

$$H_T^*(\nu, \nu - s_0(\{v_{s+1}\})) \cong H_T^*\langle t \rangle.$$

As $s_0^*(j^*(t)) = e_T(\nu)$ (by definition), the image of $s_0^* \circ j^*$ is therefore the submodule of $H_T^*(\{v_{s+1}\})$ generated by $e_T(\nu)$, and $s_0^* \circ j^*$ is also injective, since $e_T(\nu)$ is nonzero and $H_T^*$ has no zero divisors. Note that since $v_{s+1}$ has even index, $\nu$ has even real dimension; hence $H_T^k(\overline{M}^{c_{s+1}+\epsilon}, \overline{M}^{c_{s+1}-\epsilon})$ is zero whenever $k$ is odd. Since the diagram of topological maps inducing the above diagram commutes, the above diagram commutes as well, and the result follows. ∎

From the long exact sequence we now obtain that $H_T^k(\overline{M}^{c_{s+1}+\epsilon})$ is zero whenever $k$ is odd. For even $k$, the inclusion $J : (F^{c_{s+1}}, F^{c_s}) \to (\overline{M}^{c_{s+1}}, \overline{M}^{c_s})$ induces a morphism of short exact sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H_T^k(\overline{M}^{c_{s+1}+\epsilon}, \overline{M}^{c_{s+1}-\epsilon}) & \longrightarrow & H_T^k(\overline{M}^{c_{s+1}+\epsilon}) & \longrightarrow & H_T^k(\overline{M}^{c_{s+1}-\epsilon}) & \longrightarrow & 0 \\
 & & \downarrow J^* & & \downarrow j_{s+1}^* & & \downarrow j_s^* & & \\
0 & \longrightarrow & H_T^k(F^{c_{s+1}}, F^{c_s}) & \longrightarrow & H_T^k(F^{c_{s+1}}) & \longrightarrow & H_T^k(F^{c_s}) & \longrightarrow & 0
\end{array}$$

The previous lemma shows that $J^*$ is injective, and $j_s^*$ is injective by the induction hypothesis; therefore by the Five Lemma, $j_{s+1}^*$ is injective as well.

The previous diagram may also be considered for all $k$ at the same time to obtain a short exact sequence

$$0 \longrightarrow H_T^*(\overline{M}^{c_{s+1}+\epsilon}, \overline{M}^{c_{s+1}-\epsilon}) \longrightarrow H_T^*(\overline{M}^{c_{s+1}+\epsilon}) \longrightarrow H_T^*(\overline{M}^{c_{s+1}-\epsilon}) \longrightarrow 0$$

of $H_T^*$-modules, with both $H_T^*(\overline{M}^{c_{s+1}+\epsilon}, \overline{M}^{c_{s+1}-\epsilon})$ and $H_T^*(\overline{M}^{c_{s+1}-\epsilon})$ free modules. Therefore $H_T^*(\overline{M}^{c_{s+1}+\epsilon})$ is a free $H_T^*$-module as well, with generators determined by the images of the generator of $H_T^*(\overline{M}^{c_{s+1}+\epsilon}, \overline{M}^{c_{s+1}-\epsilon})$, and lifts of the generators of $H_T^*(\overline{M}^{c_s})$. The generator for $H_T^*(\overline{M}^{c_{s+1}+\epsilon}, \overline{M}^{c_{s+1}-\epsilon})$, when restricted to $H_T^*(F^{c_{s+1}}, F^{c_s}) \cong H_T^*(\{v_{s+1}\})$, gives the equivariant Euler class $e_T(\nu)$, where $\nu$ is a negative eigenspace of the Hessian of $f$ at $v_{s+1}$. Recall that $A_{v_{s+1}}$ is the flow-up face of $v_{s+1}$; then the normal bundle of $\mu^{-1}(A_{v_{s+1}})$ at $v_{s+1}$ is (isomorphic to) $\nu$. Note that $\mu^{-1}(A_{v_{s+1}}) \cap F^{c_{s+1}} = \{v_{s+1}\}$, since $A_{v_{s+1}}$ is the flow-up face of $v_{s+1}$. Therefore, if $E_{v_{s+1}}$ is the extension of the normal bundle of $\mu^{-1}(A_{v_{s+1}})$ , then $e_T(E)|_{\overline{M}^{c_{s+1}+\epsilon}}$ defines an element of $H_T^*(\overline{M}^{c_{s+1}+\epsilon})$. This is also the image of the generator of $H_T^*(\overline{M}^{c_{s+1}+\epsilon}, \overline{M}^{c_{s+1}-\epsilon})$: we have

$$e_T(E_{v_{s+1}})|_{F^{c_{s+1}}} = e_T(\nu) \in H_T^*(F^{c_{s+1}}).$$

To see this, note that $\mu^{-1}(A_{v_{s+1}}) \cap F^{c_{s+1}} = \{v_{s+1}\}$, that the fiber of $E_{v_{s+1}}$ at $v \in F$ with $v \notin A_{v_{s+1}}$ has trivial equivariant Euler class, and that the fiber of $E_{v_{s+1}}$ at $v_{s+1}$ is that of the normal bundle of $\mu^{-1}(A_{v_{s+1}})$, which is precisely $\nu$. This together with the fact that the generator of $H_T^*(\overline{M}^{c_{s+1}+\epsilon}, \overline{M}^{c_{s+1}-\epsilon})$ is sent to $e_T(\nu)$ in $H_T^*(F^{c_{s+1}})$, and the injectivity of $j_{s+1}^* : H_T^*(\overline{M}^{c_{s+1}}) \to H_T^*(F^{c_{s+1}})$, give that the image of the generator of $H_T^*(\overline{M}^{c_{s+1}+\epsilon}, \overline{M}^{c_{s+1}-\epsilon})$ restricts to $e_T(\nu)$. On the other hand, by the induction hypothesis, we already have generators for $H_T^*(\overline{M}^{c_{s+1}-\epsilon})$ in terms of restrictions of equivariant Euler classes of vector bundles over $M$; and these directly lift to classes in $H_T^*(\overline{M}^{c_{s+1}+\epsilon})$. □

We have now completed the proof of Theorem 4.2.5. Note that we also have the following result, relating the equivariant cohomology to the ordinary cohomology of $M$.

**Theorem 4.2.7.** *Let $j : M \to M \times_T ET$ denote the inclusion. Then the map $j^* : H_T^*(M) \to H^*(M)$ is surjective, and $H^*(M)$ is generated as an abelian group by the set of $e(E_v)$, $v \in F$. Furthermore, for any element $r \in H_T^*$ of degree at least 2, we have $j^*(re_T(E_v)) = 0$.*

*Proof.* We proceed as in the proof of Theorem 4.2.5: we choose a generic direction $X \in \mathfrak{t}$, and use induction on the fixed point set $F$, which are the critical points of $f = \langle \mu, X \rangle$. For each critical value $c_s$ the map $j^* : H_T^*(\overline{M}^{c_s+\epsilon}, \overline{M}^{c_s-\epsilon}) \to H^*(\overline{M}^{c_s+\epsilon}, \overline{M}^{c_s-\epsilon})$ is surjective. To see this, note that if $\nu$ is the maximal negative-definite subspace of the Hessian of $f$ at $v_s$, then pulling back the Thom class for $\nu \times_T ET$ along $j$ gives a Thom class for $\nu$. However, any product of the Thom class for $\nu \times_T ET$ with an element of $H_T^*$ is sent to 0, because $H^*(\overline{M}^{c_s+\epsilon}, \overline{M}^{c_s-\epsilon})$ lives only in a single degree (namely the rank of $\nu$).

As before, the long exact sequences in both equivariant and ordinary cohomology for each pair $(\overline{M}^{c_s+\epsilon}, \overline{M}^{c_s-\epsilon})$ contains only zero terms in odd degrees, so that by induction on the critical points and the Five Lemma, we obtain surjectivity of $j^* : H_T^*(M) \to H^*(M)$, and the claim on products with elements of $H_T^*$ also follows. □

Therefore we now have both a surjection $H_T^*(M) \to H^*(M)$, and an injection $H_T^*(M) \to H_T^*(F)$. To compute a product of two elements in $H^*(M)$, it therefore suffices to lift these elements to $H_T^*(M)$ (which we can do by having canonical lifts of the additive generators of $H^*(M)$), compute their product in $H_T^*(F)$, and with that deduce the product in $H^*(M)$. The only obstruction to this procedure is now the fact that we do not yet know what the image of each generator $e_T(E_v)$ of $H_T^*(M)$ in $H_T^*(F)$ is, and this is a problem we solve in the next section (see Theorem 4.3.8).

