CHAPTER 4

The Homology Groups

4.1 INTRODUCTION

In this chapter we give an account of the homology groups $H_p(x)$, $p = 0, 1, 2, \ldots$, associated with a topological space $X$. Rather than consider an arbitrary space $X$ we will suppose that the space $X$ is triangulable, i.e. homeomorphic to some polyhedron $K$. The homology groups $H_p(K)$ can then be defined in terms of the simplexes of $K$ and are, for this reason, called the simplicial homology groups of the polyhedron $K$. At the end of the chapter we will briefly describe one way of defining homology groups for an arbitrary, not necessarily triangulable, space $X$.

In our discussions we will also accept without proof the following important theorems:

**Theorem**

If $X$ and $Y$ are two topological spaces of the same homotopy type then $H_p(X)$ is isomorphic to $H_p(Y)$ for all $p$.

We write:

$$H_p(X) \cong H_p(Y) \quad (4.1a)$$

An immediate implication of this theorem is that the homology groups are topological invariants. This is because homeomorphic spaces are necessarily of the same homotopy type.

**Theorem**

If $K_1$ and $K_2$ are two triangulations of the same topological space $K$ then

$$H_p(K_1) = H_p(K_2), \forall p \quad (4.1b)$$

A proof of these two theorems can be found in references [1] or [3] listed at the end of the Chapter.
We now proceed to explain the geometrical ideas underlying the simplicial homology groups. We do this by examining once again the two spaces $X_1$ and $X_2$ introduced at the beginning of Chapter 3. $X_1$ and $X_2$ were rectangular regions of $\mathbb{R}^2$. $X_1$ contained a hole while $X_2$ did not. $X_2$ is thus homeomorphic to the 2-simplex $\sigma^2$ while, in view of Theorem (4.1a), $X_1$ can be replaced by its deformation retract, the edges of $\sigma^2$. Figure 4.1 explains what is involved.

![Figure 4.1](image)

The shaded region represents the surface, the unshaded region the hole. In Chapter 3 the topological difference between $X_1$ and $X_2$ was studied using homotopy classes of loops. Now we proceed differently. Observe that the boundary of Fig. 4.1(a) is the boundary of a connected region. But Fig. 4.1(b) consists only of the edges of the triangular region $v_1, v_2, v_3$ and is not the boundary of any region. This simple observation suggests a method of spotting holes in a space. A closed two-dimensional region has a hole if its boundary, some closed curve, is not the boundary of a connected region. The usefulness of this procedure is two-fold. Firstly, the idea can be generalized to higher dimensions. Secondly, for a triangulable space, a simple algebraic definition of the boundary of the region can be given. Let us explain how this is done by considering some examples. We begin with the simplest case, namely, when our space $K$ is the 0-simplex $\sigma^0 = [v]$. Then $K$ does not have any boundary and we require:

$$
\partial[v] = 0
$$
where $\partial[v]$ represents the boundary of the 0-simplex $[v]$. What this equation means is the following: a correspondence between a positive or negative integer and $[v]$ is set up. In terms of such a correspondence $\partial[v]$ is mapped to zero.

Next suppose $K$ is the 1-simplex $\sigma^1 = [v_1, v_2]$, together with its faces. Geometrically the boundary of $\sigma^1$ consists of the two end points of $\sigma^1$, namely $[v_1]$ and $[v_2]$. We write this formally as:

$$\partial[v_1, v_2] = [v_1] + [v_2] \quad (4.2)$$

where, as before, the formal sum can be understood in terms of a correspondence set up between the simplexes $[v_1, v_2]$, $[v_1]$ and $[v_2]$ and the positive and negative integers. Proceeding in this manner the boundary of the 2-simplex $\sigma^2 = [v_1, v_2, v_3]$ can be written as:

$$\partial[v_1, v_2, v_3] = [v_1, v_2] + [v_2, v_3] + [v_1, v_3] \quad (4.3)$$

and the boundary of a $K$-simplex $\sigma^K = [v_0, \ldots, v_K]$ as:

$$\partial[v_0, \ldots, v_K] = \sum_{i=1}^{K} [v_0, \ldots, \hat{v}_i, \ldots, v_K] \quad (4.4)$$

where $[v_0, \ldots, \hat{v}_i, \ldots, v_K]$ represents the $(K-1)$-simplex $\sigma^{K-1}$ obtained from the $K$-simplex $\sigma^K$ by omitting the vertex $v_i$. Geometrically this is reasonable since such a $(K-1)$-simplex is a face of the $K$-simplex $\sigma^K$.

