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Sub-riemannian geometry of the p, q Hopf actions

by

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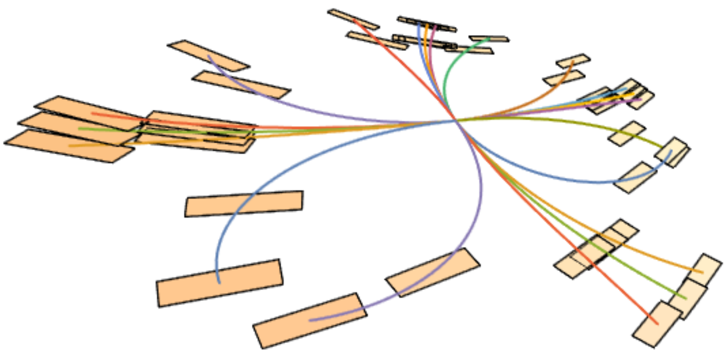

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Abstract

Sub-Riemannian geometry is the study of paths and distances in space where motion is restricted. Such spaces arise in the study of various physical systems. The aim of this text is twofold.

First, we provide an introduction at an undergraduate level to this topic and illustrate all related concepts through their immediate application to a toy problem: a physical model of a unicycle.

Second, we investigate the sub-Riemannian geometries on S^3 given by the p,q Hopf action through working out the Hamiltonian equations that define the sub-Riemannian geodesics of this geometry. A proof is given that the great circles on S^3 with a horizontal initial velocity are among the geodesics. Using numerical solutions to the Hamiltonian equations, we geodesic spheres of this geometry.

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1 Introduction

Anyone familiar with the problems of parallel parking already knows what sub-Riemannian geometry is about. When you are parking a car in a spot that is parallel to the road, you are really asking the car to do something it is by design incapable of: moving perpendicular to the forward direction. Perhaps this could be fixed by making a car with rotatable front and rear wheels but in practice there is no need for this: By moving forward and backwards while also changing the steering direction, a car can be zig-zaged into such troublesome parking spots. Of course this requires a certain skill on the driver and the process is not entirely intuitive.

This problem can be translated into one of sub-Riemannian geometry. We could find a manifold that models the location and orientation of the car and, on its tangent space, we can specify a subset of directions that the car can move in. If we then also have a suitable way of measuring the length of the permitted tangent vectors (which corresponds to speed of the car), we arrive at a notion of distance between points, only considering paths that the car can actually take. It seems plausible that for the car, the distance to a point perpendicular to the forward direction is much greater than to one just ahead, which leads to the problems with parallel parking. For this, we don't even need to consider 4 wheels, and we will use a model of a unicycle to demonstrate the precise constructions needed for sub-Riemannian geometry later on.

Another example of a space with a sub-Riemannian geometry is the Hopf-fibration. This mathematical construction on the 3 dimensional sphere is an important example of a fibre bundle, meaning that locally it has the same structure as a product space. These kinds of spaces have been studied by mathematicians since the mid 1900s (as for example in [Ste16]), but fibre bundles also are of great value within physics. The Hopf-fibration specifically has been of use for the study of magnetic monopoles [Min79], quantum physics [MD01] and several other applications [Urb03], [Pen05, Sec. 15.1]. Through its nature as a fibre bundle it can be given a sub-Riemannian structure. The action defining the Hopf-fibration can be generalized to produce other topological quotient spaces with a sub-Riemannian structure defined similarly. Their study will be the topic of the second half of this text.

2 Sub-Riemannian Geometry

2.1 A short tour of Riemannian geometry

The notion of a manifold can be traced back to the *Habilitationsschrift* (probationary essay) by Bernhard Riemann [RW19], a text "on the hypotheses which lie at the foundations of geometry". There he describes manifolds as sets that can be parameterised by a tuple of parameters. In the same text, he describes how if such a space has a way of measuring lines, one can derive various geometric properties of these spaces, a study which would later develop into the field of Riemannian geometry. Sub-Riemannian geometry contains Riemannian geometry as a special case, however developing some of the Riemannian theory first will help us in understanding the particularities of Sub-Riemannian geometry. In this text our interest in Riemannian geometry lies mostly within the definition of geodesics, which are generalizations of straight lines in Euclidean geometry and their properties regarding distance.

In what follows, we call $\gamma : I \to M$ a curve if it is a smooth map from some interval $I \subset R$ to M. It is simpler to think of I as a closed interval, however, some proofs require it to be open. It turns out that this is not a hindrance as we can extend or constrict the domain of a curve slightly as necessary [See Lee06, Page 55].

A geodesic is readily defined for a submanifold of $\mathbb{R}^n$. First, for $\mathbb{R}^n$ itself, curves of constant velocity are most straight when their acceleration is zero, so we define geodesics of $\mathbb{R}^n$ as that. On the sphere as a submanifold of $\mathbb{R}^3$ however, no nonconstant smooth curve has zero acceleration (as it is to stay on the sphere). To define geodesics for such a case, we merely require the acceleration to be orthogonal to the manifold: The acceleration is computed in $\mathbb{R}^3$ and we compute an orthogonal projection to the tangent space of the manifold. A curve where this projection is equal to zero is then called a geodesic. However, on an arbitrary manifold without a given embedding into $\mathbb{R}^n$, we have to use a different definition, of which it is the goal to develop throughout this chapter.

The standard definition of an inner product of $\mathbb{R}^n$ states that

- $\langle x, x\rangle \in \mathbb{R}$ and $\langle x, x\rangle \geq 0$;
- $\langle x, x\rangle = 0$ if and only if $x = 0$;

- $\langle ax + by, z\rangle = a\langle x, z\rangle + b\langle y, z\rangle$;
- $\langle x, y\rangle = \langle y, x\rangle$,

for any x, y and z in $\mathbb{R}^n$ and a and b in $\mathbb{R}$ [RY00, Definition 3.1]. These can be summarised by saying that $\langle\cdot,\cdot\rangle$ is a symmetric, positive-definite and bilinear function.

On $\mathbb{R}^n$, the usual inner product

$$\langle (x^i), (y^i)\rangle = \sum_i x^i y^i,$$

satisfies these properties. As we know from linear algebra on $\mathbb{R}^n$, the number $\langle x, y\rangle$ is closely related to the angle of two vectors. We can also define the length of a vector x as $\langle x, x\rangle^{\frac{1}{2}}$ and do the usual Euclidian geometry with these definitions, but there is one thing to be remarked about this common definition. Specifying the domain of $\langle\cdot,\cdot\rangle$ as $\mathbb{R}^n \times \mathbb{R}^n$ is slightly awkward. Each point in $\mathbb{R}^n$ is merely a point, and how can an angle between two points make sense? Or the length of a point? Really, the domain of $\langle\cdot,\cdot\rangle$ should be given as $T\mathbb{R}^n$ so that we see $\langle\cdot,\cdot\rangle$ as a way of comparing "directions" (represented through tangent vectors) with one another. That is, if we understand $T_x\mathbb{R}^n$ as equivalence classes of smooth curves through x, we are only computing angles and lengths of their velocity vectors as they pass through x. With $T\mathbb{R}^n$ as its domain, we have a clear geometric idea of what $\langle\cdot,\cdot\rangle$ does and we can define the notion for arbitrary manifolds.

Definition (Riemannian geometry)**.** The tuple $(M, \langle\cdot,\cdot\rangle)$ is called a Riemannian geometry if $\langle\cdot,\cdot\rangle$ is a symmetric, bilinear and positive-definite smooth 2-form on M. We also call $\langle\cdot,\cdot\rangle$ the Riemannian metric of M.

The inner product of $\mathbb{R}^n$ with domain $T\mathbb{R}^n$ is also called the standard metric of $\mathbb{R}^n$. Using it and the following two examples, we can construct metrics on many other manifolds.

Example 2.1 (Induced Metric)**.** If $\iota : M \to \tilde{M}$ is a smooth immersion so $d\iota$ is injective, and $\tilde{M}$ has a metric $\langle\cdot,\cdot\rangle_{\tilde{M}}$, we can define a metric on M using the pullback via ι. That is, we define for tangent vectors $v, w \in T_pM$,

$$\langle v, w\rangle = \langle d\iota v, d\iota w\rangle_{\tilde{M}}.$$

As $d\iota$ is a smooth map from TM to $T\tilde{M}$, this defines a smooth 2-form that is clearly symmetric and bilinear. Since $d\iota$ is injective, it is positive definite.

□

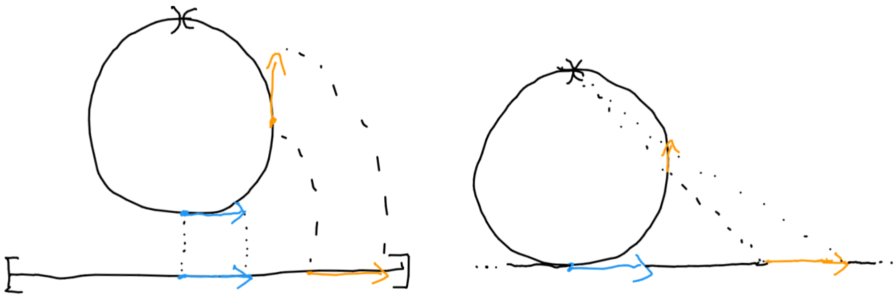


Figure 2.1: Translation in coordinates depends on the chart.

Example 2.2 (Product metric)**.** If M_1 and M_2 are two Riemannian manifolds, M_i with metric $\langle \cdot, \cdot \rangle_{M_i}$, we can define a metric on their product $M_1 \times M_2$. Since $T_{(p_1,p_2)}(M_1 \times M_2) \cong T_{p_1}M_1 \oplus T_{p_2}M_2$, we can rewrite an arbitrary tangent vector in $T(M_1 \times M_2)$ as the sum $w_i + v_i$ with $w_i \in TM_1$ and $v_i \in TM_2$. Then we define

$$\langle v_1 + w_1, v_2 + w_2 \rangle = \langle v_1, v_2 \rangle_{M_1} + \langle w_1, w_2 \rangle_{M_2}$$

to be the metric on the product: it satisfies the required properties. □

We turn back to the problem of defining acceleration of curves on arbitrary manifolds. The usual identification of $T\mathbb{R}^n$ with R^n explains why on $\mathbb{R}^n$, we usually define the inner product on $\mathbb{R}^n$ directly; we associate a tangent vector at a point x with a point in $\mathbb{R}^n$ by translating x to the origin and then describing the endpoint of the tangent vector in $\mathbb{R}^n$. This identification also provides us with an obvious way of comparing tangent vectors at different points with one another. On an arbitrary manifold, this does not work. We could try to use coordinates to work with the structure of $\mathbb{R}^n$ but then the translation depends on the choice of coordinates as Figure 2.1 illustrates. There, S^1 is seen with angular coordinates on the left and under the stereographic projection on the right. The blue tangent vector is to be translated to the right in coordinates, resulting in the orange vector. In the case of the stereographic projection, the resulting tangent vector on S^1 is shorter than the original vector, whereas in the angular coordinates, both have the same length.

This problem becomes relevant to us as to compute acceleration in $\mathbb{R}^n$, we would form a difference quotient

$$\ddot{\gamma}(t) = \lim_{h \to 0} \frac{\dot{\gamma}(t+h) - \dot{\gamma}(t)}{h}, \tag{2.1}$$

however $\dot{\gamma}(t+h)$ is in a tangent space at a different point than $\dot{\gamma}(t)$. To compute this, again, we use the identification of the various tangent spaces of $\mathbb{R}^n$. On other manifolds, we resolve this problem through the concept of a connection. Our presentation of them will closely follow the exposition by Do Carmo [Do 92]. Let $\mathfrak{X}(M)$ be the set of all smooth vector fields on M, that is, the smooth sections of TM.

Definition (Linear Connection)**.** A linear connection ∇ is a mapping $\mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ that satisfies

- $\nabla_{fX+gY} Z = f\nabla_X Z + g\nabla_Y Z$;
- $\nabla_X(Y + Z) = \nabla_X Y + \nabla_X Z$;
- $\nabla_X(fY) = f\nabla_X Y + X(f)Y$,

where $X, Y, Z \in \mathfrak{X}(M)$ and f, g are smooth functions on M.

Such maps exist: for example, $\nabla_X Y = XY$ satisfies the required properties. As the next proposition shows, ∇ is used to define a derivative of vector fields along curves.

Proposition 2.3. *Let M be a manifold with linear connection ∇. There exists a unique map D_t that maps a vector field V along a curve γ to another vector field D_tV, such that*

(a) $D_t(V + W) = D_tV + D_tW$;

(b) $D_t(fV) = \frac{df}{dt}V + fD_tV$;

(c) If $V_t = Y_{\gamma(t)}$ *with* $Y \in \mathfrak{X}(M)$*, then* $D_tV = \nabla_{\dot\gamma} Y$.

The map D_t is called the covariant derivative.

The full proof of this proposition can be found in [Do 92, Proposition 2.2]. Essentially it works by expressing a vector field V along γ as $V^i\partial_{x^i}$ in coordinates x^i. Then, because ∂_{x^i} is not just a vector field along $\gamma = (\gamma^1, \dots, \gamma^n)$ but everywhere within the coordinate chart, we can use (a), (b) and (c) to find the expression

$$D_tV = \frac{dV^i}{dt}\partial_{x^i} + \frac{d\gamma^j}{dt}V^i\nabla_{\partial_{x^j}}\partial_{x^i} \tag{2.2}$$

for D_tV in terms of ∇. This shows that if such a map exists, it has to be given by this expression and is therefore unique. On the other hand, if we take (2.2) as the definition, all of the properties are satisfied. Since D_t is unique if it exists, this shows that this definition has to be valid on any chart of M.

In coordinates, the vector field $\nabla_{\partial_{x^j}}\partial_{x^i}$ completely determines the covariant derivative. Its component functions are denoted as Γ^k_{ij} and called the Christoffel symbols of the connection. (Actually, the Γ_{ij}^k also completely determine ∇ as [Lee06, Lemma 4.4] shows.)

Using D_t, we can define the acceleration of a curve γ as $D_t\dot{\gamma}$. What's more, if v is a tangent vector at $\gamma(t)$ of some curve on M, there exists a unique vector field V along γ such that $V_t = v$ and $D_tV = 0$ for all t. [See Do 92, Proposition 2.6.] With that, we could now go ahead and reinterpret the difference quotient (2.1). Let $V^{(h)}$ be the unique vector field such that $V_{t+h}^{(h)} = \dot{\gamma}(t+h)$ and $D_tV^{(h)} = 0$. Then, $V_t^{(h)} \in T_{\gamma(t)}M$, so Equation (2.1) can be computed (provided the limit exists). However,

$$D_t\dot{\gamma}(t) = \lim_{h\to 0} \frac{V_t^{(h)} - \dot{\gamma}(t)}{h},$$

so there is no necessity to do this [See Lee06, Exercise 4.12.]. This shows that the map D_t can be used to compute the acceleration of a curve connecting the tangent spaces $T_{\gamma(t)}M$ and $T_{\gamma(t+h)}M$.

Now, defining a linear connection seems like adding an additional choice to a Riemannian geometry. However, for such a geometry, there is only one connection that goes along with the metric in the following way.

Theorem 2.4 (Levi-Civita connection). *For a Riemannian geometry $(M, \langle\cdot,\cdot\rangle)$, there exists a unique linear connection ∇ such that for all $X, Y \in \mathfrak{X}(M)$, $[X,Y] = \nabla_XY - \nabla_YX$ and for V and W vector fields along γ, $\langle V_t, W_t\rangle$ is constant if $D_tV = D_tW = 0$.*

For a proof, see [Do 92, Theorem 3.6]. This is the connection we usually work with in Riemannian geometry and with which we define the notion of a Riemannian geodesic.

Definition. Let D_t be the covariant derivative with respect to the Levi-Civita connection. A curve γ is called a geodesic if $D_t\dot{\gamma} = 0$.

In principle, one can define geodesics using arbitrary connections. However, the length minimizing properties of Riemannian geodesics are tied to the properties of the Levi-Civita connection. The condition that $[X,Y] = \nabla_XY = \nabla_YX$ is used in the necessary proofs.

Riemannian Geodesics are smooth, but they minimize length over curves that are not allways smooth. We call a map $\gamma : I \to M$ a piecewise curve if it is smooth on $I - A$, where $A = \{t_1, t_2, \ldots, t_m\}$ is a finite set of points in $\mathbb{R}$ and $\gamma(I)$ is connected. Of course, a curve is a piecewise curve with $A = \emptyset$. Define the length of a piecewise curve $\gamma : I \to M$ to be

$$L(\gamma) = \int_{I-A} \langle\dot{\gamma}, \dot{\gamma}\rangle^{1/2}\, dt. \tag{2.3}$$

Definition (Minimizing curve)**.** We call a piecwise curve γ a minimizing curve if for all other piecwise curves $c : I \to M$ with the same start and end point as γ,

$$L(\gamma) \leq L(c).$$

Theorem 2.5 (Minimality of Riemannian geodesics)**.** *If γ is a Riemannian geodesic then any sufficiently short arc of γ is, up to reparametrization, the unique minimizing curve joining its endpoints.*

Conversely, if γ is a minimizing piecewise curve between its endpoints and $\langle \dot\gamma, \dot\gamma \rangle$ is constant (where it is defined), γ is a geodesic.