## 4.3. Delzant's construction and vector bundles

In this section, we describe Delzant's explicit construction of a symplectic toric manifold from its moment polytope, and use it to construct the bundles that are required for the proof of Theorem 4.2.5. For more information, we refer the reader to [7]. Let $C \subset (\mathbb{R}^n)^*$ be a closed convex polytope, satisfying Delzant's conditions in Theorem 2.2.6. Let $F_1, \ldots, F_k$ be the codimension 1 faces of $C$; then Delzant's conditions imply that there exist vectors $v_1, \ldots, v_k \in \mathbb{Z}^n$ and real numbers $\lambda_1, \ldots, \lambda_k$ such that

1. The entries of each $v_i$ have greatest common divisor 1,
2. $F_i = C \cap \{x \in (\mathbb{R}^n)^* : \langle x, v_i \rangle = \lambda_i\}$, where $\langle \cdot, \cdot \rangle : (\mathbb{R}^n)^* \times \mathbb{R}^n \to \mathbb{R}$ is the dual pairing,
3. and
$$C = \bigcap_{i=1}^{k} \{x \in (\mathbb{R}^n)^* : \langle x, v_i \rangle \leq \lambda_i\}.$$

For each $i$, the second condition determines $v_i$ and $\lambda_i$ up to a scalar; the first and third conditions then uniquely define $v_i$ and $\lambda_i$ by determining the correct multiple. Therefore it suffices to establish existence of $v_i$ and $\lambda_i$ satisfying the above conditions. Let $c \in F_i$ be a corner, and let $u_1, \ldots, u_n \in \mathbb{Z}^n \subset (\mathbb{R}^n)^*$ be such that the edges at $c$ are of the form $c - tu_j$, $t \geq 0$, where $u_1, \ldots, u_n$ form a $\mathbb{Z}$-basis for $\mathbb{Z}^n$ (see Theorem 2.2.6). Assume without loss of generality that $c - tu_j \in F_i$ for $j = 2, \ldots, n$, and let

$w_1, \ldots, w_n \in \mathbb{Z}^n$ be the dual basis to $u_1, \ldots, u_n$. Note that this dual basis is in $\mathbb{Z}^n$ by virtue of $u_1, \ldots, u_n$ forming a $\mathbb{Z}$-basis for $\mathbb{Z}^n \subset (\mathbb{R}^n)^*$. Then we claim that $v_i = w_1$ and $\lambda_i = \langle c, w_1 \rangle$ satisfy the above conditions. The first condition follows directly from the fact that $w_1, \ldots, w_n \in \mathbb{Z}^n$ form a $\mathbb{Z}$-basis. For the second condition, note that for $x \in C$, we have $x \in F_i$ if and only if $x - c$ is in the span of $u_2, \ldots, u_n$, which holds if and only if $\langle x - c, w_1 \rangle = 0$, i.e., $\langle x, w_1 \rangle = \langle c, w_1 \rangle$. The last condition determines the sign of $v_i$; by assumption, $c - tu_1 \in C$ for small $t \geq 0$, and hence $\langle c - tu_1, w_1 \rangle = \lambda_i - t \leq \lambda_i$.

Note that we use a different sign convention for the inequality from Delzant, as we use different sign conventions for the definition of a moment map. We now proceed with the actual construction of the corresponding symplectic toric manifold. $\pi : T^k \to T^n$ by viewing $T^1$ as $\mathbb{R}/(2\pi\mathbb{Z})$, and setting

$$\pi([e_i]) = [v_i],$$

where $e_1, \ldots, e_k$ denotes the standard basis for $\mathbb{R}^k$. This does in fact define a map $T^k \to T^n$ since each $v_i$ is a vector with integral entries. Note that $d\pi : \mathfrak{t}^k \to \mathfrak{t}^n$ is now given by the matrix with columns $v_1, \ldots, v_n$, i.e., $(d\pi)(e_i) = v_i$. We let $T^k$ act on $\mathbb{C}^k$ in the standard way, that is, for $u = (u_1, \ldots, u_k) \in T^k$ and $z = (z_1, \ldots, z_k)$, we set

$$(u_1, \ldots, u_k)(z_1, \ldots, z_k) = (u_1 z_1, \ldots, u_k z_k).$$

This action is Hamiltonian with moment map $\mu_{T^k} : \mathbb{C}^k \to (\mathbb{R}^k)^*$ given by

$$\mu_{T^k}(z_1, \ldots, z_k) = (\lambda_1, \ldots, \lambda_k) - \frac{1}{2}\left(|z_1|^2, \ldots, |z_k|^2\right),$$

where we have represented vectors in $(\mathbb{R}^k)^*$ with respect to the standard basis $e_1^*, \ldots, e_k^*$. The image of $\mu_{T^k}$ is the cone

$$\operatorname{im} \mu_{T^k} = \{(w_1, \ldots, w_k) \in (\mathbb{R}^k)^* : w_j \leq \lambda_j \text{ for all } j\}.$$

Now let $N$ denote the kernel of $\pi$, let $\mathfrak{n}$ be its Lie algebra, and let $\varphi : \mathfrak{n} \to \mathbb{R}^k$ be the inclusion map. Its dual $\varphi^*$ is a map $(\mathbb{R}^k)^* \to \mathfrak{n}^*$, and the action of $N$ on $\mathbb{C}^k$ is Hamiltonian with moment map $\mu_N = \varphi^* \circ \mu_{T^k}$. Furthermore, the kernel of $\varphi^*$ is precisely the image of $(d\pi)^*$, and we have

$$\ker \varphi^* \cap \operatorname{im} \mu_{T^k} = \operatorname{im}(d\pi)^* \cap \operatorname{im} \mu_{T^k} = (d\pi)^*(C).$$

**Lemma 4.3.1.** *Let $w = (w_1, \ldots, w_k) = (d\pi)^*(w')$ for $w' \in C$. Then $w_j = \lambda_j$ if and only if $w' \in F_j$.*

*Proof.* Note that $w' \in F_j$ if and only if

$$\langle w', v_j \rangle = \lambda_j,$$

and

$$\langle w', v_j \rangle = \langle w', (d\pi)(e_j) \rangle = \langle (d\pi)^*(w'), e_j \rangle = \langle w, e_j \rangle$$

and the latter is the $j$-th coordinate of $w$. □

**Lemma 4.3.2.** *The point* $0 \in \mathfrak{n}^*$ *is a regular value of* $\mu_N$*, so* $\overline{M} = \mu_N^{-1}(0) = \mu_{T^k}^{-1}((d\pi)^*(C))$ *is a submanifold of* $\mathbb{C}^k$*.*

*Proof.* For $p \in \overline{M}$, we know $\mu_{T^k}(p)$ is in the kernel of $\varphi^*$ by definition of $\mu_N$. Denote the coordinates on $\mathbb{C}^k$ are by $z_j = x_j + iy_j$, and denote the coordinates on $\mathbb{R}^k$ by $(w_1, \ldots, w_k)$, so that

$$(d\mu_{T^k})_p = \sum_{j=1}^{k} (-x_j\, dx_j - y_j\, dy_j) \frac{\partial}{\partial w_j}$$

which we view as a map $T_p\mathbb{C}^k \to T_{\mu_{T^k}(p)}\mathbb{R}^k$. Therefore the image of $(d\mu_{T^k})_p$ is the span of those $\partial/\partial w_j$ for which $z_j \neq 0$, as it suffices to have at least one of $x_j$ and $y_j$ non-zero. We must now show that $(d\mu_N)_p$ is surjective.

Recall that $\dim \mathfrak{n} = k - n$, since it is the kernel of $d\pi : \mathbb{R}^k \to \mathbb{R}^n$ (which is surjective), and hence it suffices to show that the rank of $(d\mu_N)_p$ is $k-n$. Since $(d\mu_N)_p = (d\varphi^*)_{\mu_{T^k}(p)} \circ (d\mu_{T^k})_p$, by the rank-nullity theorem,

$$\begin{aligned}\dim \operatorname{im}(d\mu_N)_p &= \dim \operatorname{im}\left((d\varphi^*)_{\mu_{T^k}(p)}|_{\operatorname{im}(d\mu_{T^k})_p}\right)\\ &= \dim \operatorname{im}(d\mu_{T^k})_p - \dim(\operatorname{im}(d\mu_{T^k})_p \cap \ker(d\varphi^*)_{\mu_{T^k}(p)}).\end{aligned}$$

Since $\mu_{T^k}(p) \in (d\pi)^*(C)$, we have $\mu_{T^k}(p) = (d\pi)^*(w')$ for some $w' \in C$, and hence

$$\ker(d\varphi^*)_{\mu_{T^k}(p)} = \operatorname{im} d((d\pi)^*)_{w'}$$

as $\varphi : \mathfrak{n} \to \mathbb{R}^k$ is the inclusion of the kernel of $d\pi : \mathbb{R}^k \to \mathbb{R}^n$. Since $(d\pi)^*$ is a linear map, we identify $d((d\pi)^*)$ with $(d\pi)^*$.