Does the boundary operator $\partial$ defined by equation (4.4) give the boundary of a polyhedron? Let us consider the simple polyhedron shown in Fig. 4.2.

![Figure 4.2](image)

From the figure it is clear that

$$\partial K = [v_1] + [v_3] \quad (4.5)$$

It is also clear that $K$ is obtained by joining the two 1-simplexes $[v_1, v_2]$ and $[v_2, v_3]$. It is tempting to write:

$$K = [v_1, v_2] + [v_2, v_3] \quad (4.6a)$$

where the formal sum is understood in terms of a correspondence between the simplexes of $K$ and the integers. In order to use equation (4.4) to determine the boundary of $K$ we have to assume that the boundary operator
\( \partial \) acts linearly on the simplexes of \( K \), i.e.
\[
\partial K = \partial [v_1, v_2] + \partial [v_2, v_3]
\]
\[
= [v_1] + [v_2] + [v_3]
\] (4.6b)
which is different from (4.5). There are two ways of saving the situation. One is to formally set \([v_2] + [v_2] = 0\). The other way is to modify the definition of \( \partial \) so that the two factors of \([v_2]\) appear with opposite signs and hence cancel. Since \( \partial \) was defined in terms of simplexes, this suggests that each simplex should be given a sign. How should this be done? Consider the \( K \)-simplex \( \sigma^K = [v_0, v_1, \ldots, v_k] \), we can clearly write \( \sigma^K \) in a variety of different ways merely by permuting the vertices \( v_0, v_1, \ldots, v_k \). Is there any way of separating all of these different possible representations into two classes? There is: namely any given representation of \( \sigma^K = [v_{i_0}, v_{i_1}, \ldots, v_{i_k}] \), where \( 0 < i_1 < \ldots < i_k \) be obtained from the `standard’ form \([v_0, v_1, \ldots, v_k]\) by an even or odd number of permutations. We could thus assign a positive sign to a member of the even permutation class and a negative sign to a member of the odd permutation class. What we are really doing is orienting the simplexes in the sense of Chapter 2. To see this let us consider the case of \( \sigma^2 = [v_1, v_2] \). According to the convention we have just described: \( +\sigma = [v_1, v_2] \), and \( -\sigma^2 = [v_2, v_1] \). On the other the basis vectors \( v_1, v_2 \) can be transformed to \( v_2, v_1 \) by means of the matrix \( \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \) and the determinant of this matrix is equal to minus one. The orientations of \([v_1, v_2]\) and \([v_2, v_1]\) are thus different.

Let us next see how the boundary operator \( \partial \) should be modified in order to make it consistant with the sign convention introduced. We still expect \( \partial \sigma^K \) to be a formal linear combination of the faces \( \sigma^{K-1} \) of \( \sigma^K \) but now each of the simplexes has to be given a sign. Consider the case of \( \sigma^2 \). We write:
\[
\partial \sigma^2 = \partial [v_1, v_2] = [\hat{v}_1, v_2] + [v_1, \hat{v}_2]
\] (4.7)
where, as before, \( \hat{v}_i \) indicates that the corresponding vertex \( v_i \) is to be removed. Writing: \( [\hat{v}_1, v_2] = [v_2] \) leads us to expect:
\[
[v_1, \hat{v}_2] = (-)[\hat{v}_2, v_1] = (-)[v_1]
\]
Thus we get:
\[
\partial [v_1, v_2] = [v_2] - [v_1]
\] (4.8)
In terms of the modified boundary operator we have:
\[
\partial K = \partial [v_1, v_2] + \partial [v_2, v_3]
\]
\[
= ([v_2] - [v_1]) + ([v_3] - [v_2])
\]
\[
= [v_3] - [v_1], \text{ as required}
\]