This theorem is a reformulation of Corollary 3.9 and Proposition 3.6 in Do Carmo's book [Do 92]. However, it goes beyond the scope of this thesis to include the necessary definitions and arguments to prove it. It might seem mysterious why only sufficiently short arcs of a geodesic γ are length minimizing. To see that this restriction is really necessary, consider a curve $\gamma : I \to S^1$ on the circle with $\langle \dot\gamma, \dot\gamma \rangle$ constant that starts at $p \in S^1$ and is surjective. The arc $\gamma([0, t])$ is length minimizing up until t is such that $\gamma(t)$ equals the antipodal point of p. For greater t, there is obviously a shorter curve. How could one specify a neighbourhood of M such that every geodesic γ contained within it is length minimizing? Such neighbourhoods are called totally normal neighbourhoods and it turns out that M is covered by them [Do 92, Theorem 3.7].

In order to do explicit computations of geodesics within a coordinate chart, we again turn to Equation (2.2). By substituting V with $\dot\gamma$, we find

$$0 = \ddot\gamma^k \partial_{x^k} + \dot\gamma^j \dot\gamma^i \nabla_{\partial_{x^j}} \partial_{x^i}. \tag{2.4}$$

Then, using the Christoffel symbols $\Gamma^k_{i,j}$, we find that (2.4) is satisfied if and only if

$$0 = \ddot\gamma^k + \dot\gamma^j \dot\gamma^i \Gamma^k_{ij}, \tag{2.5}$$

for all k. Note that this is a second-order differential equation and can be rewritten to a first-order differential equation by introducing the variables $v^i = \dot\gamma^i$. As that we can view it as a differential equation on TM where v^i is the coordinate of ∂_{x^i}. Then, because solutions to such differential equations can be proven to exist uniquely, we conclude that a geodesic is completely specified by providing an initial point $\gamma(0) \in M$ and initial velocities $\dot\gamma(0) \in T_{\gamma(0)}M$ [Lee06, Theorem 4.10]. This property gives rise to a map from T_pM to M.

Definition (The exponential map)**.** Given $p \in M$ and let $B_\epsilon(0) \subset T_pM$ be a ball of radius ϵ (with respect to $\langle \cdot, \cdot \rangle$) in the vector space T_pM. Define the exponential map at p

$$\exp_p : B_\epsilon(0) \to M$$

by taking a tangent vector v to $\gamma(1)$, where γ is the geodesic with $\gamma(0) = p$ and $\dot{\gamma}(0) = v$.

From Theorem 2.5 we know that if ϵ is small enough, $\exp_p$ should be a bijection. Even stronger, by computing $d/dt|_{t=0} \exp_p(tv) = v$, we see that $d\exp_p$ is the identity map around the origin of T_pM, so $\exp_p$ is a local diffeomorphism [See Do 92, Proposition 2.9]. However, as ϵ gets too large, the geodesics behind the exponential map might loose their minimizing property and the exponential map ceases to be a diffeomorphism.

2.2 The definitions of sub-Riemannian geometry

In the example given in the introduction, we saw how physical problems can involve motion under constraints. Once the possible configurations of a physical problem have been modeled as a manifold, we encode the directions admissible as pointwise subspaces of the tangent bundle. Given such a pointwise restriction, one may ask if a shortest curve between any two points exists and if it is possible to find it. To tackle this problem, we first need to define what we mean by restricting the number of possible directions on a manifold. This is done through a smooth subbundle of the tangent space.

Definition (smooth subbundle)**.** Let M be a manifold. If E is a smooth vector bundle $\pi_E : E \to M$, D is an embedded submanifold of E such that it is a vector bundle $\pi_D : D \to M$ with $\pi_D = \pi_E|_D$ and $D \cap E_p$ is a linear subspace of E_p, then D is called a smooth subbundle of E.

Smooth subbundles of TM are usually called distributions. A set of vector fields $\{X_a\}$ defined on some subset U of M is called a frame for $\mathcal{H}$ if $\{X_a|_p\}$ spans $\mathcal{H}$ at each p in U.

Definition (Sub-Riemannian geometry)**.** Let M be a smooth manifold. A sub-Riemannian geometry is a triple $(M, \mathcal{H}, \langle\cdot,\cdot\rangle)$ where $\mathcal{H} \subset TM$, is a smooth subbundle of the tangent space of M and $\langle\cdot,\cdot\rangle$ an inner product on $\mathcal{H}$. We call $\mathcal{H}$ the horizontal distribution, $\langle\cdot,\cdot\rangle$ the sub-Riemannian metric and the dimension of $\mathcal{H}_p$ the rank of $\mathcal{H}$ at p.

Remark. Usually, $\mathcal{H}$ is allowed to have varying rank at different points of the manifold. However, we define a subbundle of TM to be a topological subspace of TM that is also a vector bundle. Since a vector bundle has constant rank on each connected component of M and we only deal with connected manifolds, the rank of $\mathcal{H}$ will not depend on the point.

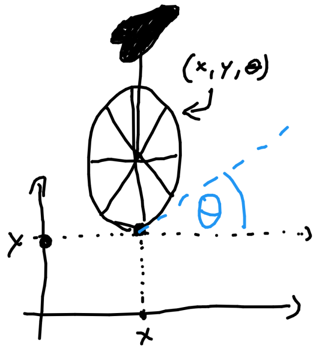


Figure 2.2: Configuration space of a unicycle.

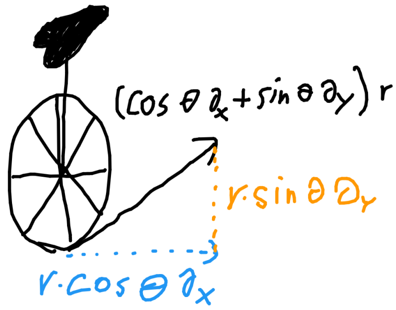


Figure 2.3: Tangent vectors of the forward motion.

To make this definition tangible, we look at the following example that will follow us throughout this chapter.

Example 2.6 (model of a unicyle)**.** If we model the earth as an infinite plane, we can describe the configurations of a unicycle in space (assuming the driver never loses balance) using the manifold $M = \mathbb{R}^2 \times S^1$. We use the coordinates (x, y, θ) so that x and y describe the location of the unicycle and θ tells us in which direction it is currently heading as sketched in Figure 2.2. The angle θ is defined as the angle from the positive direction of the x axis to the forward direction of the unicycle. In physics, the manifold M is called the configuration space.

A unicycle can only move in specific directions. If the driver was to paddle forward for r meters, he would move from (x, y, θ) to $(x + r\cos\theta, y + r\sin\theta, \theta)$. If we consider the infinitesimal generator of that motion, we get

$$\frac{d}{dr}(x + r\cos\theta, y + r\sin\theta, \theta)\Big|_{r=0} = \cos\theta\,\partial_x + \sin\theta\,\partial_y = X_1.$$

This is illustrated in Figure 2.3. Similarly if we consider rotation on the spot, we find the infinitesimal generator $X_2 = \partial_\theta$ for rotating counterclockwise. This gives us the horizontal distribution

$$\mathcal{H}_{(x,y,\theta)} = \operatorname{span}\{\cos\theta\,\partial_x + \sin\theta\,\partial_y, \partial_\theta\}.$$

In order to obtain a metric on $R^2 \times S^1$, we use Examples 2.1 and 2.2. With $\iota : S^1 \to \mathbb{R}^2$ being the standard inclusion, we see that there is a metric on S^1 with the property that $\langle \partial_\theta, \partial_\theta \rangle = 1$. As TS^1 is one-dimensional, this completely specifies the metric. If we use the standard metric on $\mathbb{R}^2$, the product metric of the two gives us an inner product on $\mathcal{H}$. This example is also considered in [Jea14, Example 1.1]. □

In the example and the other sub-Riemannian geometries we will consider in this thesis, the sub-Riemannian metric is a Riemannian metric that is restricted from TM to $\mathcal{H}$. In that case we can decompose T_pM into $\mathcal{H}_p$ and $\mathcal{V}_p := \mathcal{H}_p^\perp$ and we will refer to $\mathcal{V}$ as the vertical distribution. It is indeed a distribution as the following lemma shows.

Lemma 2.7 (Orthogonal complements of distributions)**.** *If $\mathcal{V}$ is a distribution on a Riemannian manifold M, then $\mathcal{V}^\perp$ is also a distribution.*

Proof. Let $n = \dim M$. As $\mathcal{V}$ is a rank k distribution, hence a smooth subbundle, at each point of M there is a neighbourhood U such that within, $X_1, \dots X_k$ form a frame for $\mathcal{V}$. Then, we can complete this frame by vector fields $X_{k+1}, \dots, X_n$ to a local frame of TM by [Lee12, Proposition 8.11]. The Gram-Schmidt process provides us with an orthogonal frame $E_1, \dots E_n$, defined inductively by

$$E_j = \frac{X_j - \sum_{i=1}^{j-1} \langle X_j, E_i \rangle E_i}{|X_j - \sum_{i=1}^{j-1} \langle X_j, E_i \rangle E_i|},$$

where if $v \in T_pM$, $|v| = \langle v, v \rangle^{1/2}$. With that, for each $p \in U$ and $m \leq n$, $\operatorname{span}\{E_1|_p, \dots, E_m|_p\} = \operatorname{span}\{X_1|_p, \dots, X_m|_p\}$. By orthogonality, $E_1|_p, \dots, E_k|_p$ span $\mathcal{V}_p$ and $E_{k+1}|_p, \dots, E_n|_p$ span $\mathcal{V}_p^\perp$. This shows that $\mathcal{V}^\perp$ satisfies a local frame criterion [Lee12, Lemma 10.32] and is therefore a smooth subbundle of TM. $\square$

In examples coming from mechanics as the one before, motion between configurations of the system can be described by curves on the configuration space. To ensure that we only consider paths that are mechanically possible, we have to make sure they are tangent to the horizontal directions.

Definition (Horizontal curve)**.** A curve $\gamma : I \to M$ is called horizontal if $\dot{\gamma} \in \mathcal{H}$ for all t.

For horizontal curves, we get a notion of their length. This works just like in the Riemannian case, so we define the length $L(\gamma)$ again by Equation (2.3). Using it, we can define the distance of two points on M to be the infimum of the lengths of all horizontal curves connecting them.

At this point however, it is not yet clear if any two points on M can be connected by a horizontal curve. For example, we could define a sub-Riemannian geometry on $\mathbb{R}^2$ with horizontal distribution $\mathcal{H}_{(x,y)} = \operatorname{span}\{\partial_x\}$ and restrict the usual inner product to it. Clearly no horizontal curve will be able to connect the point $(0, 1)$ with the origin, as for any horizontal curve $dy(\dot{\gamma}(t)) = 0$, so projecting γ to the y axis yields a constant function. This is because here the horizontal distribution is so restrictive, it only allows motion along horizontal slices of R^2. The situation is depicted in Figure 2.4. Fortunately, there is a simple criterion as to when a

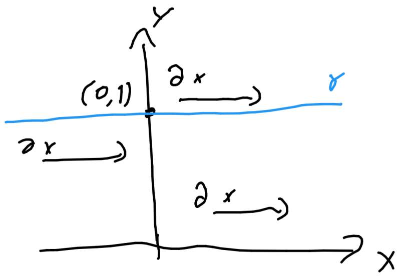


Figure 2.4: An example of an integrable distribution.

horizontal distribution has enough freedom to connect all points on a manifold using horizontal curves.

Definition (bracket generating)**.** We call a distribution $\mathcal{H} \subset TM$ bracket generating if the union of any local frame X_i of $\mathcal{H}$ around p with its iterated Lie brackets $[X_i, X_j]$, $[X_k, [X_i, X_j]]$, etc. spans T_qM for all q in some neighbourhood of p.

This condition could also be called non-integrability. A distribution $\mathcal{D}$ on M is called integrable if every point is contained in some immersed submanifold $N \subset M$ such that $T_pN \subset \mathcal{D}_p$. By Frobenius' Theorem, this happens if and only if the Lie bracket $[X, Y]$ of any two vector fields X, Y in $\mathcal{D}$ is again in $\mathcal{D}$ [Lee12, Thm. 19.12]. For bracket generating distributions, we get the following result:

Theorem 2.8 (Chow's theorem)**.** *If $\mathcal{H} \subset TM$ is bracket-generating, and M a connected manifold, then any two points of M can be connected with a horizontal curve.[Mon02, Theorem 2.2]*

This theorem tells us that for a bracket generating distribution, distance of points on a connected manifold will always be finite. In our examples, this turns out to be the case.

Example 2.9 (motions of a unicycle)**.** If we compute the Lie bracket of the global frame of $\mathcal{H}$ of our unicycle example, we find

$$V := [\cos\theta\,\partial_x + \sin\theta\,\partial_y, \partial_\theta] = \partial_\theta(\cos\theta)\,\partial_x + \partial_\theta(\sin\theta)\,\partial_y = -\sin\theta\,\partial_x + \cos\theta\,\partial y$$

and as this vector is linearly independent with the two vectors X_1 X_2, we see that the three of them span TM. Therefore, any two configurations of the unicycle can be reached by driving around. (This isn't very surprising considering that we allowed the unicycle to be able to turn on the spot.) $\square$

Now that we have established when exactly horizontal paths between arbitrary points on a manifold exist, we turn to the problem of finding the shortest such paths.

2.3 Singular Sub-Riemannian geodesics

Unlike in Riemannian geometry, where all length minimizing curves come from the geodesic equations, in sub-Riemannian geometry there are length minimizing curves that do not come from the differential equations analogous to the geodesic equaitons. We call such length minimizing curves *singular geodesics*. It is one of the open questions of sub-Riemannian geometry whether the signular geodesics are smooth, so formally we can not call them curves [Mon02, Chapter 10.1]. A proof of the existence of singular geodesics was only published in 1994 by Montgomery [Mon94]. (See also [Mon94, Section 3.9] for a review of the subject.)

In other words, in sub-Riemannian geometry, there is no analog to the second part of Theorem 2.5. However, it turns out that if $\mathcal{H}$ is a contact distribution, there are no singular geodesics and the situation is again like in the Riemannian case [See Mon02, Theorem 5.3 and 5.8].

Definition (Contact distribution)**.** Let M be a manifold of dimension $2k+1$. We call a rank $2k$ distribution $\mathcal{H}$ a contact distribution if in some neighbourhoods of each $p \in M$, there exists a 1-form ν in T^*M such that for each q in the neighbourhood of p,

$$\mathcal{H}_q = \ker \nu_q$$

and

$$\nu \wedge \underbrace{d\nu \wedge \cdots \wedge d\nu}_{k \text{ times}} \neq 0$$

All the examples of sub-Riemannian geometries we will encounter in this thesis are of this type.

Example 2.10. For the unicyle consider the 1-form

$$\nu = -\sin\theta\, dx + \cos\theta\, dy = \iota_V \langle \cdot, \cdot \rangle,$$

with $V = -\sin\theta\partial_x + \cos\theta\partial_y$ as before. Then, $\nu(X_1) = -\sin\theta\cos\theta + \sin\theta\cos\theta = 0$ and $\nu(X_2) = 0$, so $\mathcal{H} \subset \ker\nu$. As $\nu(-\sin\theta\,\partial_x + \cos\theta\,\partial_y) = 1$ at every point of M, and the three tangent vectors X^1, X^2 and $-\sin\theta\,\partial_x + \cos\theta\,\partial_y$ form a basis of T_pM, we therefore see that $\ker\nu = \mathcal{H}$.

If we compute the exterior derivative of ν, we find

$$d\nu = -\cos\theta\, d\theta \wedge dx - \sin\theta\, d\theta \wedge dy,$$

and

$$\nu \wedge d\nu = \sin^2\theta\, dx \wedge d\theta \wedge dy - \cos^2\theta\, dy \wedge d\theta \wedge dx = -(\sin^2\theta + \cos^2\theta)\, dx \wedge dy \wedge d\theta \neq 0$$

so $\mathcal{H}$ is a contact distribution and there are no singular geodesics on this sub-Riemannian geometry. $\square$

2.4 Normal Sub-Riemannian geodesics

Riemannian geodesics are defined using the Levi-Civita connection of the Riemannian metric, however the construction needs the Riemannian metric to be defined on the entire tangent space. (This can, for example, be seen in [Lee06, Equation 5.4], where an explicit formula for the Christoffel symbols of the Levi-Civita connection is given.) We can avoid this by rewriting the geodesic equations on the cotangent bundle T^*M.