Note that for the point $p = (z_1, \ldots, z_k) \in \overline{M}$, $z_j \neq 0$ if and only if the $j$-th coordinate of $\mu_{T^k}(p)$ is $\lambda_j$ (by the definition of $\mu_{T^k}$), and by definition of $\pi$, this is equivalent to $w'$ being in the $j$-th face $F_j$ of $C$. Now let $j_1, \ldots, j_l$ be the coordinates $j$ for which $z_j = 0$, and choose $j_s$ for $s = l+1, \ldots, n$ such that $\{v_{j_s} : s = 1, \ldots, n\}$ forms a basis for $\mathbb{R}^n$. Denote by $v_{j_1}^*, \ldots, v_{j_n}^*$ the corresponding dual basis for $(\mathbb{R}^n)^*$; we now claim that

$$V = \{(d\pi)_{w'}^* v_{j_s}^* : s = l+1, \ldots, n\}$$

is a basis for $\operatorname{im}(d\mu_{T^k})_p \cap \operatorname{im}(d\pi)_{w'}^*$. Since $(d\pi)_{w'}^*$ is injective it suffices to check that the intersection is the span of $V$. Observe that for any standard basis vector $e_{j_s} \in \mathbb{R}^k$ and $s' = 1, \ldots, n$, we have

$$\langle (d\pi)_{w'}^* v_{j_{s'}}^*, e_{j_s} \rangle = \langle v_{j_{s'}}^*, v_{j_s} \rangle = \delta_{ss'}$$

so the $j_s$'th coordinate of $(d\pi)_{w'}^* v_{j_{s'}}^*$ is $\delta_{ss'}$. As the image of $(d\mu_{T^k})_p$ consists of all vectors whose $j_1, \ldots, j_l$'th coordinates are zero, $V$ is contained in $\operatorname{im}(d\mu_{T^k})_p$. Conversely, for any vector $w \in \operatorname{im}(d\mu_{T^k})_p \cap \operatorname{im}(d\pi)_{w'}^*$, say $w = (d\pi)_{w'}^*(\bar{w})$ and $s = 1, \ldots, l$, we have

$$0 = \langle w, e_{j_s} \rangle = \langle \bar{w}, v_{j_s} \rangle$$

so $\bar{w}$ must be in the span of $\{v_{j_s}^* : s = l+1, \ldots, n\}$.

Thus we have determined that if $l$ is the number of coordinates of $p$ that is zero, then

$$\begin{aligned}&\dim \operatorname{im}(d\mu_{T^k})_p = k - l\\&\dim(\operatorname{im}(d\mu_{T^k})_p \cap \ker(d\varphi^*)_{\mu_{T^k}(p)}) = n - l\end{aligned}$$

and hence $\dim \operatorname{im}(d\mu_N)_p = k - n$, so $(d\mu_N)_p$ is surjective. □

**Lemma 4.3.3.** *Let $\overline{M}$ be as in Lemma 4.3.2. Then $N$ acts freely on $\overline{M}$, so that $M = \overline{M}/N$ is a symplectic manifold of dimension $2n$. Furthermore, the remaining $(T^k/N)$-action on $M$ is Hamiltonian with moment map $\mu : M \to (\mathbb{R}^n)^*$ defined by $\mu([z_1, \ldots, z_k]) = w$, where $w \in (\mathbb{R}^n)^*$ is the unique $w \in C$ such that $(d\pi)^*(w) = \mu_{T^k}(z_1, \ldots, z_k)$. The quotient $T^k/N$ is identified with $T^n$ via $\pi$, and hence $M$ is a symplectic toric manifold, with moment polytope $C$.*

*Proof.* The fact that $M$ is symplectic is an application of symplectic reduction, also known as the Marsden–Weinstein–Meyer theorem [6, Ch. 23]. The freeness of the $N$-action on $\overline{M}$ is established in [7, 3.2]. □

We now construct a $T^n$-equivariant vector bundle over $M$ which restricts to the normal bundle of a face $F_j$, and compute the $T^n$-action of the fiber over each fixed point.

**Proposition 4.3.4.** *Let $F_1, \ldots, F_k$ be the faces of the polytope as before, and fix $i \in \{1, \ldots, n\}$. Then there exists a $T^n$-equivariant rank one complex vector bundle $E \to M$ such that*

1. *$E|_{\mu^{-1}(F_i)} \cong \nu_{\mu^{-1}(F_i)}$, where $\nu_{\mu^{-1}(F_i)}$ denotes the normal bundle of $\mu^{-1}(F_i)$ in $M$, and*

2. *if $v \in F$ is a fixed point of the $T^n$-action, and $v \notin \mu^{-1}(F_i)$, then $T^n$ acts trivially on $E|_v$.*

3. *If $v \in F$ is a fixed point with $v \in \mu^{-1}(F_{i_1} \cap F_{i_2} \cap \cdots \cap F_{i_n})$, where $i_1 < i_2 < \cdots < i_n$ and $i = i_s$, then if $d_1, \ldots, d_n$ denotes the standard basis of $\mathbb{R}^n$ and we write*

$$d_j = \sum_{l=1}^{n} b_{jl} v_{i_l},$$

*the weight vector of the $T^n$-action on $E|_v = \mathbb{C}$ is $(b_{1s}, b_{2s}, \ldots, b_{ns})$.*

Before we prove the above proposition, we first prove a small result concerning Delzant's construction of $M$, which is essentially an extension of Lemma 4.3.1.

**Lemma 4.3.5.** *Let $p \in M$ be a point in $M$, and suppose $p = [z_1, \ldots, z_k]$. Then $\mu(p) \in F_i$ if and only if $z_i = 0$, and this is independent of choice of representative of $p$.*

*Proof.* Note that $\mu(p) \in F_i$ if and only if $\mu_{T^k}(z_1, \ldots, z_k) \in (d\pi)^*(F_i)$. Now suppose $x = (x_1, \ldots, x_k) \in (d\pi)^*(C)$ is in $(d\pi)^*(F_i)$. Then $x = (d\pi)^*(y)$ for some $y \in F_i$, and hence

$$\langle x, e_i \rangle = \langle (d\pi)^*(y), e_i \rangle = \langle y, (d\pi)^*(e_i) \rangle = \langle y, v_i \rangle = \lambda_i$$

since $y \in F_i$. Therefore if $\mu_{T^k}(z_1, \ldots, z_k) \in (d\pi)^*(F_i)$, its $i$-th coordinate is $\lambda_i$, and hence

$$\lambda_i - \frac{1}{2}|z_i|^2 = \lambda_i$$

by definition of $\mu_{T^k}$, showing $z_i = 0$.

Conversely, assume $z_i = 0$. Then the $i$-th coordinate of $\mu_{T^k}(z_1, \ldots, z_k)$ is $\lambda_i$. Since $\mu_{T^k}(z_1, \ldots, z_k) = (d\pi)^*(\mu(p))$, we see that

$$\langle \mu(p), v_i \rangle = \langle \mu_{T^k}(z_1, \ldots, z_k), e_i \rangle = \lambda_i,$$

and since we assume that

$$F_i = C \cap \{x \in (\mathbb{R}^n)^* : \langle x, v_i \rangle = \lambda_i\},$$

it immediately follows that $\mu(p) \in F_i$. □

Now we proceed to the proof of Proposition 4.3.4. We will explicitly construct $E$ as a quotient of a vector bundle over $\mu_N^{-1}(0)$ (recall that $M = \mu_N^{-1}(0)/N$), and then use the previous lemma and an explicit description of $N$ to compute the $T^n$-action on the fiber over each fixed point.

*Proof of Proposition 4.3.4.* Assume without loss of generality (by reordering the faces) that $i = 1$. Let $\bar{E} = \mathbb{C} \times \mu_N^{-1}(0) \overset{p_2}{\to} \mu_N^{-1}(0)$ be the trivial vector bundle with $p_2$ the projection on the second coordinate, and let $T^k$ act on $\bar{E}$ by

$$(u_1, \ldots, u_k) \cdot (w, (z_1, \ldots, z_k)) = (u_1 w, (u_1 z_1, \ldots, u_k z_k)).$$

Then $\bar{E}$ is a $T^k$-equivariant complex vector bundle of rank 1. Then define $E = \bar{E}/N$; by Lemma A.2, this is a vector bundle over $M = \mu_N^{-1}(0)/N$.

First we show that $E$ restricted to $\mu^{-1}(F_1)$ is isomorphic to the normal bundle of $\mu^{-1}(F_1)$ in $M$. Since $\mu_N^{-1}(0)$ is a submanifold of $\mathbb{C}^k$, its tangent bundle is a subbundle of $T\mathbb{C}^k = \mathbb{C}^k \times \mathbb{C}^k$. Let $\phi : T\mu_N^{-1}(0) \to \bar{E}$ then be defined as the restriction of the map $p_1 \times \mathrm{id}_{\mathbb{C}^k} : T\mathbb{C}^k = \mathbb{C}^k \times \mathbb{C}^k \to \mathbb{C} \times \mathbb{C}^k$, given by projecting onto the first coordinate in the first factor, and the identity in the second factor. Then $\phi$ is $T^k$-equivariant, since the action on $T\mathbb{C}^k = \mathbb{C}^k \times \mathbb{C}^k$ is just the diagonal action. Therefore $\phi$ descends to a map $\psi : (T\mu_N^{-1}(0))/N \to E$. Since $(T\mu_N^{-1}(0))/N \cong T(\mu_N^{-1}(0)/N) = TM$, it now suffices to show that for $p \in \mu^{-1}(F_1)$, $\psi_p : T_pM \to E_p$ is surjective and has kernel $T_p\mu^{-1}(F_1)$. The kernel of $\psi_p$ certainly contains $T_p\mu^{-1}(F_1)$, since $\mu^{-1}(F_1)$ consists of those $q = [z_1, \ldots, z_k] \in M$ with $z_1 = 0$.