82  TOPOLOGY AND GEOMETRY FOR PHYSICISTS
We can now return to our original problem of distinguishing the space \( X_1 \) from the space \( X_2 \) in an algebraic manner using the boundary operator \( \partial \). \( X_1 \) is a polyhedron made up of the following simplexes:
\[
X_1 = \{ [v_1, v_2]; [v_2, v_3]; [v_1, v_3] \} \text{ and their faces}
\]
while
\[
X_2 = \{ [v_1, v_2, v_3]; \text{ and all its faces} \}
\]
We observe that in general if \( K \) is a closed region then \( \partial K \), the boundary of \( K \), is expected to be a closed surface. \( \partial K \) should not itself have any boundary, i.e. \( \partial(\partial K) = 0 \). Thus once we have defined a boundary operator \( \partial \) it is straightforward to spot a boundary surface \( b \). We just verify that \( \partial b = 0 \). It is also possible to tell if the boundary \( b \) is the boundary of a hole or of a connected region. This is because if \( b \) were the boundary of some connected region then we expect to find a \( K \) such that \( b = \partial K \). Let us now apply these ideas to the spaces \( X_1 \) and \( X_2 \). We note that
\[
b = [v_2, v_3] - [v_1, v_3] - [v_1, v_2]
\]
in \( X_1 \) is a boundary since \( \partial b = 0 \). On the other hand there are no higher dimensional simplexes in \( X_1 \), i.e. \( b \neq \partial K \). Thus \( X_1 \) contains a hole. For \( X_2 \), \( b \) is also a boundary but it is precisely the boundary of the 2-simplex \( \sigma^2 = [v_1, v_2, v_3] \). Thus \( X_2 \) does not contain a hole. Using the genuine boundary \( b \) of \( X_1 \) an abelian group can be generated simply by noting that if \( b \) is a boundary then so is \( \pm Nb \), where \( N \) is an integer. This abelian group is the homology group \( H_1(X_1) \). Our discussion suggests that \( H_1(X_1) \) is isomorphic to \( \mathbb{Z} \), the group of positive and negative integers under addition.

Let us now make precise the intuitive ideas discussed.

### 4.2 ORIENTED SIMPLEXES AND THE DEFINITION OF THE HOMOLOGY GROUPS

**Definition**

An oriented \( p \)-simplex \( p > 1 \) is obtained from a \( p \)-simplex \( \sigma^p = [v_0, \ldots, v_p] \) by choosing an ordering for its vertices. The equivalence class of even permutations of the chosen ordering determines the positively oriented simplex \( +\sigma^p \), while the equivalence class of odd permutations determines the negatively oriented simplex \( -(+)\sigma^p \). A simplicial complex whose simplexes have been assigned an orientation is called an oriented simplicial complex. (4.9)
Example

For the 2-simplex, if we choose \( +\sigma^2 = [v_0, v_1, v_2] \), then
\[
+\sigma^2 = [v_0, v_1, v_2] = [v_1, v_2, v_0] = [v_2, v_0, v_1]
\]
while
\[
-\sigma^2 = [v_0, v_2, v_1] = [v_2, v_1, v_0] = [v_1, v_0, v_2]
\]

Our next step is to associate with each \( p \)-simplex \( \sigma_p^i \) \((p = 0, 1, \ldots, n)\), of a simplicial complex \( K \), an abelian group \( C_p(K) \) called the chain group. Once this is done the geometric notion of a boundary which we discussed can be changed into an algebraic statement involving the chain group.

Definition

Let \( K \) be a \( n \)-dimensional simplicial complex containing \( l_p \) \( p \)-simplexes. The \( p \) chain of \( K \), \( C_p(K) \) is the free abelian group generated by the orientated \( p \)-simplexes of \( K \). What this means is the following: an arbitrary element \( c_p \in C_p(K) \) can be written as the formal sum:

\[
c_p = \sum_{i=1}^{l_p} f_i \sigma_p^i, f_i \in \mathbb{Z}
\]

where

\[
\sigma_p^i + (-\sigma_p^i) = 0, \forall i, p
\]

and

\[
\sum_{i=1}^{l_p} f_i \sigma_p^i + \sum_{i=1}^{l_p} g_i \sigma_p^i = \sum_{i=1}^{l_p} (f_i + g_i) \sigma_p^i (f_i, g_i \in \mathbb{Z})
\]

The statement that \( K \) is \( n \)-dimensional simply means that \( P = 0, 1, \ldots, n \). It is often convenient to define \( C_p(K) = \{0\} \) for \( P > n \)

The boundary operator \( \partial \) can now be properly defined.

Definition

The boundary operator \( \partial_p \) is the map:

\[
\partial_p : C_p(K) \rightarrow C_{p-1}(K)
\]

with the following properties

i. It is linear: \( \partial_p (\sum f_i \sigma_p^i) = \sum f_i \partial \sigma_p^i \)

ii. For an oriented \( p \)-simplex

\[
\sigma_p = [v_0, \ldots, v_p]
\]

\[
\partial[v_0, \ldots, v_p] = \sum_{j=0}^{p} (-1)^j[v_{0, \ldots, \hat{j}, \ldots, v_p}]
\]
where \([v_0, \ldots, \hat{v}_i, \ldots, v_p]\) represents the \((p-1)\)-simplex \(\sigma^{p-1}\) obtained from the \(p\)-simplex \(\sigma^p\) by omitting the vertex \(v_i\).

iii. The boundary of every zero chain is defined to be zero.