Rewriting the geodesic equations is closely tied to two formalisms of classical mechanics: the Hamiltonian and Lagrangian formulation of mechanics. They are in some sense equivalent and the description of geodesics given so far followed the Lagrangian approach. It states that the evolution of a mechanical system is completely determined by a function $L : TM \to \mathbb{R}$ (called the Lagrangian) and the initial positions and velocities in the configuration space. Then, second order differential equations called the Euler-Lagrange equations determine the evolution of the system from this point onward. [SH14, Chapter 8]

It can be shown that Equation (2.5) are the Euler-Lagrange equations corresponding to the Lagrangian function $L : TM \to R$ given by

$$L(v) = \frac{1}{2}\langle v, v\rangle. \tag{2.6}$$

However, instead of showing this explicitly, we note that a solution to the Euler-Lagrange equations minizies the action $\int L(\gamma, \dot{\gamma})\, dt = \int \frac{1}{2}\langle \dot{\gamma}, \dot{\gamma}\rangle\, dt$. This is equivalent to minimizing length as shown in [Mon02, Page 6], which short arcs of geodesics minimize. The principle of least action states that minimizing the action is exactly what characterizes the solutions to the Euler-Lagrange equations, so Riemannian geodesics solve them.

Alternatively, we could have defined Riemannian geodesics using the Hamiltonian approach, which consists of a function $H : T^*M \to \mathbb{R}$ (called the Hamiltonian) and corresponding differential equations. The two aproaches can be shown to yield the same geodesics. In the sub-Riemannian case, the Lagrangian L above can not be used, as $\langle \cdot, \cdot\rangle$ is not defined for all tangent vectors. We will therefore first construct a Hamiltonian unique to a sub-Riemannian geometry and then show how it corresponds to a Lagrangian very similar to L. To do this, we construct the following map between T^*M and TM.

Proposition 2.11. *For a sub-Riemannian geometry on M with distribution $\mathcal{H}$ and inner product $\langle \cdot, \cdot\rangle$, there exists a unique map $\beta : T^*M \to TM$ such that*

(a) $\operatorname{im} \beta_p = \mathcal{H}_p$,

(b) for $w \in \mathcal{H}_p$, $\nu \in T_p^*M$, $\nu(w) = \langle \beta_p(\nu), w\rangle$.

Proof. Suppose β' is another such map satisfying the two conditions. Then, for each point of M, there exists a neighbourhood U on which X_a, $a = 1, \dots, k$ is a local frame that locally spans $\mathcal{H}$. By applying the Gram-Schmidt process to the vectors X_a, we obtain a local frame E_a that is orthonormal and spans $\mathcal{H}$ [Lee12, Lemma 8.13]. Now, if ν is an arbitrary cotangent vector,

$$\langle \beta(\nu), E_a \rangle = \nu(E_a) = \langle \beta'(\nu), E_a \rangle =: c^a$$

for all a and therefore

$$\beta(\nu) = c^a E_a = \beta'(\nu), \tag{2.7}$$

so the two maps are equal in every such neighbourhood, so they agree on all of M.

To show that such a map exists, we can construct it on T^*U using the frame E_a. We define $\beta_U(\nu) = \sum_a^k \nu(E_a)E_a$ in this neighbourhood, which is clearly a function with codomain $\mathcal{H}$. Then, we also have $\langle \beta_U(\nu), E_i \rangle = \nu(E_i)$, so the properties of β are satisfied. As we know that β is unique, this local description defines β globally. □

Definition (cometric)**.** The fiber-bilinear function $(\cdot, \cdot) : T^*M \times T^*M \to \mathbb{R}$ associated to the sub-Riemannian geometry M, defined by

$$(\nu, \mu) := \nu\beta(\mu) = \langle \beta(\mu), \beta(\nu) \rangle$$

is called the cometric of M.

Using the expression in (2.7) and given an orthonormal frame, it is easy to compute the cometric explicitly.

Example 2.12 (cometric of the unicycle)**.** Let $X_1 = \cos\theta\,\partial_x + \sin\theta\,\partial_y$ and $X_2 = \partial_\theta$ be the frame spanning $\mathcal{H}$ as before. If we use the coordinates x, y and θ for M as before, we get the coordinates x, y, θ, dx, dy and $d\theta$ for T^*M. The vectors $X_1 = \cos\theta\partial_x + \sin\theta\partial_y$ and $X_2 = \partial_\theta$ are orthonormal and a global frame for $\mathcal{H}$. We can write the map $\beta_{(x,y,\theta)}$ explicitly as

$$\beta_{(x,y,\theta)}(a\,dx + b\,dy + c\,d\theta) = (a\cos\theta + b\sin\theta)X_1 + cX_2.$$

□

The function $(\cdot, \cdot)$ is like an inner product on the cotangent space but is always degenerate if $\mathcal{H} \neq TM$. Assume $\mathcal{H}$ is of rank k and we let E_a be a local orthonormal frame for $\mathcal{H}$. We can complete the frame to a local frame X_i spanning TM, with $X_1 = E_1, \dots, X_k = E_k$. Then, if ξ^i is the covector field dual to X_i, for $i > k$, we have

$$\beta(\xi^i) = \sum_a^k \xi^i(E_a)E_a = \sum_a^k \delta^i_a E_a = 0.$$

As $\xi^i \neq 0$, this shows that that $(\xi^i, \nu) = \nu(0) = 0$ for any convector field ν, so $(\cdot, \cdot)$ is degenerate.

Example 2.13. Let $V = \cos\theta\,\partial_y - \sin\theta\,\partial_x$ be a non-horizontal tangent vector of $R^2 \times S^1$. We can define an associated covector field $\nu = \cos\theta\,dy - \sin\theta\,dx$. Now, for the two vector fields X_1 and X_2 spanning $\mathcal{H}_{(x,y,\theta)}$ from before, we have

$$\nu(X_1) = \cos\theta\sin\theta - \sin\theta\cos\theta = 0 \quad \text{and} \quad \nu(X_2) = 0.$$

Therefore, $(\nu, \nu) = 0$. $\square$

The cometric is up to a factor the Hamiltonian function we needed.

Definition (Sub-Riemannian Hamiltonian)**.** The function $H : T^*M \to \mathbb{R}$ defined by $H(\nu) = \frac{1}{2}(\nu, \nu)$ is called the Hamiltonian.

Given a Lagrangian L, one can under certain assumptions transform them into a Hamiltonian system using the Legendre transform [See Mon02, Appendix A.3]. The equation for the transformation of $\tilde{L}$ is given by

$$T(x, \nu) = \sup_v \{\nu(v) - \tilde{L}(x, v)\}, \tag{2.8}$$

where the supremum is taken over all $v \in T_xM$. If we now take the Lagrangian

$$\tilde{L} = \begin{cases} \frac{1}{2}\langle v, v\rangle & v \in \mathcal{H} \\ \infty & \text{else,} \end{cases}$$

we can show that its Legendre transform yields the sub-Riemannian Hamiltonian. Note the similarity to the Lagrangian in Equation (2.6) and that if $\mathcal{H} = TM$, the two coincide.

We can immediately see that in Equation (2.8), the supremum may be taken only over vectors in $\mathcal{H}$, for otherwise the quantity in question equals $-\infty$. Then, if E_a is an orthonormal local frame for $\mathcal{H}$, let ξ^a be the covector corresponding to E_a. If $v = v^a E_a|_p$ is a horizontal vector at p, the expression to be maximized in Equation (2.8) is

$$\nu(v) - \frac{1}{2}\langle v, v\rangle = v^a \nu(E_a|_p) - \frac{1}{2}\sum_a (v^a)^2.$$

By derivating with respect to each v^a, we find that this quantity has a critical point at $v^a = \nu(E_a|_p)$. As the Hessian is equal to minus the identity matrix (so negative definite), this critical point is a maximum. This is the only extreme value of the function, so the maximum is global. Therefore, $T(p, \nu) = \frac{1}{2}\sum_a \nu(E_a|_p)^2$, which using (2.7) equals the expression for $H(\nu)$ in the frame E_a. In conclusion, the Hamiltonian we have constructed is the Legendre transform of the Lagrangian above.

To finally arrive at the geodesics of a sub-Riemannian geometry, we need one additional ingredient, which is the Hamiltonian version of the Euler-Lagrange equations.

Definition (Hamilton's Equations)**.** We call the differential equations

$$\dot{x}^i = \frac{\partial H}{\partial y_i}, \qquad \dot{y}_i = -\frac{\partial H}{\partial x^i} \tag{2.9}$$

on T^*M the Hamilton's equations, where H is the Hamiltonian of M and y_i is the ith coefficient of ν when using the basis dx^i of T^*M.

The projection of solutions to this differential equation on T^*M are sub-Riemannian geodesics.

Definition (Sub-Riemannian Geodesics)**.** If $(\gamma, \kappa) : I \to T^*M$ is a solution curve to Hamilton's equations, then $\gamma(t) : I \to M$ is called a sub-Riemannian geodesic.

Similarly to Riemannian case, these curves are locally minimizing.

Theorem 2.14 (Normal Geodesics)**.** *If $\gamma(t) : I \to M$ is a sub-Riemannian geodesic, then every sufficiently short arc of γ is the unique minimizing curve joining its endpoints.*

For a proof of this theorem, see [Mon02, Section 1.9]. Examining Equation (2.9), we see that for an n-dimmensional manifold there are $2n$ equations and the same number of parameters. However, a geodesic starting at a point p of M may only have $\dim \mathcal{H}_p$ degrees of freedom for its initial direction. As we shall see later in the examples, the remaining degrees of freedom determine how "twisted" a curve becomes after a short time.

However if γ is a geodesic, there is an explicit relation between the initial cotangent vector $w \in T_p^*M$ and $\dot{\gamma}(0)$. That is

$$\beta_p(w) = \dot{\gamma}(0).$$

To see this, we again use the coordinates x^i and a local orthogonal frame E_a for $\mathcal{H}$ around p. Then,

$$dx^i\beta(w) = \langle \beta(w), \beta(dx^i)\rangle = \left\langle \sum_a w(E_a)E_a, \sum_a dx^i(E_a)E_a \right\rangle = \sum_a w(E_a)dx^i(E_a),$$

but also

$$\begin{aligned}
\frac{\partial H}{\partial y_i} &= \frac{1}{2}\frac{\partial\langle \beta(w), \beta(w)\rangle}{\partial y_i} \\
&= \frac{1}{2}\frac{\partial \sum_a (y_j dx^j(E_a))^2}{\partial y_i} \\
&= \sum_a \frac{1}{2}\frac{\partial (y_j dx^j(E_a))^2}{\partial y_i} = \sum_a y_j dx^j(E_a)dx^i(E_a),
\end{aligned}$$

which evaluated at $y_j dx^j = w$ equals the expression for $dx^i\beta(w)$. Therefore the two agree.

Example 2.15 (Hamilton's equations of the unicycle)**.** Using the explicit formula for the cometric of this sub-Riemannian geometry, we compute

$$\begin{aligned} H(a\,dx + b\,dy + c\,d\theta) &= \frac{1}{2}\langle (a\cos\theta + b\sin\theta)X_1 + cX_2, (a\cos\theta + b\sin\theta)X_1 + cX_1\rangle \\ &= \frac{1}{2}\pi_\theta(a,b)^2 + \frac{1}{2}c^2, \end{aligned}$$

with $\pi_\theta(a,b) = a\cos\theta + b\sin\theta$. Then, since we have

$$\frac{\partial \pi_\theta}{\partial a} = \cos\theta \qquad \text{and} \qquad \frac{\partial \pi_\theta}{\partial b} = \sin\theta,$$

Hamilton's equations can be computed to be

$$\begin{aligned} &\dot{x} = \cos\theta\pi_\theta(a,b) \qquad &&\dot{y} = \sin\theta\pi_\theta(a,b) \qquad &&\dot{\theta} = c \\ &\dot{a} = 0 &&\dot{b} = 0 &&\dot{c} = \pi_\theta(a,b)(-a\sin\theta + b\cos\theta). \end{aligned}$$

□

In the same way as we defined the exponential map $\exp_p$ for a Riemannian geometry, we can define it for sub-Riemannian geometry, with the slight alteration that the domain is now a subset of the cotangent bundle.

Definition (The exponential map)**.** Given $p \in M$ and let $B_\epsilon(0) \subset T_p^*M$ be a ball of radius ϵ (with respect to $\langle\cdot,\cdot\rangle$) in the vector space T_pM. Define the exponential map at p

$$\exp_p : B_\epsilon(0) \to M$$

by taking a cotangent vector w to $\gamma(1)$, where (γ, κ) is the solution to Hamilton's equations with $\gamma(0) = p$ and $\kappa(0) = w$.

In the Rimannian case, this map was diffeomorphism from $B_\epsilon(0)$ to a neighbourhood of M for sufficiently small ϵ. If $\mathcal{H}_p \neq T_pM$ then this can not happen in the sub-Riemannian case as then, the cometric is degenerate. This implies that there exists a covector $w \in T_p^*M$ such that $\beta(w) = 0$. Therefore, for all $c \in \mathbb{R}$, the geodesic with inital conditions $cw \in T_p^*M$ is constant, so $d\exp_p$ can not be injective and therefore $\exp_p$ is not a diffeomorphism around the origin in T_p^*M.

2.5 Computing Hamilton's equations using the Poisson bracket

There is also an alternative way of computing Hamiltonian's equations without needing to compute the cometric explicitly. For it, we need the following alternative formulation of the cometric.

Lemma 2.16. *Let X_a be a local frame spanning $\mathcal{H}$ and $g_{ab} = \langle X_a, X_b \rangle$. Define for each X_a the function $P_a : T^*M \to \mathbb{R}$ by*

$$P_a(\nu) = \nu X_a.$$

Then, for the sub-Riemannain Hamiltonian we have

$$H(\nu) = \frac{1}{2} g^{ab} P_a(\nu) P_b(\nu)$$

(with summation convention over a and b) where g^{ab} is the inverse matrix of g_{ab}.

Remark. Note that this way of writing the Hamiltonian is especially useful for when the X_i are orthogonal in the sub-Riemannian metric. Then, the matrix g_{ab} is diagonal so inverting it becomes a matter of taking reciprocals and the Hamiltonian is given as

$$H(\nu) = \sum_a \frac{1}{2g_{aa}} P_a(\nu)^2. \tag{2.10}$$

Proof. In this proof, summation per Einstein convention with the indecies a and b goes up to the rank of $\mathcal{H}$ and summation with indices i, j up to $\dim M$. We first prove that

$$\langle Y, Y \rangle = g^{ab} \langle X_a, Y \rangle \langle X_b, Y \rangle.$$

If $g_{ab} = \langle X_a, X_b \rangle$ and we write the vector field $Y = Y^i X_i$ in the local frame X_i, then $\langle Y, X_a \rangle = Y^i g_{ia}$. Therefore,

$$\langle X, X \rangle = Y^i g_{ij} Y^j = Y^i g_{ia} g^{ab} g_{bj} Y^j = \langle Y, X_a \rangle g^{ab} \langle X_b, Y \rangle = g^{ab} \langle X_a, Y \rangle \langle X_b, Y \rangle,$$

so the claim is proven.

Now if $\nu \in T^*M$, by the properties of β form Proposition 2.11 above, we see that

$$(\nu, \nu) = \nu \beta(\nu) = \langle \beta(\nu), \beta(\nu) \rangle.$$

On the other hand, again by the property of β,

$$g^{ab} \nu(X_a) \nu(X_b) = g^{ab} \langle \beta(\nu), X_a \rangle \langle \beta(\nu), X_b \rangle = g^{ab} \langle X_a, \beta(\nu) \rangle \langle X_b, \beta(\nu) \rangle.$$

Using the above claim, we have proven that $H(\nu) = \frac{1}{2}(\nu, \nu) = \frac{1}{2} g^{ab} P_a(\nu) P_b(\nu)$. $\square$

Let x^i be local coordinates of M in a neighbourhood U. If y_i are the coefficients corresponding to the frame dx^i of T^*U, $(x^i, y_i) = (x^1, \dots, x^n, y_1, \dots, y_n)$ form a set of coordinates of in a neighbourhood of T^*M. Define the Poisson bracket, denoted by $\{\cdot, \cdot\}$, taking two smooth functions f and g on T^*M to the function defined by

$$\{f, g\} = \sum_i \frac{\partial f}{\partial x^i} \frac{\partial g}{\partial y_i} - \frac{\partial g}{\partial x^i} \frac{\partial f}{\partial y_i},$$

This operation satisfies the identity

$$\{f, gh\} = g\{f, h\} + h\{f, g\}, \tag{2.11}$$

where h is another function on T^*M and it can be shown to be coordinate independent[Mon02, Section 1.7].