On the other hand, $\phi$ is surjective at $\hat{p}$, where $p = [\hat{p}]$. To show this, we wish to find a $w \in \ker(d\mu_N)_{\hat{p}} \subset \mathbb{C}^k$ such that $w_1 \neq 0$, since $\phi$ was given by projecting on the first coordinate. To do this, first observe that $T_{\hat{p}}\mu_N^{-1}(0) = \ker(d\mu_N)_{\hat{p}}$. Since $\mu_N = \varphi^* \circ \mu_{T^k}$, we have

$$\ker(d\mu_N)_{\hat{p}} = \ker d(\varphi^* \circ \mu_{T^k})_{\hat{p}} = (d\mu_{T^k})_{\hat{p}}^{-1}(\ker d(\varphi^*)_{\mu_{T^k}(\hat{p})}).$$

If $\hat{p} = (z_1, \ldots, z_k)$, $z_j = x_j + iy_j$, and $w = a + ib \in \mathbb{C}^k$ is a tangent vector to $\hat{p}$ in $\mathbb{C}^k$, then

$$(d\mu_{T^k})_{\hat{p}}\, w = -\sum_{j=1}^{k}(x_j a_j + y_j b_j)e_j^*$$

But now recall that $p \in \mu^{-1}(F_1)$, and so $z_1 = 0$, and hence $a_1, b_1 = 0$. Therefore taking $w = (1, 0, \ldots, 0) \in \mathbb{C}^k$ gives

$$(d\mu_{T^k})_{\hat{p}}\, w = 0 \in \ker d(\phi^*)_{\mu_{T^k}(\hat{p})}$$

and hence $w \in \ker(d\mu_N)_{\hat{p}}$ with $w_1 \neq 0$. Therefore $\psi_p$ is surjective as well, and we obtain fiberwise isomorphisms $T_pM/T_p\mu^{-1}(F_1) \to E_p$, i.e., an isomorphism between the normal bundle of $\mu^{-1}(F_1)$ and $E$.

We now move to the computation of the action of $T^n$ on $E$. Let $v \in M$ be a fixed point of the $T^n$-action. Then there exist indices $i_1 < \cdots < i_n$ such that $v \in \mu^{-1}(F_{i_1} \cap \cdots \cap F_{i_n})$. Fix some representative $v = [z_1, \ldots, z_k]$, and take $[w, (z_1, \ldots, z_k)] \in E|_v$. We first compute the action of $T^k/N$. Since $v \in \mu^{-1}(F_{i_l})$ for $l = 1, \ldots, n$, we know that $z_{i_l} = 0$. Now write, for $j \neq i_1, \ldots, i_n$,

$$v_j = \sum_{l=1}^{n} a_{jl} v_{i_l}.$$

For $x = (x_1, \ldots, x_k) \in \mathbb{R}^k$, we have

$$(d\pi)(x) = \sum_{i=1}^{n} x_i v_i = \sum_{j \neq i_1, \ldots, i_n} x_j \sum_{l=1}^{n} a_{jl} v_{i_l} + \sum_{l=1}^{n} x_{i_l} v_{i_l} = \sum_{l=1}^{n}\left(x_{i_l} + \sum_{j \neq i_1, \ldots, i_n} x_j a_{jl}\right) v_{i_l}.$$

Therefore if $x \in \ker(d\pi) = \mathfrak{n}$, we have for each $l = 1, \ldots, n$,

$$x_{i_l} + \sum_{j \neq i_1, \ldots, i_n} x_j a_{jl} = 0.$$

This implies that $N$ consists of those elements $(u_1, \ldots, u_k) \in T^n$ such that

$$u_{i_l} \prod_{j \neq i_1, \ldots, i_n} u_j^{a_{jl}} = 1$$

for $l = 1, \ldots, n$. Thus, if $[u_1, \ldots, u_k] \in T^k/N$, then $\eta = (\eta_1, \ldots, \eta_k)$ defined by $\eta_j = u_j$ for $j \neq i_1, \ldots, i_n$ and

$$\eta_{i_l} = \left(\prod_{j \neq i_1, \ldots, i_n} u_j^{a_{jl}}\right)^{-1}$$

lies in $N$. Therefore

$$\begin{aligned}[u_1, \dots, u_k] \cdot [w, (z_1, \dots, z_k)] &= [u_1 w, (u_1 z_1, \dots, u_k z_k)] = [u_1 w, \eta \cdot (z_1, \dots, z_k)] \\ &= [w', (z_1, \dots, z_k)]\end{aligned}$$

where

$$w' = \begin{cases} w & \text{if } i_1, \dots, i_n \neq 1 \\ u_1 w \prod_{j \neq i_1, \dots, i_n} u_j^{a_{j1}} & \text{otherwise.} \end{cases}$$

Recall here that we assumed $i_1 < \cdots < i_n$, and so it suffices to check whether $i_1$ is 1 or not, i.e., whether $v$ is in $\mu^{-1}(F_1)$ or not. Thus we have finished part 2 of the lemma: if $v \notin \mu^{-1}(F_1)$, then $T^k/N \cong T^n$ acts trivially on $E|_v$.

We now turn to the third part. From this point onwards we assume that $i_1 = 1$, so $v \in \mu^{-1}(F_1)$. Let $d_1, \dots, d_n$ be the standard basis of $\mathbb{R}^n$, and write

$$d_j = \sum_{l=1}^{n} b_{jl} v_{i_l}.$$

Define a map $s : T^n \to T^k$ by $ds(v_{i_l}) = e_{i_l}$ for $l = 1, \dots, n$; this gives rise to a well-defined map since the $v_{i_1}, \dots, v_{i_n}$ form a $\mathbb{Z}$-basis for $\mathbb{Z}^n$. Then composing $s$ with the quotient map $T^k \to T^k/N$ gives a map $S : T^n \to T^k/N$. We show that $S = \pi^{-1}$, where we view $\pi$ as a map $T^k/N \to T^n$. First let $[u_1, \dots, u_k] \in T^k/N$; then

$$S(\pi[u_1, \dots, u_k]) = S(\alpha_1, \dots, \alpha_n)$$

where

$$\alpha_i = \prod_{j=1}^{k} u_j^{(v_j)_i}$$

with $(v_j)_i$ the $i$-th coordinate of $v_j \in \mathbb{Z}^n$. Therefore $\alpha = (\alpha_1, \dots, \alpha_n)$ is the product

$$\alpha = \prod_{j=1}^{k} u_j^{v_j}$$

where $u_j^{v_j} = (u_j^{(v_j)_1}, \dots, u_j^{(v_j)_n})$, and hence

$$S(\alpha) = \prod_{j=1}^{k} S(u_j^{v_j}) = \prod_{l=1}^{n} S(u_{i_l}^{v_{i_l}}) \cdot \prod_{j \neq i_1, \dots, i_n} \prod_{l=1}^{n} S(u_j^{a_{jl} v_{i_l}}).$$

For $\theta \in T^1$ and $l = 1, \dots, n$, we have $S(\theta^{v_{i_l}}) = [1, \dots, 1, \theta, 1, \dots, 1]$ with $\theta$ in the $i_l$-th position, and hence $S(\alpha) = \eta = [\eta_1, \dots, \eta_k]$ defined by

$$\begin{aligned} \eta_j &= 1 && j \neq i_1, \dots, i_n, \\ \eta_{i_l} &= u_{i_l} \prod_{j \neq i_1, \dots, i_n} u_j^{a_{jl}} && l = 1, \dots, n \end{aligned}$$

From the earlier description of $N$, we see that $\xi = (\xi_1, \dots, \xi_k)$ with

$$\xi_j = u_j, \quad j \neq i_1, \dots, i_n, \quad \xi_{i_l} \prod_{j \neq i_1, \dots, i_n} u_j^{a_{jl}} = 1$$

is such that $\xi \in N$, and hence $\eta = [u_1, \dots, u_k]$. Thus we have shown that $S \circ \pi = \mathrm{id}_{T^k/N}$.