It is straightforward to check that \(\partial_p\) is a homomorphism from \(C_p(K)\) to \(C_{p-1}(K)\). Very often we will omit the subscript \(p\) of the boundary operator \(\partial_p\) and write simply \(\partial\). We next check that the boundary of a polyhedron \(K\) does not itself have a boundary, i.e. we have:

\[
\partial_{p-1} \circ \partial_p = 0
\]  

(4.12)

**Theorem**

\[
\partial_{p-1} \circ \partial_p = 0
\]  

**Proof**

Because of the linearity of \(\partial_p\) it is sufficient to show that

\[
\partial_{p-1} \circ \partial_p \sigma^p = 0
\]

Now

\[
\partial_{p-1} \circ \partial_p \sigma^p = \partial_{p-1} \circ \partial_p [v_0, \ldots, v_p] \\
= \partial_{p-1} \left\{ \sum_{j=0}^{p} (-1)^j [v_0, \ldots, \hat{v}_j, \ldots, v_p] \right\} \\
= \sum_{j=0}^{p} (-1)^j \partial_{p-1} [v_0, \ldots, \hat{v}_j, \ldots, v_p] \\
= \sum_{j=0}^{p} (-1)^j \left\{ \sum_{i=0}^{j-1} (-1)^i [v_0, \ldots, \hat{v}_i, \ldots, \hat{v}_n, \ldots, v_p] + \sum_{i=j+1}^{p} (-1)^{j-1} [v_0, \ldots, \hat{v}_i, \ldots, \hat{v}_n, \ldots, v_p] \right\} \\
= \sum_{i=j} (-1)^{i+j} [v_0, \ldots, \hat{v}_i, \ldots, \hat{v}_n, \ldots, v_p] + \sum_{i=j} (-1)^{i+j-1} [v_0, \ldots, \hat{v}_i, \ldots, \hat{v}_n, \ldots, v_p] \\
= \sum_{i<j} ((-1)^{i+j} + (-1)^{i+j-1}) [v_0, \ldots, \hat{v}_i, \ldots, \hat{v}_n, \ldots, v_p] \\
= 0
\]

which establishes the result.
We now proceed to first identify all the $p$-dimensional boundaries present (cycles) and then to identify the ones that are boundaries of connected regions in terms of the chain group.

**Definition**

$z_p \in C_p(K)$ is called a $p$-dimensional cycle or $p$-cycle if $\partial z_p = 0$. The family of $p$-cycles is thus the kernel of the homomorphism: $\partial : C_p \rightarrow C_{p-1}$ and is a subgroup of $C_p(K)$. This subgroup is called the $p$-dimensional cycle group of $K$ and is denoted by $Z_p(K)$. \hfill (4.13)

**Definition**

$b_p \in C_p(K)$ is called a $p$-dimensional boundary or $p$-boundary if there is a $(p + 1)$ chain $C_{p+1}$ such that $\partial C_{p+1} = b_p$. The family of $p$-boundaries is thus the homomorphic image $\partial C_{p+1}(K)$ and is a subgroup of $C_p(K)$. This subgroup is called the $p$-dimensional boundary group of $K$ and is denoted by $B_p(K)$. \hfill (4.14)

Note, because of Theorem (4.12) any element $b_p$ of $B_p(K)$ has the property that $\partial b_p = 0$. Thus $B_p(K)$ is a subgroup of $Z_p(K)$. In order to spot the $(p + 1)$-dimensional holes we have thus to weed out the elements belonging to $B_p(K)$ contained in $Z_p(K)$. This is achieved by introducing the homology group $H_p(K)$.

**Definition**

The $p$-dimensional homology group of $K$ denoted by $H_p(K)$ is the quotient group:

$$H_p(K) = Z_p(K)/B_p(K)$$ \hfill (4.15)

An element $h_p$ of $H_p(K)$ is thus an equivalent class $[z_p]$, defined by the relation $z_p^1$ is equivalent to $z_p^2$ if $z_p^1 - z_p^2 \in B_p(K)$. This equivalence relation is called homology and if $z_p^1 - z_p^2$ is in $B_p(K)$ then $z_p^1$ and $z_p^2$ are said to be homologous. The fact that $H_p(K)$ is a group is easy to verify and follows from the fact that the $Z_p(K)$ and $B_p(K)$ are both abelian groups.