The Poisson bracket is useful to us, since if $\Phi_t(x^i, y_i)$ is the flow associated to Hamilton's equations and $f : T^*M \to \mathbb{R}$,

$$\dot{f}|_p = \frac{d}{dt} f(\Phi_t(p)) = \{f, H\}|_p,$$

with $p = (x^i, y_i)$ any point in the chart. This is because if $x^i(t)$ and $y_i(t)$ are any solution to Hamilton's equations,

$$\begin{aligned}\frac{d}{dt} f(x^1(t), \ldots, x^n(t), y^1(t), \ldots, y^n(t)) &= \frac{\partial f}{\partial x^i}\frac{\partial x^i}{\partial t} + \frac{\partial f}{\partial y_i}\frac{\partial y_i}{\partial t} \\ &= \sum_i \frac{\partial f}{\partial x^i}\frac{\partial H}{\partial y_i} - \frac{\partial f}{\partial y_i}\frac{\partial H}{\partial x^i} \\ &= \{f, H\}.\end{aligned}$$

By letting f equal the coordinate functions x^i, y_i, we recover Hamiltons equations as in (2.9), so we can find them by computing Poisson brackets alone.

Given a frame X_i for TM, if we instead of y_i choose to work with the functions P_{X_i} defined as in Lemma 2.16, we have

$$\dot{x}^i = \{x^i, H\} \qquad \dot{P}_{X_i} = \{P_{X_i}, H\}. \tag{2.12}$$

Furthermore, if we are in coordinates with an orthogonal frame X_i for $\mathcal{H}$, the first half of this expression becomes particularly simple. In coordinates, the function $P_{X_i}(x^i, y_i)$ amounts to taking an inner product of the vector (y_i) with the vector of component functions of the vector field X_i. In other words, if $X = X^j \partial_{x^j}$, $P_X(y_j dx^j) = y_j X^j$. Then, since the components X_j^i of the local frame X_j do not depend on the y_i, $\frac{\partial P_{X_i}}{\partial y_j} = X_i^j$. Also, g^{jj} does not depend on y_i. Thus, using the expression in (2.10), the first half of Equations (2.12) turns out to be

$$\dot{x}^i = \sum_j g^{jj} X_j^i P_{X_j}. \tag{2.13}$$

In combination with the rule that

$$\{P_X, P_Y\} = -P_{[X,Y]}, \tag{2.14}$$

once we have computed all brackets $[X_i, X_j]$ for our orthogonal frame of choice spanning TM, we can also easily compute the other half of Equation (2.12). Equation (2.14) is verified by letting $P_X = X^j P_j$ and $P_Y = Y^j P_j$, with $P_j = P_{\partial/\partial_{x^j}}$. Then,

$$\frac{\partial P_j(y_j\, dx^j)}{\partial y_i} = \frac{\partial y_j}{\partial y_i} = \delta_i^j.$$

so since X^j does not depend on y_i,

$$\frac{\partial X^j P_j}{\partial y_i} = \frac{X^j \partial y_j}{\partial y_i} = X^j.$$

Similarly,

$$\frac{\partial X^j P_j}{\partial x^i} = P_j \frac{\partial X^j}{\partial x^i}.$$

Therefore,

$$\begin{aligned} \{P_X, P_Y\} &= \sum_i \frac{\partial (X^j P_j)}{\partial x^i} \frac{\partial (Y^j P_j)}{\partial y_i} - \frac{\partial (Y^j P_j)}{\partial x^i} \frac{\partial (X^j P_j)}{\partial y_i} \\ &= Y^i \frac{\partial X^j}{\partial x^i} P_j - X^i \frac{\partial Y^j}{\partial x^i} P_j, \end{aligned}$$

If we evaluate this function at the covectorfield dx^i, we get

$$\left(Y^i \frac{\partial X^j}{\partial x^i} P_j - X^i \frac{\partial Y^j}{\partial x^i} P_j \right)(dx^k) = Y^i \frac{\partial X^k}{\partial x^i} - X^i \frac{\partial Y^k}{\partial x^i},$$

which equals the coefficient of ∂_k in $[Y, X] = -[X, Y]$, so $\{P_X, P_Y\} = -P_{[X,Y]}$.

Example 2.17 (Hamilton's equations of the unicycle)**.** Let

$$P_1(\nu) = \nu(X_1) \qquad P_2(\nu) = \nu(X_2) \quad \text{and} \quad P_3(\nu) = \nu(X_3),$$

with $X_3 = -[X_1, X_2] = \sin\theta\, \partial_x - \cos\theta\, \partial_y$. Since X_1 and X_2 are orthonormal, we can write $H(\nu) = \frac{1}{2}(P_1(\nu)^2 + P_2(\nu)^2)$. Using the bracket relations

$$[X_1, X_2] = -X_3, \qquad [X_2, X_3] = -X_1, \qquad [X_1, X_3] = 0,$$

and the relation in Equation 2.11 on the Hamiltonian above we obtain

$$\begin{aligned} \dot{x} &= \cos\theta P_1 & \dot{y} &= \sin\theta P_1 & \dot{\theta} &= P_2 \\ \dot{P}_1 &= P_3 P_2 & \dot{P}_2 &= -P_1 P_3 & \dot{P}_3 &= -P_1 P_2. \end{aligned}$$

This system is equivalent to the differential equations previously computed under the change of coordinates $P_1 = a\cos\theta + b\sin\theta$, $P_3 = -a\sin\theta + b\cos\theta$ and $P_2 = c$.

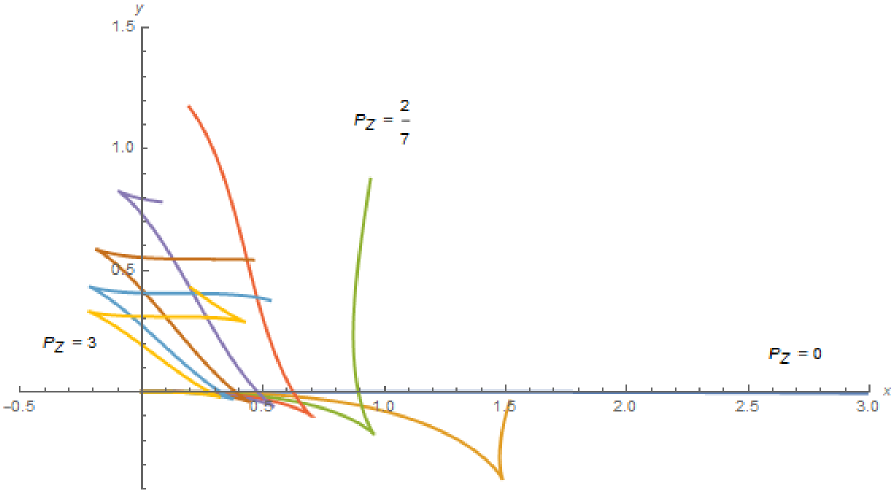


Figure 2.5: Solution curves to Hamilton's equations of the unicycle.

Note that x and y do not appear on the right side of the equations, which tells us that the system is translation invariant. Computing $\dot{x}^2 + \dot{y}^2$ and $\dot{\theta}^2$, we see that $P_1(0)$ gives us the initial forward velocity and $P_2(0)$ the initial angular velocity of the unicycle. By deriving $\dot{P}_1^2 + \dot{P}_2^2$ with respect to the time variable, one can see that the sum squares of the two velocities is constant.

If we solve this system numerically, it seems like the parameter P_3 determines how much of a "parallel-parking" path the unicycle takes. In Figure 2.5 we have used initial conditions $(x(0), y(0), \theta(0)) = (0, 0, 0)$, the initial speed $P_1(0) = 1$ and the initial angular velocity $\dot{\theta}(0) = P_2(0) = 0$ while varying $P_3 \in [0, 3]$. Solutions are drawn in the (x, y) plane for times in the interval $[0, 3]$. When looking at the Figure, note how less distance from start to end point is covered when the P_Z value is increased. Also note that instead of turning on the spot, the solution changes direction as it drives, even utilizing initial countersteering.

Lastly note that the solution curves overlap quite quickly. This is not in contradiction with Theorem 2.5 as at the point of overlap, the solutions have different values in the θ coordinate. $\square$

2.6 Computing a sub-Riemannian geodesic sphere

To get an impression of the notion of distance within a certain sub-Riemannian geometry, we can visualize its geodesic sphere. To draw all solutions to Hamilton's equations starting at p, we compute $\exp(B)$, where B is some subset of T_p^*M to be determined. However, exp satisfies the following helpful condition: If (γ, κ) is the solution to Hamilton's equations for the initial covector ν, then,

$$\gamma(t) = \exp(t\nu), \tag{2.15}$$

for all t in the domain of γ [ABB12, Corollary 8.34]. Therefore, if ∂B is the boundary of the subset of T_p^*M, we only need to draw $\exp(\partial B)$ to visualize the points reachable by all solutions.

In order for all solutions starting at p to have equal length, we need to ensure that their inital speeds $\langle\dot\gamma(0), \dot\gamma(0)\rangle$ are all equal to one. Practically, we do this as follows. Fix a point $p \in M$ and choose coordinates x^i around it. We will take p to be the center of our unit distance sphere which gives us initial conditions for the first n equations. Let X_a be an orthogonal frame for $\mathcal{H}$ in this neighbourhood. Then, using Equation (2.13) we find that the initial tangent vector $\dot\gamma(0)$ equals

$$\dot\gamma(0) = g^{jj} P_{X_j}(t) X_j.$$

Therefore, the square of its length is given as

$$\begin{aligned}\langle\dot\gamma(0), \dot\gamma(0)\rangle &= \langle g^{jj} P_{X_j}(0) X_j, g^{jj} P_{X_j}(0) X_j\rangle \\ &= (g^{jj})^2 (P_{X_j}(0))^2 \langle X_j, X_j\rangle = g^{jj} P_{X_j}(0)^2,\end{aligned} \tag{2.16}$$

where we used that X_j are orthogonal in the sub-Riemannian metric. If the X_j are additionally normal vectors, this simplifies to choosing initial conditions such that $\sum_j P_{X_j}(0)^2 = 1$. In any case, if the rank of $\mathcal{H}$ is k, this equation determines one of the parameters $P_{X_j}(0)$ and leaves us with $k-1$ parameters to choose to determine the initial (horizontal) direction. For the $n-k$ remaining $P_{X_j}(0)$ variables, we are free to choose any initial conditions, so the set B is in some sense a cylinder. This is illustrated in the next example.

Example 2.18 (The geodesic sphere of the unicycle)**.** To parametrize the initial conditions necessary to find solutions with $\dot\gamma(0)$ of unit length, we use the expressions of Example 2.17 in Equation 2.16. Therefore we obtain

$$P_{X_1}^2 + P_{X_2}^2 = 1,$$

so we can use the parametrizations $P_X = \cos\lambda$, $P_Y = \sin\lambda$. The variable P_Z is free and we sweep through it linearly over some interval $[-k, k]$.

In Figure 2.6 we draw a numerical approximation of the exponential map of initial conditions parametrized as above. The variables λ and P_Z we chosen to lie in the cylinder $(\lambda, P_Z) \in S^1, [-18, 18]$ and the plot is cut open at $x = 0.6$. The initial point is the origin (in particular also $\theta(0) = 0$). The obtained shape extends not very far in y direction. This is essentially the parallel parking problem, that starting out parallel to the x-axis, it takes more time to move in the y direction than to move in the x direction. As we have seen in the previous example, a zigzag motion provides the shortest way to do so, where our model considers rotation equally costly as forward motion. □

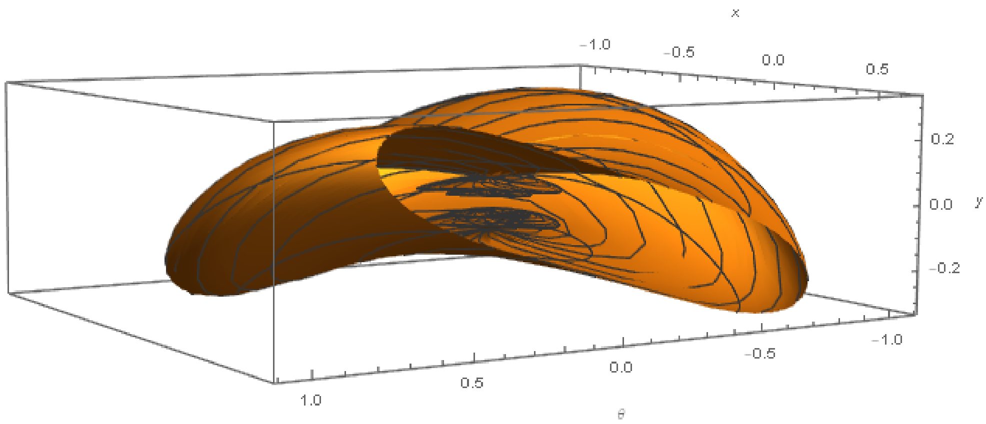


Figure 2.6: The unit sphere of the unicycle.

3 The Sub-Riemannian Geometry of the Hopf-fibration

3.1 A manifold structure on $\mathbb{C}^2$

Since the Hopf-fibration is a construction on the three dimensional sphere, which can be realized as an embedded submanifold of $\mathbb{R}^4$, we will do many of our computations in this vector space. To make this more managable, we can identify $\mathbb{R}^4$ with $\mathbb{C}^2$ in the following way. If x^i, y^i, $i = 1, \ldots, n$ are the global standard coordinates for $\mathbb{R}^{2n}$, we can define a mapping to $\mathbb{C}^n$ with coordinates u^i by

$$\psi(x^1, x^2, \ldots, x^n, y^1, y^2, \ldots, y^n) \mapsto (x^1 + iy^1, \ldots, x^n + iy^n),$$

with an inverse map given by $(u^1, \ldots, u^n) \mapsto \frac{1}{2}(u^1 + \bar{u}^1, u^2 + \bar{u}^2, \ldots, u^n + \bar{u}^n, u^1 - \bar{u}^1, u^2 - \bar{u}^2, \ldots, u^n - \bar{u}^n)$. As both of these are smooth, we find that the association ψ_4 of $\mathbb{R}^4$ with $\mathbb{C}^2$ is a diffeomorphism.

Since the properties of (sub-)Riemannian geometry are encoded on the tangent and cotangent bundles of a manifold, we will also need to understand how these work on $\mathbb{C}^2$. On $\mathbb{R}^n$ with coordinates x^i and y^i, ∂_{x^i} and ∂_{y^i} span the tangent space at each point and if we use the differential of the diffeomorphism ψ, we find the tangent vectors $d\psi\partial_{x^i} = \partial_{\mathrm{Re}\, u^i}$ and $d\psi\partial_{y^i} = i\partial_{\mathrm{Im}\, u^i}$, which when writing u^i as $x^i + iy^i$, in slight abuse of notation, we can see $\partial_{\mathrm{Re}\, u} = \partial_{x^1}$ and $i\partial_{\mathrm{Im}\, u} = i\partial_{x^2}$.

With this, we define

$$\partial_u = \frac{1}{2}(\partial_{x^1} - i\partial_{x^2}) \qquad \partial_{\bar{u}} = \frac{1}{2}(\partial_{x^1} + i\partial_{x^2}),$$

in order to get the simple identities

$$\partial_u u = \frac{1}{2}(\partial_{x^1} - i\partial_{x^2})(x_1 + ix_2) = 1, \qquad \partial_u \bar{u} = \frac{1}{2}(\partial_{x^1} - i\partial_{x^2})(x^1 - ix^2) = 0,$$
$$\partial_{\bar{u}} \bar{u} = \frac{1}{2}(\partial_{x^1} + i\partial_{x^2})(x_1 - ix_2) = 1, \qquad \partial_{\bar{u}} u = \frac{1}{2}(\partial_{x^1} + i\partial_{x^2})(x^1 + ix^2) = 0.$$

The operation ∂_u is complex linear. If $h, g : \mathbb{R}^{2n} \to \mathbb{R}$ are smooth and real valued,

$$\frac{\partial ih}{\partial x^j} = i\frac{\partial h}{\partial x^j} \qquad \frac{\partial ih}{\partial y^j} = i\frac{\partial h}{\partial y^j}$$

and if $f = g + h$,

$$\partial_{u^j} if = \frac{1}{2}(\partial_{x^j} - i\partial_{y^j})(-g + ih) = \frac{1}{2}(\partial_{x^j} g + i\partial_{y^j} g + \partial_{x^j} ih - i\partial_{y^j} ih)$$
$$= \frac{1}{2}(\partial_{y^j} h - \partial_{x^j} g + i\partial_{y^j} g + i\partial_{x^j} h) = i\partial_{u^j} f.$$

Also, the operator $\partial_{\bar{u}}$ is complex linear as a similar computation of $\partial_{\bar{u}^j} if$ shows.