Next, we show that $\pi \circ S = \mathrm{id}_{T^n}$. Let $\alpha = (\alpha_1, \dots, \alpha_n) \in T^n$, and recall that we wrote

$$d_i = \sum_{l=1}^{n} b_{jl} v_{i_l}$$

for $d_1, \dots, d_n$ the standard basis of $\mathbb{R}^n$. Then $S(\alpha) = [u_1, \dots, u_k]$ with $u_j = 1$ for $j \neq i_1, \dots, i_n$, and $u_{i_l} = \prod_{i=1}^n \alpha_i^{b_{il}}$. Therefore

$$\pi(S(\alpha)) = \prod_{l=1}^{n} u_{i_l}^{v_{i_l}} = \prod_{l=1}^{n} \left( \prod_{j=1}^{n} \alpha_j^{b_{jl}} \right)^{v_{i_l}} = \prod_{j=1}^{n} \prod_{l=1}^{n} \alpha_j^{b_{jl} v_{i_l}} = \prod_{j=1}^{n} \alpha_j^{d_j} = \alpha,$$

and we conclude that $S = \pi^{-1}$.

Now we return to the action of $T^n$ on $E|_v$. If $\alpha = (\alpha_1, \dots, \alpha_n) \in T^n$, then $S(\alpha) = [u_1, \dots, u_k]$, with $u_j = 1$ for $j \neq i_1, \dots, i_n$, and

$$u_{i_l} = \prod_{j=1}^{n} \alpha_j^{b_{jl}}.$$

Then $\alpha$ acts on $E|_v$ as

$$\alpha \cdot [w, (z_1, \dots, z_k)] = \left[ u_1 w \prod_{j \neq i_1, \dots, i_n} u_j^{a_{j1}}, (z_1, \dots, z_k) \right]$$

but all these $u_j$ are 1, so this is equal to

$$[u_1 w, (z_1, \dots, z_k)] = \left[ \prod_{j=1}^{n} \alpha_j^{b_{j1}} w, (z_1, \dots, z_k) \right].$$

Therefore the weights of the $T^n$-action on $E|_v$ are $b_{11}, \dots, b_{n1}$. $\square$

Having finished the proof of Proposition 4.3.4, we can now return to computing equivariant Euler classes.

**Lemma 4.3.6.** *Choose $i \in \{1, \dots, n\}$ and let $E \to M$ be the equivariant vector bundle constructed in Proposition 4.3.4. Let $v \in F$ be a fixed point with $v \in \mu^{-1}(F_{i_1} \cap \cdots \cap F_{i_n})$ where $i = i_s$. Furthermore, let $u_1, \dots, u_n \in \mathbb{Z}^n$ be such that the edges at $\mu(v)$ are of the form $\mu(v) - t u_l$, $t \geq 0$, ordered such that $\mu(v) - t u_l \notin F_{i_l}$ for each $t > 0$, and the greatest common divisor of the coordinates of each $u_j$ is 1. Then the weight vector of the $T^n$-action on $E|_v$ is $u_s$.*

*Proof.* It suffices to show that for each $u_j$, we have

$$\langle u_j, v_{i_l} \rangle = \delta_{jl}$$

so that $u_1, \ldots, u_n$ is the dual basis to $v_{i_1}, \ldots, v_{i_n}$, for then it follows that

$$\langle u_j, d_r \rangle = \langle u_j, \sum_{l=1}^{n} b_{rl} v_{i_l} \rangle = b_{rj}$$

and hence $u_j = (b_{1j}, b_{2j}, \ldots, b_{nj})$, as $d_1, \ldots, d_n$ was the standard basis of $\mathbb{R}^n$. But the $v_{i_l}$ are constructed as the dual basis to $u_l$ (see the discussion at the start of this section), so we are done. □

**Lemma 4.3.7.** *We write $H_T^*(\{v\}) = \mathbb{Z}[x_{vj} : j = 1, \ldots, n]$ for $v \in F$ (as in Example 4.1.5), and identify $H_T^*(F) = \oplus_{v \in F} H_T^*(\{v\})$ as rings. The equivariant Euler class of the bundle $E$ constructed in Proposition 4.3.4, restricted to the fixed point set $F \subset M$, is given by*

$$e_T(E)|_F = \sum_{v \in F \cap \mu^{-1}(F_i)} \sum_{j=1}^{n} -(u_{vi})_j x_{vj}$$

*where $u_{vi}$ is the direction of the edge at $\mu(v)$ not pointing into the face $F_i$, and $(u_{vi})_j$ is its $j$-th component.*

The above lemma gives us the last result we need to prove the following theorem.

**Theorem 4.3.8.** *Let $A = \mu^{-1}(F_{j_1} \cap \cdots \cap F_{j_m})$ be the intersection of the $m$ faces $F_{j_1}, \ldots, F_{j_m}$, and assume $A$ is non-empty. Then there exists a $T^n$-equivariant vector bundle $E \to M$ such that $E|_A$ is equivariantly isomorphic to the normal bundle $\nu$ of $A$. Furthermore, for $v \in F$ a fixed point, we have*

$$e_T(E)|_v = \begin{cases} 0 & \text{if } v \notin A \\ \prod_{l=1}^{m} \left( \sum_{i=1}^{n} -(u_{vl})_i x_{vi} \right) & \text{if } v \in A \end{cases}$$

*where $-u_{vl} \in \mathbb{Z}^n$ is the direction of the edge at $\mu(v)$ not pointing into $F_{j_l}$ (recall that our edges are of the form $\mu(v) - tu_{vl}$ with $t \geq 0$).*

*Proof.* For $l = 1, \ldots, m$, let $E_l$ be the bundle constructed in Proposition 4.3.4. Now

$$A = \bigcap_{l=1}^{m} \mu^{-1}(F_{j_l}),$$

and these intersections are pairwise transversal; hence the normal bundle $\nu$ of $A$ is the direct sum of normal bundles of each of the $\mu^{-1}(F_{j_l})$ (restricted to $A$). Setting $E = \sum_{l=1}^{m} E_l$, we obtain

$$E|_A = \left( \sum_{l=1}^{m} E_l \right) \Bigg|_A \cong \sum_{l=1}^{m} (\nu_{\mu^{-1}(F_{j_l})})|_A = \nu$$

as $T^n$-equivariant vector bundles.

To compute the equivariant Euler class of $E$ restricted to a fixed point $v$, note that the Euler class of a sum of complex line bundles is the product of Euler classes. In particular, if $v \in F$ is a fixed point not in at least one of the $F_{j_l}$, then $e_T(E_l)|_v = 0$, and hence $e_T(E)|_v = 0$. Otherwise, the above formula for $e_T(E)|_v$ follows directly from the previous lemma. □

**Example 4.3.9** ($\mathbb{CP}^1$)**.** We again compute the equivariant cohomology of $\mathbb{CP}^1$, now using Theorem 4.2.5 and Theorem 4.3.8. Let $T^1$ act on $\mathbb{CP}^1$ as before, with moment map $\mu$ defined by

$$\mu([z_0 : z_1]) = -\frac{1}{2}\frac{|z_1|^2}{|z_0|^2 + |z_1|^2}.$$

We identify $H_T^*(\{v_0, v_1\}) = \mathbb{Z}[x_0] \oplus \mathbb{Z}[x_1]$. Let $X = -1 \in \mathfrak{t} \cong \mathbb{R}$, then $f = \langle \mu, X \rangle$ is a Morse function, and Theorem 4.2.5 applies. The polytope $\mu(M)$ is $[-1/2, 0] \subset (\mathbb{R})^*$, so if $v_0 = [1 : 0]$, $v_1 = [0 : 1]$ are the fixed points of the $T^1$-action, the flow-up faces for $v_0$ and $v_1$ relative to $X$ are $A_0 = [-1/2, 0]$ and $A_1 = \{-1/2\}$, respectively. The equivariant Euler class, for the normal bundle to $\mu^{-1}(A_0)$, restricted to $\{v_0, v_1\}$, is therefore clearly the identity $(1, 1) \in H_T^*(\{v_0, v_1\})$ . By Theorem 4.3.8, if $\nu^{v_1}$ is the normal bundle to $\{v_1\}$ (which is the preimage of the flow-up face for $\{v_1\}$ under $\mu$), then

$$e_T(\nu^{v_1}) = x_1$$

(compare with Example 4.2.1). Therefore the equivariant cohomology ring $H_T^*(\mathbb{CP}^1)$ has image generated by $\{(1, 1), x_1\} \in H_T^*(\{v_0, v_1\})$ as $H_T^*$-module, and we have succeeded in computing the ring structure.