From the way it is defined it might seem quite remarkable that the groups $H_p(K)$ do not depend on the triangulation of $K$ (Theorem 4.1b). This is because the groups $C_p(K)$, $Z_p(K)$ and $B_p(K)$ certainly do depend on the triangulation of $K$. On the other hand from the geometrical ideas which
motivate the definition of $H_p(K)$. Theorem (4.1b) is understandable. Different triangulations of $K$ are certainly expected to lead to different cycle groups and boundary groups since the number of boundaries introduced depends on triangulation. But $H_p(K)$, which depends on the number of $(p + 1)$-dimensional holes present in the space, should not.

Using Theorem (4.1a) and the definition of the homology group $H_p(K)$ just introduced we can prove our first general Theorem.

**Theorem**

If $K$ is a contractible space i.e. has the homotopy type of a single point then

$$H_p(K) = \begin{cases} \{0\}, & p \neq 0 \\ \mathbb{Z}, & p = 0 \end{cases} \quad (4.16)$$

**Proof**

From Theorem (4.1b) it follows that if $K$ and the point $v_0$ have the same homotopy type then

$$H_p(K) = H_p([v_0]), \forall p$$

Thus, as far as the homology groups are concerned, the complex corresponding to $K$ is the 0-simplex $\sigma^0 = [v^0]$. The dimension of $K$ is thus zero, $C_p(K) = \{0\}$ for $p > 0$ by definition and hence $H_p(K) = \{0\}$ for $p \neq 0$. For $p = 0$ we note that $Z_0(K) = C_0(K)$ since all elements $C_0$ of $C_0(K)$ have property $\partial C_0 = 0$. On the other hand, as there are no higher dimensional simplexes present, $B_0(K) = \{0\}$. So that

$$H_0(K) = Z_0(K) = \{z_0 | z_0 = f[v_0], f \in \mathbb{Z}\}$$

$$= \mathbb{Z}$$

which completes the proof.

A few examples illustrating how the homology groups $H_p(K)$ are calculated might be useful.

**Example 1.** Let $K = \sigma^2 = [v_0, v_1, v_2]$. We will calculate $H_k(K)$ for $k = 0, 1, 2, \ldots$. We first note that $\dim K = 2$. Thus, by definition $C_0(K) = \{0\}$ for $p > 2$, hence $H_k(K) = \{0\}$ for $k > 2$. Thus we only have to calculate $H_0(K), H_1(K)$, and $H_2(K)$.

**Calculation of $H_0(K)$.** Since $H_0(K) = Z_0(K)/B_0(K)$, we have to determine $Z_0(K)$ and $B_0(K)$ we recall that $z_0 \in Z_0(K)$ if $z_0 \in C_0(K)$ and $\partial z_0 = 0$ from definition. (4.11) it follows that $C_0(K) = Z_0(K)$ since all zero chains
have zero boundaries. Also any element of $C_0(K)$ can be written as:

$$a_0[v_0] + b_0[v_1] + c_0[v_2]$$

where $a_0, b_0, c_0 \in \mathbb{Z}$; $Z_0(K)$ thus has three independent generators so that:

$$Z_0(K) = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} = \mathbb{Z}^3$$

Next we study $B_0(K)$ we recall that $b_0 \in B_0(K)$ if $b_0 \in C_0(K)$ and $b_0 = \partial C_1$ where $C_1 \in C_1(K)$. Any element of $C_1(K)$ can be written as:

$$C_1 = a_1[v_0, v_1] + b_1[v_0, v_2] + c_1[v_1, v_2]$$

Thus

$$\partial C_1 = a_1([v_1] - [v_0]) + b_1([v_2] - [v_0]) + c_1([v_2] - [v_1])$$

$$= (a_1 - c_1)[v_1] - (a_1 + b_1)[v_0] + (b_1 + c_1)[v_2]$$

Hence any element $b_0$ of $B_0(K)$ can be written as:

$$b_0 = a_0[v_0] + b_0[v_1] + c_0[v_2]$$

with $a_0 + b_0 + c_0 = 0$. This means that $B_0(K)$ has two independent generators. Hence $B_0(K) = \mathbb{Z} \oplus \mathbb{Z}$. Finally if $h_0 \in H_0(K)$ we can write $h_0$ as the coset:

$$h_0 = z_0 + B_0(K),$$

$$= a_0[v_0] + b_0[v_1] + c_0[v_2] + [-a_0[v_0] - b_0[v_1] + (a_0 + b_0)[v_2]]$$

$$= (a_0 + b_0 + c_0)[v_2] = d_0[v_2], d_0 \in \mathbb{Z}$$

Thus $H_0(K)$ has only one independent generator and $H_0(K) = \mathbb{Z}$

**Calculation of $H_1(K)$**. Since $H_1(K) = Z_1(K)/B_1(K)$, we have to determine $Z_1(K)$ and $B_1(K)$. Now $z_1 \in Z_1(K)$ if $z_1 \in C_1(K)$ and $\partial z_1 = 0$. From our previous calculation we know that if $z_1 \in C_1(K)$ then $z_1 = a_1[v_0, v_1] + b_1[v_0, v_2] + c_1[v_1, v_2]$ and $\partial z_1 = (a_1 - c_1)[v_1] - (a_1 + b_1)[v_0] + (b_1 + c_1)[v_2]$. Hence the requirement $\partial z_1 = 0$ means that

$$z_1 = a_1[v_0, v_1] - a_1[v_0, v_2] + a_1[v_1, v_2]$$

i.e.

$$Z_1(K) = \mathbb{Z}$$

Next we have to study $B_1(K)$. We note that $b_1 \in B_1(K)$ if $b_1 \in C_1(K)$ and $b_1 = \partial C_2$ where $C_2 \in C_2(K)$. Any element of $C_2(K)$ can be written as:

$$C_2 = a_2[v_0, v_1, v_2]$$

Then

$$\partial C_2 = a_2([v_1, v_2] - [v_0, v_2] + [v_0, v_1])$$
Hence any element \( b_1 \) of \( B_1(K) \) can be written as:

\[
b_1 = a_2[[v_1, v_2] - [v_0, v_2] + [v_0, v_1]]
\]

This means that \( B_1(K) = \mathbb{Z} \). Finally if \( h_1 \in H_1(K) \) we write \( h_1 = z_1 + B_1(K) \), and note that \( z_1 \in B_1(K) \) so that \( H_1(K) = \{0\} \).

**Calculation of \( H_2(K) \).** Since \( H_2(K) = Z_2(K)/B_2(K) \), we have to determine \( Z_2(K) \) and \( B_2(K) \). As before \( z_2 \in Z_2(K) \) if \( z_2 \in C_2(K) \) and \( \partial z_2 = 0 \).

From our previous calculation it follows that \( Z_2(K) = \{0\} \), since \( \partial z_2 = 0 \), implies \( a_2 = 0 \). Again \( b_2 \in B_2(K) \) if \( b_2 \in C_2(K) \) and \( b_2 = \partial c_3 \). Since there are no 3-simplexes in \( K \), \( b_2 = 0 \). So that \( H_2(K) = \{0\} \).

Thus when \( K = \sigma^2 = [v_0, v_1, v_2] \)

\[
H_0(K) = \mathbb{Z}, \quad H_k(K) = \{0\}, \quad k \neq 0.
\]

(4.17)

**Example 2**

\[
K = S^1 = \partial \sigma^2
\]

Now \( \dim K = 1 \), so that \( C_k(S^1) = \{0\} \) for \( k > 1 \). Hence \( H_k(S^1) = \{0\} \), for \( k > 1 \). Thus we only have to calculate \( H_0(S^1) \) and \( H_1(S^1) \).

**Calculation of \( H_0(S^1) \).** Since \( H_0(S^1) = \mathbb{Z} / B_0(S^1) \) we have to determine \( Z_0(S^1) \) and \( B_0(S^1) \). Again \( z_0 \in Z_0(S^1) \) if \( z_0 \in C_0(S^1) \) and \( \partial z_0 = 0 \). From Example 1 it follows that \( H_0(S^1) = \mathbb{Z} \) since the 0-simplex and 1-simplex structure of \( \sigma^2 \) and \( \partial \sigma^2 \) are the same.

**Calculation of \( H_1(S^1) \).** Since \( H_1(S^1) = Z_1(S^1) / B_1(S^1) \) we have to determine \( Z_1(S^1) \) and \( B_1(S^1) \). From Example 1 it follows that \( Z_1(S^1) = \mathbb{Z} \) since \( Z_1(S^1) \) only involves the 1-simplexes. However \( B_1(S^1) = \{0\} \), since there are no 2-simplexes in \( S^1 \). So that \( H_1(S^1) = \mathbb{Z} \).