At this point we should note that if the tangent space $T\mathbb{C}^n$ consists of all linear combinations of $2n$ linearly independent vectors, we should end up with a $2n$ dimensional vector space. However, for our purposes we only consider vector fields of the form $f^j\,\partial_{u^j} + \bar{f}\,\partial_{\bar{u}^j}$, with the f^j being complex valued functions on $\mathbb{C}^n$. This is because if $f^j = g^j + ih^j$ (with the g^j and h^j real valued and smooth), then

$$f^j\,\partial_{u^j} + \bar{f}^j\,\partial_{\bar{u}^j} = (g^j + ih^j)\frac{1}{2}(\partial_{x^j} - i\partial_{y^j}) + (g^j - ih^j)\frac{1}{2}(\partial_{x^j} + i\partial_{x^j})) = g^j\,\partial_{x^j} + h^j\,\partial_{x^j},$$

so in writing the tangent vector $f^j\,\partial_{u^j} + \bar{f}^j\,\partial_{\bar{u^j}} = 2\,\mathrm{Re}\,f\,\partial_{u^j}$, we obtain a real valued derivative and we have a convenient description of the real vector field $g^j\,\partial_{x^j} + h^j\,\partial_{x^j}$ on $\mathbb{R}^{2n}$. Using this, we can also associate a real vector field on $\mathbb{R}^4$ with one on $\mathbb{C}^2$ by writing

$$X^j\,\partial_{x^j} + Y^j\,\partial_{y^j} \leftrightarrow 2\,\mathrm{Re}((X^j + iY^j)\partial_{u^j}). \tag{3.1}$$

It has to be noted that we do not require the component functions of a vector field on $\mathbb{C}$ to be holomorphic. For example, $u \mapsto \bar{u}$ on $\mathbb{C}$ gives us the vector field $2\,\mathrm{Re}(\bar{u}\partial_u) = \bar{u}\,\partial_u + u\partial_{\bar{u}}$ which corresponds to the smooth, real vector field $x^1\,\partial_{x^1} - y^1\,\partial_{y^1}$.

The covectors on $\mathbb{C}$ can be defined analogously as $du = dx^1 + idx^2$ and $d\bar{u} = dx^1 - idx^2$ in order to obtain the familiar relations

$$\begin{aligned} du\partial_u &= 1 \qquad & du\partial_{\bar{u}} &= 0 \\ d\bar{u}\partial_u &= 0 \qquad & d\bar{u}\partial_{\bar{u}} &= 1. \end{aligned}$$

Therefore, given a real covector field $\nu_1 dx^1 + \nu_2 dx^2$ on $\mathbb{R}^2$, we obtain the associated complex covector field $\bar{z}\,du + z\,d\bar{u}$ by letting $z = \frac{1}{2}(\nu_1 + i\nu_2)$, as can be seen by computing

$$(\bar{z}\,du + z\,d\bar{u}) = \frac{1}{2}(\nu_1 - i\nu_2)(dx^1 + idx^2) + \frac{1}{2}(\nu_1 + i\nu_2)(dx^1 - idx^2) = \nu_1 dx^1 + \nu_2 dx^2.$$

Therefore, between R^4 and $\mathbb{C}^2$ we have the association

$$\nu_i\,dx^i \leftrightarrow \frac{1}{2}\left[(\nu_1 - i\nu_2)\,du + (\nu_1 + i\nu_2)\,d_{\bar{u}} + (\nu_3 - i\nu_4)\,dv + (\nu_3 + i\nu_4)\,d_{\bar{v}}\right]. \tag{3.2}$$

The standard inner product of $\mathbb{R}^4$ can also be carried over to $\mathbb{C}^2$. If φ is the invers map of ϕ form $\mathbb{C}^n$ to $\mathbb{R}^{2n}$ given by

$$\varphi(u^i) = (\operatorname{Re} u^i, \operatorname{Im} u^i),$$

we can compute its differential to be the fiber linear map determined by

$$d\varphi\partial_{u^j} = \frac{1}{2}(\partial_{x^j} - \partial_{y^j}), \qquad d\varphi\partial_{\bar{u}^j} = \frac{1}{2}(\partial_{x^j} + \partial_{y^j}), \tag{3.3}$$

The map φ is a diffeomorphism, so an immersion, so we can define a metric on $\mathbb{C}^2$ as the pullback of the standard metric on $\mathbb{R}^4$. That is, if X and Y are vector fields on $\mathbb{C}^2$, we define

$$\langle X, Y\rangle_{\mathbb{C}} = \langle d\varphi X, d\varphi \bar{Y}\rangle.$$

For vectorfields of the form as in Equation (3.1), we have a particularly easy expression for their inner product.

Lemma 3.1 (An inner product formula on $\mathbb{C}^2$)**.** *We have that*

$$\langle 2\operatorname{Re}(f^l\partial_{u^l}), 2\operatorname{Re}(g^j\partial_{u^j})\rangle_{\mathbb{C}} = \sum_j \operatorname{Re}(f^j\bar{g}^j)$$

Proof. We write

$$2\operatorname{Re}(f^l\partial_{u^l}) = f^l\partial_{u^l} + \bar{f}^l\partial_{\bar{u}^l} \qquad \text{and} \qquad 2\operatorname{Re}(g^j\partial_{u^j}) = g^j\partial_{u^j} + \bar{g}^j\partial_{\bar{u}^j}.$$

Next, note that $\langle\partial_{u^l}, \partial_{\bar{u}^j}\rangle = \delta_{lj}/2 = \langle\partial_{\bar{u}^l}, \partial_{u^j}\rangle$ and $\langle\partial_{u^j}, \partial_{u^l}\rangle = 0 = \langle\partial_{\bar{u}^j}, \partial_{\bar{u}^l}\rangle$ for all j and l, which can be seen by computing with the identities in Equation 3.3. Therefore, if we expand the inner product,

$$\langle f^l\partial_{u^l} + \bar{f}^l\partial_{\bar{u}^l}, g^j\partial_{u^j} + \bar{g}^j\partial_{\bar{u}^j}\rangle = f^j\bar{g}^j\frac{1}{2} + \bar{f}^j g^j\frac{1}{2} = \frac{1}{2}(f^j\bar{g}^j + \overline{f^j\bar{g}^j}) = \sum_j \operatorname{Re} f^j\bar{g}^j.$$

This shows the claimed equality. ☐

3.2 The Hopf-fibration

Let (u, v) be global coordinates on $\mathbb{C}^2$. We construct the Hopf fibration as follows: If we identify S^1 with the unit circle in $\mathbb{C}$ and S^3 with the subset of points (u, v) of $\mathbb{C}^2$ such that $u\bar{u} + v\bar{v} = 1$, we can define an action $S^1 \times \mathbb{C}^2 \to \mathbb{C}^2$ by

$$(e^{it}, (u, v)) \mapsto (e^{it}u, e^{it}v).$$

If we compute $\langle (e^{it}u, e^{it}v), (e^{it}u, e^{it}v) \rangle = e^{it}e^{-it}(u\bar{u} + v\bar{v}) = 1$, we see that the action maps points in $\mathbb{C}^2$ with norm one to themselves, so we will consider it as an action $S^1 \times S^3 \to S^3$. This action is smooth, for if it is the restriction of the smooth map $\mathbb{C} \times \mathbb{C}^2 \to \mathbb{C}^2$ mapping $(z, (u, v)) \mapsto (zu, zv)$.

It is easy to see that this action is free, for otherwise $u = e^{it}u$ for a nonzero t, which would imply that $e^{it} = 1$, so t is a multiple of 2π. To verify that it is a proper action, we use that a continuous action by a compact Lie group on a manifold is proper [See Lee12, Corollary 21.6], and as S^1 is compact, this shows that the action above is free, proper and smooth. This in turn tells us that the quotient space S^3/S^1 has a unique smooth structure such that the projection $\pi : S^3 \to S^3/S^1$ is a smooth submersion [See Lee12, Theorem 21.10]. We denote the space S^3/S^1 as $\mathbb{CP}^1$.

Looking back to the start of the chapter, we still have to argue how this construction constitutes a fibration. In this light, we first give the precise definition in the context of manifolds.

Definition (smooth fiber bundle)**.** If M and F are smooth manifolds, a *fiber bundle* over M with model fiber F is a manifold E with a smooth surjective map $p : E \to M$ with the property that for each $x \in M$, there exists a neighborhood U of x in M and a diffeomorphism $\phi : p^{-1}(U) \to U \times F$, called a local trivialization of E over U, such that the following diagram commutes:

$$\begin{tikzcd} p^{-1}(U) \arrow[r, "\phi"] \arrow[dr, "p"'] & U \times F \arrow[d, "\pi_1"] \\ & U. \end{tikzcd}$$

[See Lee12, p. 268]

Now, as we have seen that the map π is a smooth submersion that is clearly surjective onto $\mathbb{CP}^1$, by the Ehresmann Lemma [Ehr50], S^3 is a fibre bundle over $\mathbb{CP}^1$ with model fiber S^1.

It turns out that the space $\mathbb{CP}^1$ is diffeomorphic to S^2: Define a map $p : (u, v) \mapsto (2u\bar{v}, u\bar{u} - v\bar{v})$ mapping $S^3 \to \mathbb{C} \times \mathbb{R}$. The computation

$$\begin{aligned} \langle (2u\bar{v}, u\bar{u} - v\bar{v}), (2u\bar{v}, u\bar{u} - v\bar{v}) \rangle &= 4u\bar{u}v\bar{v} + (u\bar{u} - v\bar{v})^2 \\ &= 4u\bar{u}v\bar{v} + (u\bar{u})^2 + (v\bar{v})^2 - 2u\bar{u}v\bar{v} \\ &= (u\bar{u} + v\bar{v})^2 = 1 \end{aligned}$$

using the product metric shows that the range of p is actually a subset of $S^2 = \{(u, r) \in \mathbb{C}^2 \times \mathbb{R} \mid u\bar{u} + r^2 = 1\}$. To see that it is surjective, we need to solve

$2u\bar{v} = z$ and $u\bar{u} - v\bar{v} = s$ for a given z in $\mathbb{C}$ and s in $\mathbb{R}$. To do so, write $z = r_z e^{it_z}$ and $u = r_u e^{it_u}$, $v = r_v e^{it_v}$. Then, the first equation becomes

$$r_z e^{i\theta} = 2r_u r_v e^{i(t_u - t_v)},$$

and we can clearly find t_u and t_v such that $t_u - t_v = t_z$. For the variables r_u and r_v, we will need to satisfy

$$2r_u r_v = r_z, \qquad r_u^2 - r_v^2 = s \quad \text{and} \quad r_u^2 + r_v^2 = 1,$$

as we are trying to find a solution in S^3. By subtracting and adding the last two equations, we find $r_u = \sqrt{(1+s)/2}$ and $r_v = \sqrt{(1-s)/2}$. This also satisfies the first equation by using that $s^2 + r_z^2 = 1$. Therefore $(r_u e^{it_u}, r_v e^{it_v})$ maps to (z, r) under p and therefore p is surjective. Computing the dp, we see that

$$dp = \begin{bmatrix} 2\bar{v} & 0 & 0 & 2u \\ \bar{u} & u & \bar{v} & v \end{bmatrix}$$

as written in the basis ∂_u, $\partial_{\bar{u}}$, ∂_v and $\partial_{\bar{v}}$ is of full rank. Therefore p is a surjective submersion. Clearly, for $\lambda \in S^1$, $p(u,v) = p(\lambda u, \lambda v)$. If $p(r_u e^{it_u}, r_v e^{it_v}) = p(r_a e^{it_a}, r_b e^{it_b})$, we know that $r_a^2 - r_b^2 = r_u^2 - r_v^2$ which when added and subtracted from the equation $r_a^2 + r_b^2 = r_u^2 + r_v^2$ shows that $r_a = r_u$ and $r_b = r_v$. We also know that $t_a - t_b = t_u - t_v$, so $(a,b) = (\lambda a, \lambda b)$ for $\lambda \in S^1$. Therefore p and π are surjective smooth submersions such that they are constant on each others fibers, so $\mathbb{CP}^1$ and S^2 are diffeomorphic (see [Lee12, Thm. 4.31]). With this we might also view the Hopf-fibration as a fiber bundle of S^3 over S^2 with model fiber S^1.

This fibration locally "splits" the tangent space of S^3 in two parts: $\ker d\pi$, which, as $d\pi$ is surjective, is a 1-dimensional subspace of the tangent space, and a remaining perpendicular part which we will make precise later on.

3.3 Sub-Riemannian structures on S^3 from the Hopf p, q action

As we have seen in the previous section, the Hopf-fibration is described by the smooth action $S^1 \times S^3 \to S^3$,

$$e^{it}.(u,v) = (e^{it}u, e^{it}v).$$

Then, we also saw that $p : S^3 \to \mathbb{CP}^1 \cong S^2$ is a fibre bundle of S^3 over S^2. Therefore, we have the local trivialisations $\phi : p^{-1}(U) \to U \times F$ such that $\pi_1 \circ \phi = p$, where $\pi_1 : U \times S^1 \to U$ is the projection onto the first component. This implies that $\ker dp = \ker d\pi_1 \circ \ker d\phi$, so as ϕ is a diffeomorphism, we see that

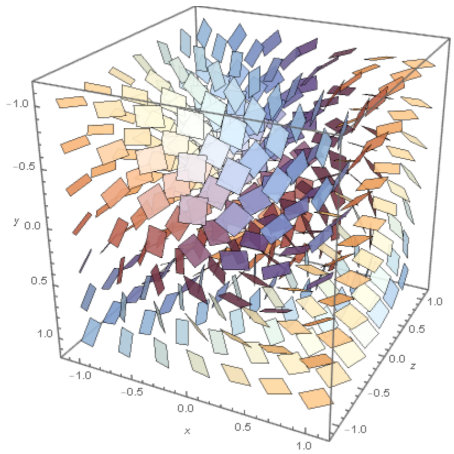


Figure 3.1: The plane field of the distribution $\mathcal{H}^{1,1}$.

$\ker dp \cong \ker d\pi_1 = 0 \times TS^1 \subset TU \times TS^1$. On the other hand, since p is constant on the fibers of the action,

$$0 = \lim_{t\to 0} \frac{p(e^{it}u, e^{it}v) - p(u,v)}{t} = dp|_{(u,v)} \circ \left(\frac{d}{dt}\bigg|_{t=0} e^{it}.(u,v) \right),$$

by the chain rule. From this we conclude that the infinitesimal generator of the group action is a basis for $\ker p$.

When using the induced metric of $\iota : S^3 \to \mathbb{R}^4$ on S^3, this allows us to decompose the tangent space of S^3 into $\ker dp \oplus (\ker dp)^\perp$. To obtain a sub-Riemannian geometry on S^3, we will define $\mathcal{H}$ as $(\ker dp)^\perp$ and restrict the induced metric to it.

More generally, on the p, q Hopf-fibration with the smooth action $S^1 \times S^3 \to S^3$

$$e^{it}.(u,v) = (e^{pit}u, e^{qit}v),$$

we do not have the structure of a fiber bundle. However, we can still define a sub-Riemannian geometry on S^3 as before: we take $\mathcal{H}^{p,q}$ to be the orthogonal complement to the sub bundle spanned by $Z = d/dt|_{t=0}e^{it}.(u,v)$. The horizontal distribution of the $p = q = 1$ action is illustrated in Figure 3.1.

The vector field Z is smooth as it can be constructed in the following way. If $F : S^1 \times S^3 \to S^3$ denotes the smooth action from above,

$$dF|_{(t,(u,v))} : T_tS^1 \times T_{(u,v)}S^3 \to T_{F(t,(u,v))}S^3$$

is a smooth map. The infinitesimal generator of the action can now be given as the restriction

$$\tilde{F}(u,v) = dF_{(0,(u,v))}(\partial_t, 0),$$

so it is the restriction of a smooth map. In the ambient space $\mathbb{C}^2$, the Z is given by the formula

$$Z = \frac{d}{dt}\bigg|_{t=0} e^{it}.(u,v) = \frac{d}{dt}\bigg|_{t=0} (e^{ipt}u, e^{iqt}v) = (ipu, iqv). \tag{3.4}$$

As the vector field is nonvanishing, this shows that $\mathcal{V} = \operatorname{span}\{Z\}$ is a smooth subbundle of TM. Lemma 2.7 then tells us that $\mathcal{H}^{p,q} = \mathcal{V}^{\perp}$ is a distribution.