**Example 4.3.10** ($\mathbb{CP}^2$)**.** We now finish Example 4.2.2. Let $M = \mathbb{CP}^2$ be the projective plane, with $v_0 = [1 : 0 : 0]$, $v_1 = [0 : 1 : 0]$, $v_2 = [0 : 0 : 1]$ as before, and choose $X \in \mathfrak{t}$ such that $\{\exp(tX) : t \in \mathbb{R}\}$ is dense in $T^2$, and such that for $f = \langle \mu, X \rangle$ has $f(v_0) < f(v_1) < f(v_2)$ (it suffices to perturb $(0, -1) \in \mathfrak{t}$ slightly in a clockwise fashion). The polytope is the triangle with corners $(0, 0)$, $(-1/2, 0)$ and $(0, -1/2)$, which are the images of $v_0, v_1, v_2$ respectively. For each $i = 0, 1, 2$, let $A_i$ be the flow-up face for $v_i$ relative to $X$, $\nu^i$ the normal bundle to $\mu^{-1}(A_i)$, and $E_i$ its extension as a bundle to $M$ (from Theorem 4.3.8). Explicitly, $A_0$ is the entire moment polytope, $A_1$ is the edge from $(-1/2, 0)$ to $(0, -1/2)$, and $A_2$ consists of just the point $(0, -1/2)$. By Theorem 4.2.5, $H_T^*(M)$ is a free $H_T^*$-module on $e_T(E_i)$, $i = 0, 1, 2$, and the inclusion-induced map $H_T^*(M) \to H_T^*(F)$ is injective, with $F = \{v_0, v_1, v_2\}$ the set of fixed points. By Theorem 4.3.8 we have equivariant Euler classes $e_T(E_i)|_F \in H_T^*(F) = \oplus_{i=0}^2 \mathbb{Z}[x_{i1}, x_{i2}]$ (with the obvious $H_T^* = \mathbb{Z}[x_1, x_2]$-module structure), given by

$$e_T(E_0)|_F = (1, 1, 1), \quad e_T(E_1)|_F = x_{11} + x_{22}, \quad e_T(E_2)|_F = (x_{22})(x_{22} - x_{21}).$$

To see that these are the equivariant Euler classes, note that if $p_i = \mu(v_i)$, the blue arrows in Figure 4.2 indicate the directions of the vectors $u_i$. For $p_1$, for instance, the only negative edge (with respect to our choice of $X$) corresponds to the direction

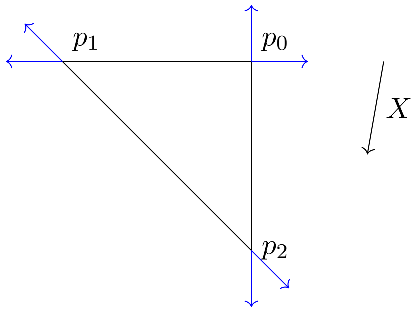


Figure 4.2.: The moment polytope for the complex projective plane $\mathbb{CP}^2$. The blue arrows at each corner $p_i$ indicate the direction of the vectors $u \in \mathbb{Z}^2$ such that the edges are of the form $p_i - tu$, $t \geq 0$. The flow-up faces relative to the direction $X$

$(-1, 0)$, as the flow-up face of $p_1$ is the edge from $p_1$ to $p_2$. Therefore, the equivariant Euler class $e_T(E_1)$ restricted to $v_1$ must be $-(-1)x_{11} - (0)x_{12} = x_{11}$. Similarly, $e_T(E_1)$ restricted to $v_2$ must be $-(0)x_{12} - (-1)x_{22}$, as the only negative edge at $v_2$ corresponds to the direction $(0, -1)$. When restricting $e_T(E_2)$ to $v_0$ and $v_1$ we obtain 0, as neither of these fixed points is in the flow-up face of $v_2$; but $e_T(E_2)$ restricted to $v_2$ is equal to $(-(-x_{22}))(-(x_{21}) - (-x_{22}))$, as the vectors at $v_2$ are $(0, -1)$ and $(1, -1)$. Thus we have now determined the image of $H_T^*(M) \to H_T^*(F)$, and can explicitly compute products of the generators of $H_T^*(M)$. As an example, we take

$$(e_T(E_1)|_F)^2 = x_{11}^2 + x_{22}^2 = e_T(E_2)|_F + x_{22}x_{21} + x_{11}^2 = e_T(E_2)|_F + x_1(e_T(E_1)|_F).$$

Therefore in $H_T^*(M)$, we have

$$(e_T(E_1))^2 = e_T(E_2) + x_1(e_T(E_1)).$$

By Theorem 4.2.7, the Euler classes $e(E_i)$ form an additive basis for $H^*(M)$. Note that $e(E_0) = 1$, and the above computation shows that

$$(e(E_1))^2 = e(E_2)$$

since the product $x_1(e_T(E_1))$ vanishes when passing to cohomology. Now

$$e_T(E_1)e_T(E_2) = (x_{11} + x_{22})(x_{22})(x_{22} - x_{21}) = x_2 e_T(E_2)$$

since $x_{11}x_{22} = 0 = x_{11}x_{21}$, and hence $e(E_1)e(E_2) = 0$ in $H^*(M)$. We have thus recovered the well-known isomorphism

$$H^*(\mathbb{CP}^2) \cong \mathbb{Z}[y]/(y^3)$$

by sending $e(E_1)$ to $y$ and $e(E_2)$ to $y^2$, where we consider $y$ to be of degree 2.

**Example 4.3.11** (Hirzebruch surfaces)**.** Let $W_k$ be the $k$-th Hirzebruch surface, corresponding to the polytope with vertices $(0, 0)$, $(-(k+1)/2, 0)$, $(0, -1/2)$, $(-1/2, -1/2)$,

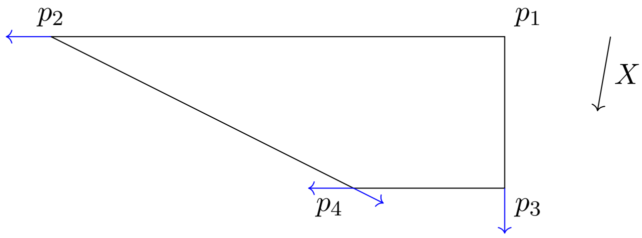


Figure 4.3.: The moment polytope for the $k$-th Hirzebruch surface $W_k$. The blue arrows indicate the directions of the edges whose coefficients we use for computing the equivariant Euler classes of the normal bundles of the flow-up faces, relative to $X$.

with corresponding fixed points labelled $v_1$ up to $v_4$, respectively, and let $p_i = \mu(v_i)$ for $i = 1, \ldots, 4$. Let $X \in \mathfrak{t}$ be a small clockwise perturbation of $(0, -1)$. Then the basis elements for the image of $H_T^*(W_k) \to H_T^*(F) = \oplus_{i=1}^4 \mathbb{Z}[x_{i1}, x_{i2}]$ are

$$a_1 = (1,1,1,1), \quad a_2 = x_{21} + x_{41}, \quad a_3 = x_{32} - kx_{41} + x_{42}, \quad a_4 = (-kx_{41} + x_{42})(x_{41}),$$

that is, $H_T^*(W_k)$ is isomorphic to the free $H_T^*$-submodule of $H_T^*(F)$ given by

$$H_T^*(W_k) \cong H_T^*\langle a_1, a_2, a_3, a_4 \rangle \subseteq H_T^*(F).$$

To see this, recall from Example 4.2.4 that the flow-up face for $p_1$ is the entire polytope, the flow-up face for $p_2$ is the edge from $p_2$ to $p_4$, the flow-up face for $p_3$ is the edge from $p_3$ to $p_4$, and the flow-up face for $p_4$ consists just of $p_4$ itself. In Figure 4.3, the directions of the negative edges at each point $p_i$ are shown. The generator $a_3$, for instance, is the equivariant Euler class corresponding to $p_3$: at $p_3$, the vector determining the edge not pointing into the flow-up face of $p_3$ is $(0, -1)$, and hence $a_3$ restricted to $v_3$ is given by $-(0x_{31} - 1x_{32}) = x_{32}$. Similarly, at $p_4$, the vector determining the edge not pointing into the flow-up face of $p_3$ is $(k, -1)$, and hence $a_3$ restricted to $v_4$ is $-(kx_{41} - 1x_{42})$. The rank of $H^*(M)$ is now 1, 2 and 1 in degrees 0, 2 and 4, respectively. To compute the ring structure, note for instance that

$$\begin{aligned} a_3^2 = (x_{32} - kx_{41} + x_{42})^2 &= x_{32}^2 + (-kx_{41} + x_{42})(-kx_{41} + x_{42}) \\ &= x_{32}^2 + x_{42}(-kx_{41} + x_{42}) + (-kx_{41})(-kx_{41} + x_{42}) \\ &= x_2(x_{32} - kx_{41} + x_{42}) - k(x_{41})(-kx_{41} + x_{42}) \\ &= x_2 a_3 - k a_4 \end{aligned}$$

The first term disappears when passing to ordinary cohomology; hence the image of $a_3$ in $H^2(W_k)$ squares to $-k$ times the image of $a_4$ in $H^4(W_k)$. We also have the relations

$$\begin{aligned} &a_1 a_i = a_i && a_2 a_3 = a_4, && a_2^2 = 0, \\ &a_3^2 = -ka_4 + x_2 a_3, && a_2 a_4 = x_1 a_4, \text{ and} && a_3 a_4 = -(kx_1 - x_2)a_4. \end{aligned}$$