Thus \( H_0(S^1) = \mathbb{Z} \), \( H_1(S^1) = \mathbb{Z} \)

\[
H_k(S^1) = \{0\}, \quad k > 1
\]

(4.18)

In all of these examples one notes that \( H_0(K) = \mathbb{Z} \). The reason for this is the following Theorem which, again, we will not prove:

**Theorem**

If \( K \) is a connected polyhedron then

\[
H_0(K) = \mathbb{Z}
\]

(4.19)

A proof of the theorem can be found in references [1] or [3] listed at the end of the Chapter.

From the fact that \( H_0 \) is the quotient group of two abelian groups \( Z_\phi \) and \( B_\phi \) a few remarks regarding the general structure of the homology groups


$H_n(K)$ can be made. To do this a few results from the theory of abelian groups are needed which we state without proof.

## 4.3 Abelian Groups

### Definition

Let $G$ be an abelian group. A set $\{g_i\}$ of elements $G$ is called a set of generators of $G$ if every element $g \in G$ can be expressed in the form of a finite sum

$$\sum_{i=1}^{K} n_i g_i$$

where $n_i \in \mathbb{Z}$. \hfill (4.20)

### Definition

The set $\{g_i\}$ of Definition (4.20) freely generates $G$ if for each $g \in G$ the expression

$$g = \sum_{i=1}^{K} n_i g_i$$

is unique i.e. the elements $\{g_i\}$ are linearly independent over $\mathbb{Z}$. An abelian group $G$ which is freely generated by a set of generators is called a free abelian group and a free generating set is called a basis. \hfill (4.21)

### Definition

An abelian group is said to be finitely generated if it has a set generators consisting of a finite number of elements. \hfill (4.22)

It is a theorem in linear algebra that the number of elements in a basis of a free finitely generated abelian group is independent of the choice of the basis. This number is called the rank of the group. The two theorems on finitely generated abelian groups that we need can now be stated.

### Theorem

Let $F$ be a free finitely generated abelian group and let $R$ be a subgroup of $F$. Then $R$ is free and finitely generated. \hfill (4.23)
Theorem

Let $A$ be a finitely generated (not free) abelian group generated by $n$ generators, say. Then

$$A = F/R = G + Z_{h_1} + \ldots + Z_{h_m}$$

where $F$ and $R$ are free finitely generated abelian groups with $R \subseteq F$, $G$ is a free abelian group of rank $(n-m)$ and $Z_{h_i}$ is cyclic of order $h_i$. The rank $(n-m)$ of $G$ and the numbers $h_1, h_2, \ldots, h_m$ are uniquely determined by $A$. Very often we write

$$A = G \oplus T, \text{ where } T = Z_{h_1} \oplus Z_{h_2} \ldots Z_{h_m} \quad (4.24)$$

and call $T$ the torsion subgroup of $A$.

Theorems (4.19, 4.20) immediately imply that the boundary groups $B_p(K)$ and cycle groups $Z_p(K)$ are free finitely generated abelian groups while

$$H_p(K) = Z_p(K)/B_p(K) = G_p \oplus T_p \quad (4.25)$$

where $G_p$ is a free finitely generated abelian group and $T$ the torsion subgroup of $H_p(K)$. This result is interesting. We introduced the homology group $H_p(K)$ as an algebraic object which could spot the $(p+1)$ dimensional holes present in $K$. This feature of $H_p(K)$ is reflected in its free finitely generated abelian group part $G_p$. The rank $R_p(K)$ of $G_p(K)$ counts the number of such holes and is called the $p^{th}$ Betti number of $K$. But as we now discover $H_p(K)$ contains in it even more information. For even if $G = \{0\}$, $H_p(K) \neq \{0\}$, if $T_p \neq 0$. The group $T_p(K)$ contains information about the manner in which the space $K$ is twisted as the following example illustrated.

Example

The projective plane $P$. We recall that the projective plane is obtained from a finite disc by identifying each pair of diametrically opposite points. A triangulation of $P$ was given in Example 6, Chapter 3. For convenience the triangulation is given below (Fig. 4.3)

We note that $\dim P = 2$, so that $C_p(P) = \{0\}$, for $p > 2$. Hence $H_p(P) = \{0\}$ for $p > 2$ and we only have to calculate $H_0(P)$, $H_1(P)$ and $H_2(P)$.

Calculation of $H_2(P)$. Since $H_2(P) = Z_2(P)/B_2(P)$ we have to determine $Z_2(P)$ and $B_2(P)$.