This distribution is a contact distribution. If we write the vector in Equation (3.4) in standard coordinates $x^1, \dots, x^4$ of $\mathbb{R}^4$ and take $\omega = \iota_Z\langle\cdot,\cdot\rangle$, we get

$$\omega = -px^2\,dx^1 + px^1\,dx^2 - qx^4\,dx^3 + qx^3\,dx^4.$$

To turn this into a one-form on S^3, we use the smooth embedding $\iota : S^3 \to \mathbb{R}^4$, and define $\tilde{\omega} = \iota^*\omega$ as the pullback of ω. Then, by the properties of the pullback [Lee12, Lemma 14.16] and by the naturality of the exterior derivative [Lee12, Proposition 14.26],

$$\tilde{\omega} \wedge d\tilde{\omega} = \iota^*\omega \wedge d\iota^*\omega = \iota^*\omega \wedge \iota^* d\omega = \iota^*(\omega \wedge d\omega).$$

This in combination with the fact that $d\iota$ is injective, it suffices to show that $\omega \wedge d\omega$ is nonzero. The computations show that

$$d\omega = 2p\,dx^1 \wedge dx^2 + 2q\,dx^3 \wedge dx^4,$$

and therefore

$$\begin{aligned}\omega \wedge d\omega = 2pq(x^1\,dx^2 \wedge dx^3 \wedge dx^4 - x^2\,dx^1 \wedge dx^3 \wedge dx^4 \\ + x^3\,dx^1 \wedge dx^2 \wedge dx^4 - x^4\,dx^1 \wedge dx^2 \wedge dx^3) \neq 0.\end{aligned}$$

Clearly, $H^{p,q} = \iota^* \ker\omega$ as the round metric on S^3 is equal to the induced metric of the immersion ι. Thus, S^3 with the horizontal distribution $\mathcal{H}^{p,q}$ admits no singular geodesics.

In the case $p = q = 1$, the fact that the hopf-Fibration is a fiber bundle gives us additional structure for computing sub-Riemannian geodesics. In Montgomery's book, the notion of a constant bi-invariant metric is defined on principle bundles where the metric is the restriction of a Riemannian metric. Essentially, the requirement is that the differential of the action of any group element is an isometry, that is, it leaves the value of the metric unchanged when applied to both arguments of $\langle\cdot,\cdot\rangle$. In a sub-Riemannian geometry on a principal bundle of constant bi-invariant type, the sub-Riemannian geodesics can be obtained from the Riemannian geodesics [Mon02, Theorem 1.26]. However, for the general p, q action, we neither have that the space is a fiber bundle, nor does the action satisfy the constant bi-invariant condition.

With this result, the sub-Riemannian geometry of the case $p = q = 1$ has been thoroughly studied [CMV11], [MM12]. However, as far as we know, all of our results for the general p, q action are original.

In order to do any explicit computations on this sub-Riemannian geometry, we will need to find a frame spanning the horizontal distribution. We could do this in local coordinates, however the generator of the action Z is best described in the ambient coordinates. Of course, in doing so we have to ensure that the vectors we work with are orthogonal to $N = x^1\partial_{x^1} + \ldots x^4\partial_{x^4}$, the outwards pointing normal vector. We find a basis for the horizontal distribution by finding vector fields X_1 and X_2 such that the equations

$$\langle X, N\rangle = 0 \qquad \text{and} \qquad \langle X, Z\rangle, \tag{3.5}$$

are satisfied for all points in $p \in \mathbb{R}^4$ with $\|p\| = 1$. As there are four component functions of each, X_1 and X_2 to be determined, this system is underdetermined. In working these out, I have come across two pairs of solutions that are still manageable to work with explicitly.

The first one is given by the vectors

$$\begin{pmatrix} -x^1x^4p + x^2x^3q \\ -x^2x^4p - x^1x^3q \\ 0 \\ ((x^1)^2 + (x^2)^2)\,p \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} -x^1x^3p - x^2x^4q \\ -x^2x^3p + x^1x^4q \\ ((x^1)^2 + (x^2)^2)\,p \\ 0 \end{pmatrix}$$

using the standard basis $\partial_{x^1} \ldots \partial_{x^4}$ for $T\mathbb{R}^4$. These vectors have a number of disadvantages . First, they are not orthogonal, so computing the Hamiltonian with them would include additional terms. They vanish when $x^1 = x^2 = 0$, so they cannot be used to describe the distribution on all of S^3. Lastly, I was not able to find a simple expression for corresponding vectors in $\mathbb{C}^2$, which would help us when computing the Lie bracket of the two vectors. For that reason we will not show that they satisfy Equation (3.5) and only include them for completeness.

The other solution is given in $\mathbb{R}^4$ by the vectors

$$\begin{pmatrix} -x^1((x^3)^2 + (x^4)^2) \\ -x^2((x^3)^2 + (x^4)^2) \\ x^3((x^1)^2 + (x^2)^2) \\ x^4((x^1)^2 + (x^2)^2) \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} qx^2((x^3)^2 + (x^4)^2) \\ -qx^1((x^3)^2 + (x^4)^2) \\ -px^4((x^1)^2 + (x^2)^2) \\ px^3((x^1)^2 + (x^2)^2) \end{pmatrix}$$

which can be conveniently rewritten on $\mathbb{C}^2$ as

$$X = 2\,\mathrm{Re}(-uv\bar{v}\partial_u + vu\bar{u}\partial_v) \qquad Y = 2\,\mathrm{Re}(-iquv\bar{v}\partial_u + ipvu\bar{u}\partial_v).$$

To see that these satisfy (3.5), we rewrite N and Z on $\mathbb{C}^2$ as $N = 2\,\mathrm{Re}(u\partial_u + v\partial_v)$ and $Z = 2\,\mathrm{Re}(ipu\partial_u + iqv\partial_v)$. Since we defined the metric on S^3 as the induced metric from its ambient space, we use the regular inner product on $\mathbb{C}^2$. Then, using Lemma 3.1, we compute

$$\begin{aligned}
\langle X, N\rangle &= \operatorname{Re}(-uv\bar{v}\,\bar{u} + vu\bar{u}\,\bar{v}) = 0 \\
\langle X, Z\rangle &= \operatorname{Re}(-uv\bar{v}\,(-i)p\bar{u} + vu\bar{u}\,(-i)q\bar{v}) = \operatorname{Re}(i(p-q)u\bar{u}v\bar{v})) = 0 \\
\langle Y, N\rangle &= \operatorname{Re}(-iquv\bar{v}\,\bar{u} + ipvu\bar{u}\,\bar{v}) = \operatorname{Re}(i(p-q)u\bar{u}v\bar{v}) = 0 \\
\langle X, Z\rangle &= \operatorname{Re}(-iquv\bar{v}\,(-i)p\bar{u} + ipvu\bar{u}\,(-i)q\bar{v}) = pqu\bar{u}v\bar{v} - pqu\bar{u}v\bar{v} = 0.
\end{aligned}$$

Lastly, we check that X and Y are orthogonal:

$$\langle X, Y\rangle = \operatorname{Re}(-uv\bar{v}\,iq\bar{u}\bar{v}v + vu\bar{u}\,(-i)p\bar{v}\bar{u}u) = \operatorname{Re}((-i)u\bar{u}v\bar{v}(q+p)) = 0$$

3.4 The Hamiltonian and Hamilton's equations

We will use

$$X = 2\operatorname{Re}(-uv\bar{v}\partial_u + vu\bar{u}\partial_v) \qquad\qquad Y = 2\operatorname{Re}(-iquv\bar{v}\partial_u + ipvu\bar{u}\partial_v).$$

spanning $\mathcal{H}^{p,q}$ to compute the Hamiltonian for $S^3 \cap \{u \neq 0\} \cap \{v \neq 0\}$. Since the vectors are orthogonal to each other in the sub-Riemannian metric, we may use the simplified formula of Equation 2.10 for the Hamiltonian. To do so, we compute first compute the inner products $\langle X, X\rangle$ and $\langle Y, Y\rangle$ using Lemma 3.1. That is,

$$g_{11} = \langle X, X\rangle = \operatorname{Re}(uv\bar{v}\overline{uv\bar{v}} + uv\bar{u}\overline{uv\bar{u}}) = u\bar{u}v\bar{v}(v\bar{v} + u\bar{u}) = u\bar{u}v\bar{v},$$

(since on S^3, $u\bar{u} + v\bar{v} = 1$) and

$$g_{22} = \langle Y, Y\rangle = q^2 uv\bar{v}\overline{uv\bar{v}} + p^2 vu\bar{u}\overline{vu\bar{u}} = u\bar{u}v\bar{v}(q^2 v\bar{v} + p^2 u\bar{u}).$$

Then, the Hamiltonian is given as

$$H = \frac{g^1}{2}P_X^2 + \frac{g^2}{2}P_Y^2,$$

with the functions $P_X, P_Y : T^*M \to \mathbb{R}$ defined like before as $P_X(\nu) = \nu(X)$ and $g^i = 1/g_{ii}$. If we use (u, v, w, z) as coordinates for $T^*\mathbb{C}^2$, working with the association of $T^*\mathbb{C}^2$ and $T^*\mathbb{R}^4$ as in Equation 3.2, the functions P_X and P_Y can be expressed as follows:

$$\begin{aligned}
P_X(\bar{w}\,du + w\,d\bar{u} + \bar{z}\,dv + z\,d\bar{v}) &= (\bar{w}\,du + w\,d\bar{u} + \bar{z}\,dv + z\,d\bar{v})(X) \\
&= -\bar{w}uv\bar{v} - w\bar{u}v\bar{v} + \bar{z}vu\bar{u} + z\bar{v}u\bar{u} \\
&= -v\bar{v}(\bar{w}u + w\bar{u}) + u\bar{u}(\bar{z}u + z\bar{u})
\end{aligned}$$

and similarly we find

$$P_Y(\bar{w}\,du + w\,d\bar{u} + \bar{z}\,dv + z\,d\bar{v}) = v\bar{v}qi(w\bar{u} - \bar{w}u) + u\bar{u}pi(\bar{z}v - z\bar{v}).$$

With this, one can write down an explicit formulation of H in these coordinates.

We could normalize the vectors X and Y to $\tilde{X}$ and $\tilde{Y}$ which would simplify our expressions even further. However, the result of the Lie bracket $[\tilde{X}, \tilde{Y}]$ is a very large expression and less manageable than $[X, Y]$. We are going to use the relation in Equation (2.14) to compute Hamilton's equations, so having manageable expressions for $[X, Y]$ will be of great importance. The necessary brackets for Hamilton's equations are

$$[X, Y] = -2v\bar{v}u\bar{u}\begin{pmatrix} iqu \\ ipv \end{pmatrix} \qquad [X, Z] = 0 \qquad [Y, Z] = 0.$$

Since computations of these are quite long, we defer them to Appendix A.1.

Using these brackets, we compute the second half of Hamilton's equations. To do so, we will make repeated use of the linearity of P_X in the subscript, that is

$$P_{\alpha Y + \beta Z}(\nu) = \alpha\nu(Y) + \beta\nu(Z) = (\alpha P_Y + \beta P_Z)(\nu).$$

Let $s^i = \sqrt{g^i}$ If F is a function on the contangent bundle, we have that

$$\begin{aligned} \{F, H\} &= \frac{1}{2}\{F, g^1 P_X^2\} + \frac{1}{2}\{F, g^2 P_Y^2\} \\ &= \frac{1}{2}\{F, P_{s^1 X}^2\} + \frac{1}{2}\{F, P_{s^2 Y}^2\} \\ &= s^1 P_X\{F, s^1 P_X\} + s^2 P_Y\{F, s^2 P_Y\}, \end{aligned}$$

by using the identity in 2.11. If $F = P_Y$, we can now use the identity of Equation 2.14 and the standard formula $[X, fY] = f[X, Y] - X(f)Y$ [See Lee12, Proposition 8.28d)] to see that

$$\{P_Y, s^1 P_X\} = -P_{[Y, s^1 X]} = -s^1 P_{[Y,X]} - Y(s^1)P_X.$$

If $F = P_X$, this bracket yields

$$\{P_X, s^1 P_X\} = -P_{[X, s^1 X]} = -s^1 P_{[X,X]} - X(s^1)P_X = -X(s^1)P_X.$$

With this and similar computations for the bracket $\{\cdot, s^2 P_Y\}$, we see that the second half of Hamiltonian's equations are given as

$$\begin{aligned} \dot{P}_X &= -g^2 P_Y P_{[X,Y]} - s^2 X(s^2)P_Y^2 - s^1 X(s^1)P_X^2 \\ \dot{P}_Y &= -g^1 P_X P_{[Y,X]} - s^1 Y(s^1)P_X^2 - s^2 Y(s^2)P_Y^2 \\ \dot{P}_Z &= -s^1 Z(s^1)P_X^2 - s^2 Z(s^2)P_Y^2 \end{aligned}$$

To simplify this, we need to express $[X,Y]$ in terms of the other vector fields, that is we need to find the real coefficients α and β depending on u and v such that

$$\alpha Y + \beta Z = \begin{pmatrix} iqu \\ ipv \end{pmatrix}.$$

This system is easily solvable using the fact that $u\bar{u} + v\bar{v} = 1$ and gives the coefficients

$$\alpha = \frac{p^2 - q^2}{p^2 u\bar{u} + q^2 v\bar{v}} \qquad \beta = \frac{pq}{p^2 u\bar{u} + q^2 v\bar{v}}.$$

We combine this with the fact that that P_X is linear in the subscript to get that $P_{[X,Y]} = -2v\bar{v}u\bar{u}(\alpha P_Y + \beta P_Z)$.

Combining this with the computation in Equation 2.13 gives the system

$$\begin{cases} \dot{u} & = -uv\bar{v}g^1 P_X + -iquv\bar{v}g^2 P_Y; \\ \dot{v} & = vu\bar{u}g^1 P_X + ipvu\bar{u}g^2 P_Y; \\ \dot{P}_X & = \frac{2}{p^2 u\bar{u}+q^2 v\bar{v}} P_Y(\alpha P_Y + \beta P_Z) - s^2 X(s^2) P_Y^2 - s^1 X(s^1) P_X^2 \\ \dot{P}_Y & = -2P_X(\alpha P_Y + \beta P_Z) - s^1 Y(s^1) P_X^2 - s^2 Y(s^2) P_Y^2 \\ \dot{P}_Z & = -s^1 Z(s^1) P_X^2 - s^2 Z(s^2) P_Y^2. \end{cases} \tag{3.6}$$

It is immediate that the vector field $\dot{u}\partial_u + \dot{\bar{u}}\partial_{\bar{u}} + \dot{v}\partial_v + \dot{\bar{v}}\partial_{\bar{v}} = g^1 P_X X + g^2 P_Y Y$ is orthogonal to the outwards pointing normal $2\,\mathrm{Re}(u\partial_u + v\partial_v)$ as it is a linear combination of the vectors X and Y which are both tangent to S^3. We may simplify this system by using the expressions for $\langle X,X\rangle$ and $\langle Y,Y\rangle$ to

$$\begin{cases} \dot{u} & = -\frac{u}{u\bar{u}} P_X + \frac{-iqu}{u\bar{u}(p^2 u\bar{u}+q^2 v\bar{v})} P_Y; \\ \dot{v} & = \frac{v}{v\bar{v}} P_X + \frac{ipv}{v\bar{v}(p^2 u\bar{u}+q^2 v\bar{v})} P_Y; \\ \dot{P}_X & = \frac{2}{p^2 u\bar{u}+q^2 v\bar{v}} P_Y(\alpha P_Y + \beta P_Z) - s^2 X(s^2) P_Y^2 - s^1 X(s^1) P_X^2 \\ \dot{P}_Y & = -2P_X(\alpha P_Y + \beta P_Z) - s^1 Y(s^1) P_X^2 - s^2 Y(s^2) P_Y^2 \\ \dot{P}_Z & = -s^1 Z(s^1) P_X^2 - s^2 Z(s^2) P_Y^2. \end{cases} \tag{3.7}$$

The system in Equation (3.7) has two parts: the equations for $\dot{u}$ and $\dot{v}$ which determine the motion on the manifold and the equations for P_X, P_Y and P_Z. In Figure 3.2 we provide a vector plot of the second half in the P_X, P_Y plane. To do so, we fixed $P_Z = 3$ and we are using $p = 3$, $q = 1$. The u and v coordinates are $u = v = 1/2 + 1/2i$. As you can see, the vectors indicate a rotational motion around the the point $P_X = P_Y = 0$. When linearizing the corresponding equations in (3.7) around $P_X = P_Y = 0$, we find

$$\begin{cases} \dot{P}_X & = \frac{2}{p^2 u\bar{u}+q^2 v\bar{v}} P_Y \beta P_Z \\ \dot{P}_Y & = -2P_X \beta P_Z, \end{cases}$$

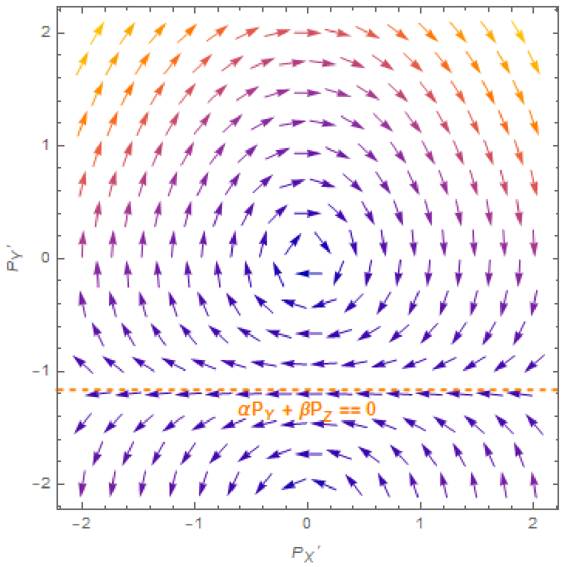


Figure 3.2: The vector field of $\dot{P}_X$ and $\dot{P}_Y$. Color indicates the vector length from blue smallest to red largest.

so as long as P_Z is nonzero, we will find our solution circling alternating along the X and Y direction.