Hence the cohomology ring of $W_k$ is given by

$$H^*(W_k) \cong \mathbb{Z}[b_1, b_2]/(b_1^2, b_2^2 + kb_1b_2).$$

by sending (the restriction to $H^*(W_k)$ of) $a_2$ to $b_1$ and $a_3$ to $b_2$, where $b_1$ and $b_2$ are considered to be of degree 2. Note that in the latter ring, we also have $b_2^3 = 0$, since

$$b_2^3 = b_2(-kb_1b_2) = -kb_1(-kb_1b_2) = 0.$$

While the cohomology ring may appear to be distinct for each $k$, this is actually not the case; whenever $k \equiv k' \mod 2$, $H^*(W_k)$ and $H^*(W_{k'})$ are isomorphic. If $k' = k + 2l$ with $l \in \mathbb{Z}$, an explicit isomorphism is given by

$$\begin{aligned} H^*(W_k) \cong \mathbb{Z}[b_1, b_2]/(b_1^2, b_2^2 + kb_1b_2) &\to \mathbb{Z}[c_1, c_2]/(c_1^2, c_2^2 + k'c_1c_2) \cong H^*(W_{k'}) \\ b_1 &\mapsto c_1 \\ b_2 &\mapsto c_2 + lc_1. \end{aligned}$$

To see this is actually well-defined, note that $b_2^2 + kb_1b_2$ is sent to $c_2^2 + 2lc_1c_2 + kc_1c_2 = c_2^2 + k'c_1c_2 = 0$. More is actually true: $W_k$ and $W_{k'}$ are known to be diffeomorphic whenever $k \equiv k' \mod 2$, with $W_2 \cong \mathbb{CP}^1 \times \mathbb{CP}^1$ and $W_1 \cong \mathbb{CP}^2 \# \overline{\mathbb{CP}}^2$, and $W_1$ and $W_2$ are not homeomorphic [5, Ch. II]. Thus $W_1$ and $W_2$ are the two distinct $\mathbb{CP}^1$-bundles over $\mathbb{CP}^1$.

It is worth noting that there is a more geometric interpretation of $b_2$ squaring to $-kb_1b_2 \in H^4(W_k)$. If $A_3$ denotes the preimage of the flow-up face of $v_3$ under the moment map, then its fundamental class $[A_3] \in H_2(W_k)$ is the Poincare dual to $b_2$. The intersection product $[A_3] \cap [A_3]$ is then Poincare dual to $b_2^2$, and so $[A_3] \cap [A_3] = -k[v]$ where $v \in W_2$ is any point, and $[v]$ represents the generator of $H_0(W_k)$. Therefore, if two embeddings $\phi, \psi : A_3 \to W_2$ intersect transversally, and both are isotopic to the standard inclusion $A_3 \to W_2$, then the fundamental class of the intersection $[\phi(A_3) \pitchfork \psi(A_3)]$ is homologous to $-k$ times a point. We now use the moment polytope description of $W_k$ to actually produce such $\phi, \psi$ for which $\phi(A_3) \pitchfork \psi(A_3)$ consists of precisely $k$ points (and we ignore the matter of orientations). Denote $\mu(W_k) = \Delta$, and let $s : \Delta \to W_k$ be a section of the moment map, that is, a continuous map such that $\mu \circ s = \mathrm{id}_\Delta$.

The existence of such a map follows from the Delzant construction of a symplectic toric manifold: with the notation as in the construction, one may define $s : \pi^*(C) \subset (\mathbb{R}^k)^* \to M = \overline{M}/N$ by

$$s(w_1, \ldots, w_k) = [\sqrt{2\lambda_1 - 2w_1}, \ldots, \sqrt{2\lambda_k - 2w_k}].$$

The moment map in the Delzant construction is

$$\mu(z_1, \ldots, z_k) = (\lambda_1, \ldots, \lambda_k) - \frac{1}{2}(|z_1|^2, \ldots, |z_k|^2),$$

so we certainly have $\mu \circ s = \mathrm{id}$. Note that $s$ cannot be "smooth" on the boundary of $\Delta$, as then the chain rule would apply, but $d\mu$ is not surjective at points with non-trivial stabilizer.

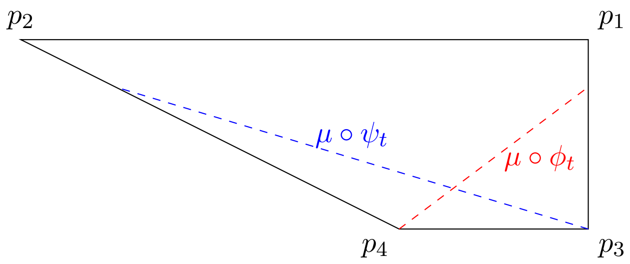


Figure 4.4.: The moment polytope of the $k$-th Hirzebruch surface $W_k$, along with line segments indicating the images of $\mu \circ \phi_t$ and $\mu \circ \psi_t$.

Returning to $W_k$, Define subtori

$$T_1 = \{(1, u) : u \in T^1\}, \quad T_2 = \{(u^k, u) : u \in T^1\}$$

of $T^2$; then both $T_1$ and $T_2$ act transitively on each fiber $\mu^{-1}(c)$, with $c$ in the interior of the flow-up face of $v_3$. Furthermore, $T_1$ and $T_2$ are distinct subtori because $k > 0$. We then define the families of smooth maps

$$\phi_t : A_3 \to W_k$$
$$(1, u) \cdot s(r, -1/2) \mapsto (1, u) \cdot s(r, -1/2 + (1/2 + r)t)$$

and

$$\psi_t : A_3 \to W_k$$
$$(u^k, u) \cdot s(r, -1/2) \mapsto (u^k, u) \cdot s(r - kt/2, -1/2 - tr)$$

for $0 \leq t \leq 1$ and $-1/2 \leq r \leq 0$. These families correspond to "sliding the edge from $(-1/2, -1/2)$ to $(-1/2, 0)$" along the edge from $(-1/2, 0)$ to $(0, 0)$ for $\phi_t$, and along the edge from $(-1/2, -1/2)$ to $(-(k+1)/2, 0)$ for $\psi_t$: the image of $\mu \circ \phi_t$ is the line segment $L_t$ from $(-1/2, -1/2)$ to $(0, -1/2 + t/2)$, whereas the image of $\mu \circ \psi_t$ is the line segment $L'_t$ from $(-1/2 - kt/2, -1/2 + t/2)$ to $(0, -1/2)$, as shown in Figure 4.4. The image of $\phi_t$ is the set of all $u \cdot s(p)$ with $p \in L_t$ and $u \in T_1$, whereas the image of $\psi_t$ is the set of all $u \cdot s(p)$ with $p \in L'_t$ and $u \in T_2$. The well-definedness of $\psi_t$ (and similarly for $\phi_t$) depends on the choice of the subtori that act, in the following sense: if $u, u' \in T_1$ are distinct elements such that $u \cdot s(r, -1/2) = u' \cdot s(r, -1/2)$, so $u(u')^{-1}$ is in the stabilizer of $s(r, -1/2)$, then we must have

$$u \cdot s(r, -1/2 + (1/2 + r)t) = u' \cdot s(r, -1/2 + (1/2 + r)t),$$

that is, $u(u')^{-1}$ must be in the stabilizer of $s(r, -1/2 + (1/2 + r)t)$. The only points in $A_3$ that have non-trivial stabilizer (with respect to the $T_1$-action) are the fixed points $v_3 = s(p_3) = s(0, -1/2)$ and $v_4 = s(p_4) = s(-1/2, -1/2)$. However, $s(-1/2, -1/2 + (1/2 - 1/2)t)$ is constant (as a function of $t$), and $s(0, -1/2 + t/2)$ has stabilizer $T_1$ with respect to the $T^2$-action for all $0 < t < 1$, and so $\phi_t(v_3)$ is well-defined.

Checking (and smoothness) of $\phi_t$ and $\psi_t$ is left as an exercise for the reader, and can be done explicitly in local coordinates on $W_k$ and $A_3$. The point is now that for $0 < t < 1$, $\phi_t(A_3)$ and $\psi_t(A_3)$ intersect transversally, in exactly $k$ points: if the line segments $L_t$ and $L'_t$ intersect in an interior point $p \in \Delta$, $\phi_t(A_3)$ and $\psi_t(A^3)$ intersect in $(T_1 \cap T_2) \cdot s(p)$, and $T_1 \cap T_2$ consists of exactly $k$ points.

# A. Vector bundles and quotients

In this appendix, we prove the following theorem, which allows us to take quotients of equivariant vector bundles.