The figure on the cover page shows geodesics obtained from numerically solving (3.7) alongside the planes spanning $\mathcal{H}^{5,1}$. They appear parallel. This is not surprising as the derivatives $\dot{u}$ and $\dot{v}$ in Hamilton's equations are given by linear combinations of vectors in the horizontal distribution. We see how for small initial values of P_Z the solutions stay relatively straight and spiral for larger values.

3.5 Geodesic spheres of the Hopf-fibration

The Hamiltonian we derived is only valid on $S^3 \cap \{u \neq 0\} \cap \{v \neq 0\}$. As a solution approaches $u = 0$ or $v = 0$, we can read off from Equation 3.7 that $\dot{u}$ or $\dot{v}$ will grow very large and numerical solution methods fail. Therefore we can only draw geodesic spheres with initial conditions such that $\gamma(1)$ is still within this region. For the starting point $u = v = 1/2 + 1/2i$, we found that this is satisfied for $\dot{\gamma}(0)$ has length $\langle \dot{\gamma}(0), \dot{\gamma}(0) \rangle \leq 0.16$. This corresponds to using $\dot{\gamma}(0)$ with unit length and only integrating the system from $t = 0$ until $t = 0.4$, by Equation (2.15).

To satisfy this condition on the initial conditions, we use the expression in Equation (2.16) on our Hamiltonian function to arrive at the equation

$$\frac{1}{u\bar{u}v\bar{v}} P_X^2 + \frac{1}{u\bar{u}v\bar{v}(p^2 u\bar{u} + q^2 v\bar{v})} P_Y^2 = 1 \tag{3.8}$$

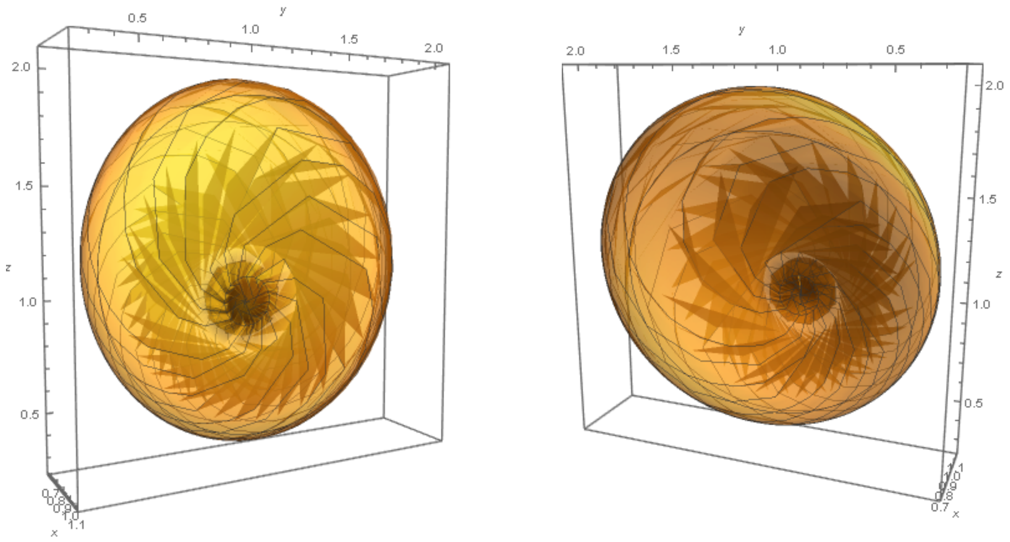


Figure 3.3: Unit sphere of the case $p = q = 1$.

when requiring $\dot{\gamma}(0)$ to be of unit length. This is solved by taking the initial conditions

$$\begin{cases} P_X(0) &= \cos\theta\sqrt{u\bar{u}v\bar{v}} \\ P_Y(0) &= \sin\theta\sqrt{u\bar{u}v\bar{v}(p^2u\bar{u} + q^2v\bar{v})}, \end{cases}$$

with the variable θ sweeping out the interval $[0, 2\pi)$. As we found through experimentation, the constant value of P_Z is best swept out in a nonlinear way. In our images, we have used the initial values $100r^3$, with r linearly sweeping out values in $[-1, 1]$.

With these parametrizations and starting at the initial point $u = v = 1/2+1/2i$, we have created the plots of the case $p = q = 1$ and of the cases where or one of p and q is 1 while the other is equal to 5. The viewpoint has been kept constant throughout to illustrate how the spheres rotate as p and q change. The case $p = 1$ $q = 5$ in Figure 3.4 is very curved and the sphere in the case $p = 5$, $q = 1$ in Figure 3.5 is very flat. The sphere of $p = q = 1$ in Figure 3.3 certainly has the roundest shape. When drawing the spheres for additional in-between values, the shape seems to be changing continuously, the cases for one of p or q equal to 2 can be seen in Figures 3.6 and 3.7.

To assess the quality of the numerical solutions, we have computed the sum of squares of the components to see if they remain on the sphere, so if it is close to 1. Furthermore, we checked that the solutions move with unit speed, i.e. that the sum of squares of the derivative of the solution is close to one. For a number of different solutions of the $p = 1$, $q = 5$ case, the deviation of these two quantities from one can be seen in Figure 3.8. The deviation is quite small, so we can assume

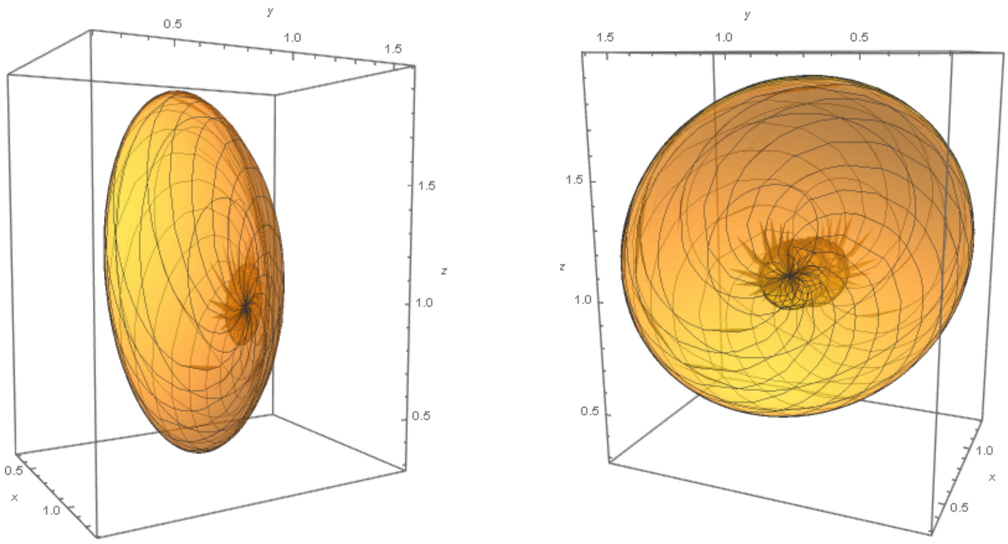


Figure 3.4: Unit sphere of the case $p = 1$, $q = 5$.

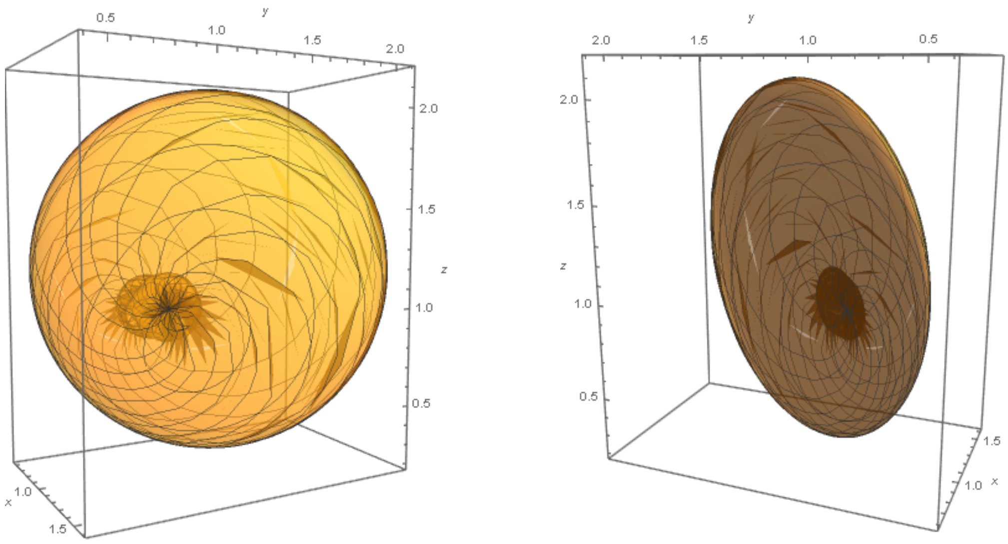


Figure 3.5: Unit sphere of the case $p = 5$, $q = 1$.

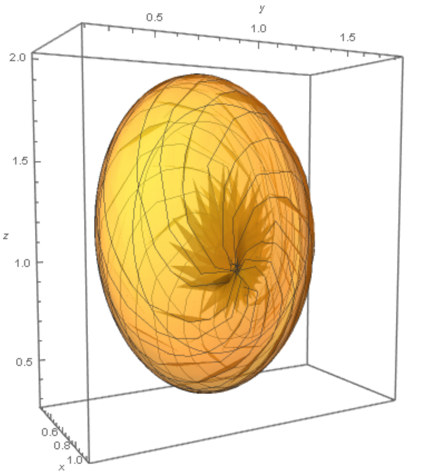


Figure 3.6: The case $p = 1$, $q = 2$.

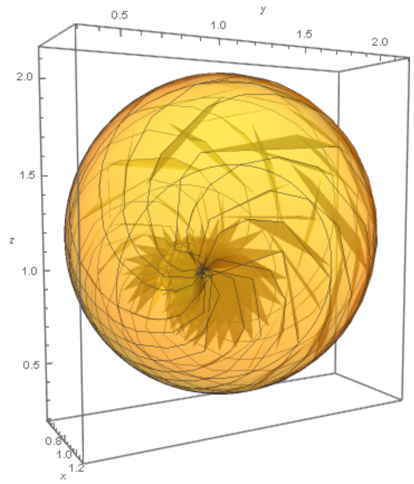
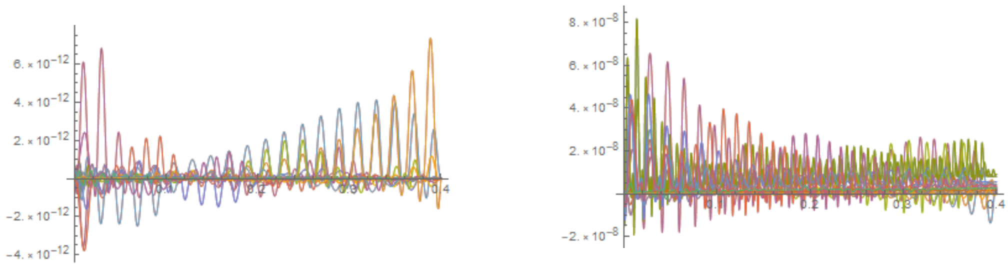


Figure 3.7: The case $p = 2$, $q = 1$.



(a) Squared norm of several solutions.

(b) Squared norm of the derivative of several solution.

Figure 3.8: Properties of the numerical solutions in the case $p = 1$ and $q = 5$.

that the numerical solution method is close to the true geodesics. Moreover, this is evidence that we have the correct expression for Hamilton's equations for this geometry.

The Mathematica [Wol] code necesseray to reproduce all images of this chapter is included in Appendix A.2.

3.6 Sub-Riemannian geodesics

Previously, we have seen that Riemannian geometry is a special case of its subriemannian cousin. With this in mind, since the Hopf-fibration is a subriemannian geometry of the Riemannian geometry S^3, we have a natural first candidate for the subriemannian geodesics of the Hopf-fibration; the geodesics of S^3. These are the great circles, that is the intersections of S^3 with planes in $\mathbb{R}^4$ that pass through the origin [Lee06, Proposition 5.13]. It turns out that if a Riemannian geodesic starts out in a horizontal direction, it remains horizontal throughout.

Lemma 3.2. *All geodesics γ of S^3 with an initial velocity $\dot{\gamma}(0) \in \mathcal{H}^{p,q}$ are horizontal curves.*

Proof. Let $\gamma(t) = \cos t\,(r,s) + \sin t\,(x,y)$ be a great circle of S^3, which means that $\|(r,s)\| = \|(x,y)\| = 1$ and that (r,s) and (x,y) are orthogonal in $\mathbb{C}^2$. Then, if we assume that $\dot{\gamma}(0) = (x,y) \in \mathcal{H}^{p,q}_{(r,s)}$, we know that $(x,y) \perp (ipr, iqs)$. Therefore, as a vector tangent to S^3 is in $\mathcal{H}^{p,q}_{\gamma(t)}$ if and only if it is orthogonal to

$$\begin{pmatrix} ipu \\ iqv \end{pmatrix}\Bigg|_{(u,v)=\gamma(t)} = \cos t \begin{pmatrix} ipr \\ iqs \end{pmatrix} + \sin t \begin{pmatrix} ipx \\ ipy \end{pmatrix},$$

it suffices to compute

$$\langle \dot{\gamma}(t), (ipu, iqv)|_{\gamma(t)} \rangle = 0.$$

This follows from distributing the inner product over the terms and noting that

$$\langle (r,s), (ipx, iqy) \rangle = \operatorname{Re} -ipr\bar{x} - iqs\bar{y} = \operatorname{Re} \overline{ipx\bar{r} + iqy\bar{s}} = -\overline{\langle (x,y), (ipr, iqs) \rangle} = 0,$$

and that for any (u,v),

$$\langle (u,v), (ipu, iqv) \rangle = \operatorname{Re} i(pu\bar{u} + qv\bar{v}) = 0.$$

Therefore, as $\dot{\gamma}$ is allways tangent to S^3 and orthogonal to vertical distribution, it must be horizontal. $\square$

This principle is illustrated in Figure 3.9 where we have drawn several great circles on S^3 that have a horizontal initial velocity. Starting points vary so the circles appear interlinked under the stereographic projection. The plane field indicates the horizontal distribution.

Theorem 3.3. *All Riemannian geodesics γ of S^3 with initial velocity $\dot{\gamma}(0) \in \mathcal{H}^{p,q}_{\gamma(0)}$ are geodesics of the (p,q) Hopf-fibration.*

Proof. The curve γ is locally length minimizing: On any sufficiently short arc $\gamma : [a,b] \to M$ of γ, suppose that a horizontal curve γ' has $\gamma'(0) = \gamma(a)$ and $\gamma'(1) = \gamma(b)$. Furthermore, suppose that γ' is shorter, so

$$\int_0^1 \langle \dot{\gamma}', \dot{\gamma}' \rangle \, dt \leq \int_a^b \langle \dot{\gamma}, \dot{\gamma} \rangle \, dt. \tag{3.9}$$

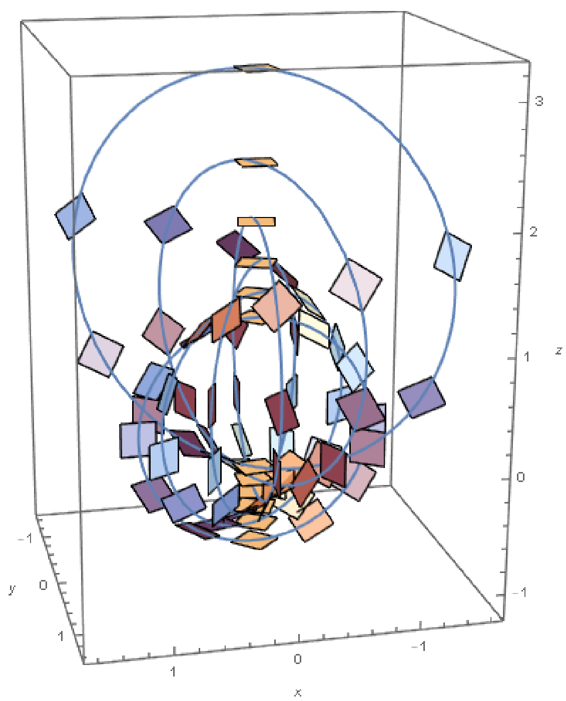


Figure 3.9: Several great circles with horizontal initial velocity.

Then, γ' can be seen as a curve on the Riemannian geometry S^3 and Equation (3.9) also holds as the inner product of the p, q hopf-fibration is merely the restriction of the usual inner product. By Theorem 2.5 we have to conclude that $\gamma'([0,1]) = \gamma([a,b])$, as γ is a Riemannian geodesic. Therefore, γ is locally length minimizing.

As the distribution $\mathcal{H}^{p,q}$ is a contact distribution, its sub-Riemannian geometry does not admit singular geodesics, so γ is a sub-Riemannian geodesic of the p, q hopf-fibration. $\square$

4 Conclusion

In this project, we have laid out the basic theory of sub-Riemannian geometry. Readers interested at studying this field further could turn to Montgomery's book [Mon02] for a panoramic overview, however its prerequisites are quite high. In the text by Agrachev, Barilari, and Boscain, more foundational material is provided but overall it is slightly more technical [ABB12]. I can nevertheless recommend it, especially since this thesis should have provided the necessary initial overview of the topic.

On a similar note, the introduction to Riemannian geometry given in Chapter 2 is a vast understatement of what the field has to offer. The two books by Do Carmo [Do 92] and Lee [Lee06] complement each other well and provide an excellent way of learning more about the topic.

We have also worked towards understanding the sub-Riemannian geometry of the p, q Hopf actions. In particular we provided a Hamiltonian valid on most of S^3 and computed Hamilton's equations from it. Perhaps this could be numerically approximated or extended analytically to all of S^3 so that an implicit description of the geodesics is available for the entire space. Also, we have provided the necessary code to plot geodesic spheres of this sub-Riemannian geometry alongside several examples. Here, an obvious next goal would be to work out for which initial conditions and time period geodesics stay length minimizing to arrive at a sub-Riemannian distance sphere.

I found working on this project sincerely interesting. Manifolds are my favourite mathematical structure and I generally find the results of differential geometry quite amazing. Sub-Riemannian geometry is especially fascinating for how applicable the subject is to real world problems and how exciting it is to learn something about recent mathematics. However, I found that the project was quite demanding. Riemannian geometry is a very large field usually placed at a graduate level and the texts on sub-Riemannian geometry I have encountered usually assume thorough knowledge of it. It took quite a bit of work to gather all necessary ingredients to understand and present the topic at an undergraduate level. The project has probably prepared me well for studying Riemannian geometry further, but I also hope to revisit sub-Riemannian geometry in the future. Until then, happy parallel parking.

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A.1 Computation of the Lie bracket of the basis vectors for $\mathcal{H}^{p,q}$ and the vertical vector field

Definition. On $\mathbb{C}^2$ we may denote a vector field as

$$\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} := f_1\,\partial_u + \bar{f}_1\,\partial_{\bar{u}} + f_1\,\partial_v + \bar{f}_2\,\partial_{\bar{v}} = 2\,\mathrm{Re}(f_1\partial_u + f_2\partial_v),$$

As we have seen in Section 3.1, this notation is really a short hand for the vector field

$$\mathrm{Re}\, f_1\,\partial_{x^1} + \mathrm{Im}\, f_1\,\partial_{x^2} + \mathrm{Re}\, f_2\,\partial_{x^3} + \mathrm{Im}\, f_2\,\partial_{x^4},$$

so we may also write

$$X^1\partial_{x^1} + X^2\partial_{x^2} + X^3\partial_{x^3} + X^4\partial_{x^4} = \begin{pmatrix} X_1 + iX_2 \\ X_3 + iX_4 \end{pmatrix} \tag{A.1}$$

Now if we start with two vector fields on $\mathbb{R}^4$ that are given in coordinates as $X^i\partial_{x^i}$ and $Y^i\partial_{x^i}$, we can define the corresponding complex valued functions $f_1 = X^1 + iX^2$, $f_2 = X^3 + iX^4$, $g_1 = Y^1 + iY^2$ and $g_2 = Y^3 + iY^4$. To compute the Lie bracket of X and Y, we can now use Equation (A.1) twice to get

$$\begin{aligned}
[X, Y] &= (XY^i - YX^i)\partial_{x^i} \\
&= \begin{pmatrix} XY^1 - YX^1 + iXY^2 - iYX^2 \\ XY^3 - YX^3 + iXY^4 - iYX^4 \end{pmatrix} \\
&= \begin{pmatrix} X(Y^1 + iY^2) - Y(X^1 - iX^2) \\ X(Y^3 + iY^4) - Y(X^3 + iX^4) \end{pmatrix} \\
&= \begin{pmatrix} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} g_1 - \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} f_1 \\ \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} g_2 - \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} f_2 \end{pmatrix},
\end{aligned}$$

an expression using the notation defined at the start of the Section.

We have the vectors

$$X = \begin{pmatrix} -uv\bar{v} \\ vu\bar{u} \end{pmatrix} \quad \text{and} \quad Y = \begin{pmatrix} -iquv\bar{v} \\ ipvu\bar{u} \end{pmatrix} \qquad C = \begin{pmatrix} ipu \\ iqv, \end{pmatrix}$$

of which we wish to compute the Lie brackets. Then,

$$[X, Y] = \begin{pmatrix} \begin{pmatrix} -uv\bar{v} \\ vu\bar{u} \end{pmatrix} (-iuqv\bar{v}) - \begin{pmatrix} -iquv\bar{v} \\ ipvu\bar{u} \end{pmatrix} (-uv\bar{v}) \\ \begin{pmatrix} -uv\bar{v} \\ vu\bar{u} \end{pmatrix} (ipvu\bar{u}) - \begin{pmatrix} -iquv\bar{v} \\ ipvu\bar{u} \end{pmatrix} (vu\bar{u}) \end{pmatrix}$$

and

$$\begin{pmatrix} -uv\bar{v} \\ vu\bar{u} \end{pmatrix} (-iuqv\bar{v}) = uv\bar{v}iqv\bar{v} - iuq\bar{v}vu\bar{u} - \bar{v}\bar{u}uiuqv$$
$$= iqu\, v\bar{v}v\bar{v} - iqu\, v\bar{v}u\bar{u} - iqu\, v\bar{v}u\bar{u},$$

$$\begin{pmatrix} -iquv\bar{v} \\ ipvu\bar{u} \end{pmatrix} (-uv\bar{v}) = iquv\bar{v}v\bar{v} - ipvu\bar{u}u\bar{v} + ip\bar{v}\bar{u}uuv$$
$$= iqu\, v\bar{v}v\bar{v} - ipu\, u\bar{u}v\bar{v} + ipu\, v\bar{v}u\bar{u},$$

$$\begin{pmatrix} -uv\bar{v} \\ vu\bar{u} \end{pmatrix} (ipvu\bar{u}) = -uv\bar{v}ivp\bar{u} - \bar{u}\bar{v}vivpu + vu\bar{u}ipu\bar{u}$$
$$= -ipv\, v\bar{v}u\bar{u} - ipv\, u\bar{u}v\bar{v} + ipv\, u\bar{u}u\bar{u},$$

and

$$\begin{pmatrix} -iquv\bar{v} \\ ipvu\bar{u} \end{pmatrix} (vu\bar{u}) = -iquv\bar{v}v\bar{u} + iq\bar{u}v\bar{v}vu + ipvu\bar{u}u\bar{u}$$
$$= -iqv\, v\bar{v}u\bar{u} + iqv\, u\bar{u}v\bar{v} + ipv\, u\bar{u}u\bar{u}.$$

By carefully studying the terms in the above expressions, we can see that the difference of the first two equals $-2iqv\,\bar{v}u\bar{u}u$ and the difference between the second two equals $-2ipu\,\bar{u}v\bar{v}v$. Therefore,

$$[X, Y] = -2iv\bar{v}u\bar{u} \begin{pmatrix} qu \\ pv. \end{pmatrix}$$

For the other brackets, we compute

$$[Y, C] == \left[\begin{pmatrix} -iquv\bar{v} \\ ipvu\bar{u} \end{pmatrix}, \begin{pmatrix} ipu \\ iqv \end{pmatrix} \right].$$

Here,

$$\begin{pmatrix} -iquv\bar{v} \\ ipvu\bar{u} \end{pmatrix} ipu - \begin{pmatrix} piu \\ qiv \end{pmatrix} (-iquv\bar{v}) = pquv\bar{v} - (-i^2pquv\bar{v} - i^2q^2uv\bar{v} + q^2i^2uv\bar{v}) = 0$$

and

$$\begin{pmatrix} -iquv\bar{v} \\ ipvu\bar{u} \end{pmatrix} iqv - \begin{pmatrix} piu \\ qiv \end{pmatrix} ipvu\bar{u} = i^2qpvu\bar{u} - (p^2i^2vu\bar{u} + p^2(-i)ivu\bar{u} + i^2qpvu\bar{u}) = 0,$$

so $[Y, C] = 0$. Also,

$$\begin{pmatrix} -uv\bar{v} \\ vu\bar{u} \end{pmatrix} ipu - \begin{pmatrix} ipu \\ iqv \end{pmatrix} (-uv\bar{v}) = -ipuv\bar{v} - (pipuv\bar{v} + iquv\bar{v} - iquv\bar{v}) = 0$$

and finally

$$\begin{pmatrix} -uv\bar{v} \\ vu\bar{u} \end{pmatrix} iqv - \begin{pmatrix} ipu \\ iqv \end{pmatrix} (vu\bar{u}) = iqvu\bar{u} - (ipvu\bar{u} - ipvu\bar{u} + iqvu\bar{u}) = 0,$$

so also $[X, C] = 0$.

A.2 Mathematica code

First we define the stereographic projection and its differentials as well as corresponding coordinate transforms.

```
sigma[{a_, b_, c_, d_}] = {a, b, d} / (1 - c);
dsigma = D[sigma[{a, b, c, d}], {{a, b, c, d}}]
isigma[{x_, y_, z_}] = {2 x, 2 y, x^2 + y^2 + z^2 - 1,
  2 z} / (x^2 + y ^2 + z^2 + 1);
disigma = Simplify[D[isigma[{x, y, z}], {{x, y, z}}]]
cc = {a → isigma[{x, y, z}] [[1]], b → isigma[{x, y, z}] [[2]],
  c → isigma[{x, y, z}] [[3]], d → isigma[{x, y, z}] [[4]]};
ccc = {x → sigma[{a, b, c, d}] [[1]],
  y → sigma[{a, b, c, d}] [[2]], z → sigma[{a, b, c, d}] [[3]]};
```

Next we define the necessary vectors as well as convert them into the format necessary for NDSolve. We also define the Jacobian matrices of the inner products in order to compute the action of the vector fields on them later on.

```
Clear[p, q];
ve = {-a (c^2 + d^2), -b (c^2 + d^2), c (a^2 + b^2), d (a^2 + b^2)};
we = {b q (c^2 + d^2), - a q (c^2 + d^2), - d p (a^2 + b^2) ,
  c p (a^2 + b^2)};
xe = {-b p, a p, -d q, c q};
changefun = {a → x1[t], b → x2[t], c → x3[t], d → x4[t]};
vx = ve /. changefun;
vy = we /. changefun;
vz = xe /. changefun;
xx = Sqrt[1 / ((a^2 + b^2) (c^2 + d^2))]
yy = Simplify[Sqrt[1 / (we . we)], a^2 + b^2 + c^2 + d^2 == 1]
Jxx = Simplify[D[xx, {{a, b, c, d}}], a^2 + b^2 + c^2 + d^2 == 1] /. changefun
Jyy = Simplify[D[yy, {{a, b, c, d}}], a^2 + b^2 + c^2 + d^2 == 1] /. changefun
xx = xx /. changefun
yy = yy /. changefun
alpha = (p^2 - q^2) / ((x1[t]^2 + x2[t]^2) p^2 +
   q^2 (x3[t]^2 + x4[t]^2));
beta = (p q) / ((x1[t]^2 + x2[t]^2) p^2 + q^2 (x3[t]^2 + x4[t]^2));
```

Up next are the differential equations. They have parameters for specifying the initial parameters on the cotangent bundle.

```
Clear[p, q];
eqns[s_, u_,
 v_] = {{x1'[t], x2'[t], x3'[t], x4'[t]} ==
  xx^2*vx*px[t] + yy^2*vy*py[t],
 px'[t] == 2 / (p^2 (x1[t]^2 + x2[t]^2) + q^2 (x3[t]^2 + x4[t]^2)) *
  py[t] (alpha*py[t] + beta*pc[t]) - yy vx.Jyy py[t]^2 - xx vx.Jxx px[t]^2,
 py'[t] == -2*
  px[t] (alpha*py[t] + beta*pc[t]) - xx vy.Jxx px[t]^2 - yy vy.Jyy py[t]^2,
 pc'[t] == -xx vz.Jxx px[t]^2 - yy vz.Jyy py[t]^2,
 {x1[0], x2[0], x3[0], x4[0]} == {1/Sqrt[4], 1/Sqrt[4], 1/Sqrt[4],
  1/Sqrt[4]},
 px[0] == s,
 py[0] == u,
 pc[0] == v};

norm = x1[t]^2 + x2[t]^2 + x3[t]^2 + x4[t]^2;
speed = px[t]^2 *xx + py[t]^2*yy;
tend = 0.4;
```

And we define the function that returns the endpoint of a geodesic. The two parameters give us an initial horizontal direction and the remaining parameter corresponding to P_Z.

```
Clear[p, q]
fun2[u_, v_] :=
 sigma[{x1[t], x2[t], x3[t], x4[t]}] /. (NDSolve[eqns[Cos[u] / 2, Sqrt[(p^2 + q^2) / 2] Sin[u] / 2 , 100*v^3], {x1, x2,
   x3, x4, px, py, pc}, {t, tend}, Method → {"Projection", Method → "ExplicitRungeKutta", "Invariants" → {norm} == 1},
   InterpolationOrder → All, MaxStepSize -> 0.02]) /. {t → tend}
```

After defining some viewing angles, this code creates the geodesic spheres seen in the thesis.

```
p = 5; q = 1;
veee1 = {1.7459078177618, -0.959751459295, 0.1751657165625};
veee2 = {-1.7943045198771543`, -0.7362242956614813`, 0.48830838250605224`};
p51 = ParametricPlot3D[fun2[u, v], {u, 0, 2 Pi}, {v, -1, 1}, PlotRange → All, PlotStyle → Opacity[0.5],
  AxesLabel → {x, y, z}, ViewPoint → veee1 ]
Show[p51, ViewPoint → veee2]


p = 1; q = 1;
p11 = ParametricPlot3D[fun2[u, v], {u, 0, 2 Pi}, {v, -1, 1}, PlotRange → All, PlotStyle → Opacity[0.5],
  AxesLabel → {x, y, z}, ViewPoint → veee1 ]
Show[p11, ViewPoint → veee2]
p = 1; q = 5;
p15 = ParametricPlot3D[fun2[u, v], {u, 0, 2 Pi}, {v, -1, 1}, PlotRange → All, PlotStyle → Opacity[0.5],
  AxesLabel → {x, y, z}, ViewPoint → veee1 ]
Show[p15, ViewPoint → veee2]
p = 2; q = 1;
p21 = ParametricPlot3D[fun2[u, v], {u, 0, 2 Pi}, {v, -1, 1}, PlotRange → All, PlotStyle → Opacity[0.5],
  AxesLabel → {x, y, z}, ViewPoint → veee1 ]
Show[p21, ViewPoint → veee2]
p = 1; q = 2;
p12 = ParametricPlot3D[fun2[u, v], {u, 0, 2 Pi}, {v, -1, 1}, PlotRange → All, PlotStyle → Opacity[0.5],
  AxesLabel → {x, y, z}, ViewPoint → veee1 ]
Show[p12, ViewPoint → veee2]
```

We create some additional solutions and we evaluate how well the solutions behaved.

```
p = 1; q = 5;
tend = 0.4;
k1 = 6;
k2 = 9;
points = Tuples[{Range[0, 2 Pi - Pi / (2 k1) , 2 Pi / k1], 100 * #^3 & /@ Range[-1, 1, 2 / k2]}];
sol = (NDSolve[eqns[Cos[#[[1]]] / 2, Sqrt[(p^2 + q^2) / 2] Sin[#[[1]]] / 2 , #[[2]]]], {x1, x2,
   x3, x4, px, py, pc}, {t, tend}, Method → {"Projection", Method → "ExplicitRungeKutta", "Invariants" → {norm} == 1},
  InterpolationOrder → All, WorkingPrecision → MachinePrecision, MaxStepSize -> 0.02]) & /@ points;

Plot[Evaluate[x1[t]^2 + x2[t]^2 + x3[t]^2 + x4[t]^2 - 1 /. sol], {t,
 0, tend}, PlotRange → All]
Plot[Evaluate[(x1'[t]^2 + x2'[t]^2 + x3'[t]^2 + x4'[t]^2 - 1) /. sol], {t, 0, tend}, PlotRange → All]
```