**Theorem A.1.** *Let $G, H$ be Lie groups, $\pi : P \to M$ a principal $G$-bundle. Suppose $H$ is compact, with a free left-action on $P$ and $M$, such that $\pi$ is $H$-equivariant. Then $P/H$ and $M/H$ admit smooth structures such that the projection maps $q_P : P \to P/H$ and $q_M : M \to M/H$ are submersions, and $\pi$ descends to a smooth map $\bar{\pi} : P/H \to M/H$. If the actions of $G$ and $H$ on $P$ commute, that is, for every $g \in G$, $h \in H$, $x \in P$,*

$$(hx)g = h(xg),$$

*then $\bar{\pi}$ gives $P/H$ the structure of a principal $G$-bundle over $M/H$.*

The argument we give for the theorem is an adaptation of the standard argument for the quotient of a manifold by a free proper Lie group admitting a unique smooth structure such that the quotient map is a surjection; see [12, Thm. 21.10].

*Proof.* Let $x_0 \in P$, and define $\alpha : G \to P$ and $\beta : H \to P$ by

$$\alpha(g) = x_0 g, \quad \text{and} \quad \beta(h) = hx_0.$$

Furthermore, we denote by $\mathfrak{g}$ and $\mathfrak{h}$ the Lie algebras of $G$ and $H$ respectively. Since $\pi$ is $G$-invariant, $(d\alpha)_{e_G}(\mathfrak{g}) \subseteq \ker \pi$. On the other hand, as the action of $H$ is free on $M$, $d(\pi \circ \beta)_{e_H}$ is injective; hence $\operatorname{im}(d\beta)_{e_H} \cap \ker \pi = \{0\}$. Therefore $(d\alpha)_{e_G}(\mathfrak{g}) \cap (d\beta)_{e_H}(\mathfrak{h}) = \{0\}$. Now choose a submanifold $S$ of $P$, containing $x_0$, such that

$$(d\alpha)_{e_G}(\mathfrak{g}) \oplus (d\beta)_{e_H}(\mathfrak{h}) \oplus T_{x_0}S = T_{x_0}P.$$

Define $f : H \times S \times G \to P$ by $f(h, x, g) = hxg$ (here we use commutativity for this to be well-defined). Then

$$df_{(e_H, x_0, e_G)}(u, v, w) = (d\beta)_{e_H}(u) + v + (d\alpha)_{e_G}(v),$$

and so $df_{(e_H, x_0, e_G)}$ is an isomorphism. We now prove that we may shrink $S$ enough so that $f$ becomes injective. Indeed, suppose not; then there exist sequences $(h_k, x_k, g_k) \neq (h'_k, x'_k, g'_k)$ such that

$$h_k x_k g_k = h'_k x'_k g'_k, \text{ and } x_k, x'_k \to x_0.$$

Applying $\pi$ and using its $G$-invariance and $H$-equivariance, we obtain

$$h_k \pi(x_k) = h'_k \pi(x'_k)$$

for each $k$. Since $H$ is compact, the sequence $d_k = (h_k)^{-1}h'_k$ contains a convergent subsequence $d_{k_n}$, with limit $d \in H$. Then $\pi(x_{k_n}) = d\pi(x'_{k_n})$, with the left-hand side converging to $\pi(x_0)$ and the right-hand side converging to $d\pi(x_0)$. By freeness of the $H$-action on $M$, we obtain $d = e_H$. Now

$$x_{k_n} = d_{k_n} x'_{k_n} g'_{k_n} (g_{k_n})^{-1}$$

converges to $x_0$, and $d_{k_n}x'_{k_n}$ converges to $e_H x_0 = x_0$. Since the $G$-action on $P$ is proper (as $\pi : P \to M$ is a smooth principal bundle), $l_{k_n} = g'_{k_n}(g_{k_n})^{-1}$ has a convergent subsequence $l_{k_{n_m}}$, with limit $l$. This limit must be $e_G$ due to freeness of the $G$-action on $P$.

Therefore, we now have $d_{k_{n_m}}$ converging to $e_H$, $l_{k_{n_m}}$ converging to $e_G$, and $x_{k_{n_m}} = d_{k_{n_m}} x'_{k_{n_m}} l_{k_{n_m}}$ converging to $x_0$. Thus $y_m = (d_{k_{n_m}}, x'_{k_{n_m}}, l_{k_{n_m}})$ converges to $(e_H, x_0, x_G)$, but the derivative of $f$ at $(e_H, x_0, e_G)$ is invertible, so $f$ is injective on a neighbourhood of $(e_H, x_0, e_G)$. Therefore $y_m$ must eventually be $(e_H, x_0, e_G)$; but reversing the roles of $x_k, x'_k$ we see that $(d^{-1}_{k_{n_m}}, x_{k_{n_m}}, l^{-1}_{k_{n_m}})$ must eventually be $(e_H, x_0, e_G)$ as well. This contradicts the assumption that the sequences $(h_k, x_k, g_k)$ and $(h'_k, x'_k, g'_k)$ were distinct.

Thus we have found, for any point $x_0 \in P$, a submanifold $S$ containing $x_0$ such that $f : H \times S \times G \to P$ defined by $f(h, x, g) = hxg$ is an embedding. The quotient map $P \to M/H$ then provides a submanifold isomorphic to $S$, containing the image of $x_0$, and whose preimage in $P/H$ is $S \times G$; since $x_0$ was arbitrary, we have now found local trivializations of $P/H \to M/H$ as a principal $G$-bundle.

Note that the $G$-action on $P/H$ is still free: for $x \in P$, $Hx \in P/H$ and $g \in G$, the $G$-action is defined as $(Hx)g = H(xg)$. Suppose now that $(Hx)g = Hx$ for $x \in P$. Then there exists an $h \in H$ such that $hx = xg$. Applying $\pi$ gives $h\pi(x) = \pi(x)$, and since $H$ acts freely on $M$, we conclude that $h = e_H$. Now $x = xg$, and from freeness of the $G$-action on $P$ we see that $g = e_G$, so the $G$-action on $P/H$ is free. □

We now translate the theorem from a statement on principal bundles into one on equivariant vector bundles.

**Lemma A.2.** *Let $G$ be a compact Lie group, $M$ a manifold and $\pi : E \to M$ a complex vector bundle of rank $k$. Assume that $G$ acts freely on $M$ and $E$, such that $\pi$ is $G$-equivariant, and such that for every $x \in M$, $g \in G$, left-multiplication by $g$ induces a linear isomorphism $E_x \to E_{gx}$. Then the principal $\mathrm{GL}_k(\mathbb{C})$-bundle $\pi : P \to M$ associated to $E$ admits a free $G$-action which commutes with the $\mathrm{GL}_k(\mathbb{C})$-action, and such that $\pi$ is $G$-equivariant. Furthermore, the vector bundle associated with the quotient $\bar{\pi} : P/G \to M/G$ is the "quotient vector bundle" $E/G \to M/G$, in the sense that $(E/G)_{Gx} \cong E_x$ for $x \in M$.*

*Proof.* We define $P$ as follows: for $x \in M$, the fiber $\pi^{-1}(x)$ consists of all linear isomorphisms $\mathbb{C}^k \to E_x$. A local trivialization of $E$ around $x$ gives rise to a local trivialization of $P$ as a $\mathrm{GL}_k(\mathbb{C})$-bundle, and therefore serves to define the topology and smooth structure of $P$. The bundle $P$ is also known as the frame bundle of $E$, and $A \in \mathrm{GL}_k(\mathbb{C})$ acts on an isomorphism $\mathbb{C}^k \to E_x$ by precomposition. Now, for $g \in G$ and $x \in M$, we obtain a linear isomorphism $L : E_x \to E_{gx}$ by left-multiplication. Therefore $g$ acts on $L' \in P_x$ by $g \cdot L' = L \circ L'$. If $A \in \mathrm{GL}_k(\mathbb{C})$, then $L' \cdot A = L' \circ A$. Since composition is associative, $(gL')A = g(L'A)$, and the $G$-action on $P \to M$ commutes with the $\mathrm{GL}_k(\mathbb{C})$-action. We conclude that the quotient $\bar{\pi} : P/G \to M/G$ is a principal $\mathrm{GL}_k(\mathbb{C})$-bundle.

To now recover from $P/G$ a vector bundle over $M/G$, let $\mathrm{GL}_k(\mathbb{C})$ act on $P/G \times \mathbb{C}^k$ by $(Gx, v)A = ((Gx)A, A^{-1}v)$, where $A \in \mathrm{GL}_k(\mathbb{C})$, $v \in \mathbb{C}^k$ and $x \in P$. Then the quotient space $(P/G \times \mathbb{C}^k)/\mathrm{GL}_k(\mathbb{C})$ admits a map $p : (P/G \times \mathbb{C}^k)/\mathrm{GL}_k(\mathbb{C}) \to M/G$ given by

$$[(Gx, v)] \mapsto G\bar{\pi}(x).$$

This is well-defined since $\bar{\pi}$ is $\mathrm{GL}_k(\mathbb{C})$-invariant. Furthermore, each fiber $p^{-1}(Gy)$ for $y \in M$ is isomorphic to $E_y$. $\square$

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