We recall that if $b_2 \in B_2(P)$, then $b_2 \in C_2(P)$ and $b_2 = \partial C_3$, where $C_3 \in C_3(P)$. Since $P$ does not contain any 3-simplexes, $b_2 = 0$ and $B_2(P) = \{0\}$. 
Again if \( z_2 \in Z_2(P) \), then \( z_2 \in C_2(P) \) and \( \partial z_2 = 0 \). There are ten 2-simplexes in \( P \) thus

\[
\begin{align*}
z_2 &= a_1 \langle v_0, v_1, v_2 \rangle + a_2 \langle v_0, v_3, v_4 \rangle + a_5 \langle v_1, v_3, v_5 \rangle + a_6 \langle v_1, v_2, v_3 \rangle \\
&\quad + a_2 \langle v_0, v_4, v_5 \rangle + a_3 \langle v_0, v_2, v_4 \rangle + a_7 \langle v_1, v_4, v_5 \rangle + a_8 \langle v_1, v_2, v_5 \rangle \\
&\quad + a_9 \langle v_2, v_3, v_5 \rangle + a_{10} \langle v_2, v_3, v_4 \rangle
d\end{align*}
\]

and

\[
\begin{align*}
\partial z_2 &= (a_1 + a_2) \langle v_1, v_2 \rangle - (a_1 - a_3) \langle v_0, v_2 \rangle + (a_1 + a_2) \langle v_0, v_1 \rangle \\
&\quad + (a_2 + a_0) \langle v_1, v_3 \rangle - (a_2 - a_3) \langle v_0, v_3 \rangle - (a_1 + a_0) \langle v_3, v_5 \rangle \\
&\quad - (a_3 + a_4) \langle v_0, v_5 \rangle + (a_4 + a_7) \langle v_4, v_5 \rangle + (a_4 - a_5) \langle v_0, v_4 \rangle \\
&\quad + (a_5 - a_{10}) \langle v_2, v_4 \rangle + (a_6 + a_{10}) \langle v_3, v_4 \rangle - (a_6 - a_7) \langle v_1, v_4 \rangle \\
&\quad - (a_7 + a_8) \langle v_1, v_5 \rangle + (a_8 - a_9) \langle v_2, v_5 \rangle + (a_9 + a_{10}) \langle v_2, v_3 \rangle
\end{align*}
\]

In order to see the geometrical significance of this expression it is convenient to write it as:

\[
\begin{align*}
\partial z_2 &= a_1 \langle v_0, v_1 \rangle + \langle v_1, v_2 \rangle + \langle v_2, v_0 \rangle \\
&\quad + a_2 \langle v_1, v_3 \rangle + \langle v_3, v_0 \rangle + \langle v_0, v_1 \rangle \\
&\quad + a_3 \langle v_0, v_3 \rangle + \langle v_3, v_5 \rangle + \langle v_5, v_0 \rangle \\
&\quad + a_4 \langle v_5, v_0 \rangle + \langle v_0, v_4 \rangle + \langle v_4, v_5 \rangle
\end{align*}
\]
\[ + a_5([v_0, v_2] + [v_2, v_4] + [v_4, v_0]) + a_6([v_1, v_3] + [v_3, v_4] + [v_4, v_1]) + a_7([v_4, v_3] + [v_1, v_2] + [v_2, v_1]) + a_8([v_1, v_1] + [v_2, v_3] + [v_3, v_1]) + a_9([v_3, v_3] + [v_2, v_2] + [v_2, v_3]) + a_{10}([v_2, v_3] + [v_3, v_4] + [v_4, v_2]) \] (4.28)

Now one can explicitly see that each term in (4.28) traces out a closed cycle through the complex. From (4.27) it follows that \( \partial z_2 = 0 \) implies 
\[ a_1 = a_5, \ a_2 = a_3, \ a_4 = a_5, \ a_5 = a_10, \ a_6 = a_7, \ a_8 = a_9, \ a_1 + a_8 = 0, \ a_2 + a_6 = 0, \ a_2 + a_8 = 0, \ a_4 + a_6 = 0, \ a_6 + a_8 = 0 \] and \( a_4 + a_8 = 0 \) which in turn means that 
\[ a_1 = a_2 = a_3 = a_4 = a_5 = a_6 = a_7 = a_8 = a_9 = a_{10} = 0 \]
Thus \( Z_2(P) = \{0\} \). Hence \( H_2(P) = \{0\} \).

**Calculation of \( H_1(P) \).** Since \( H_1(P) = Z_1(P)/B_1(P) \) we have to determine \( Z_1(P) \) and \( B_1(P) \).

We note that \( b_1 \in B_1(P) \) means that \( b_1 \in C_1(P) \) and \( b_1 = \partial C_2 \), where \( C_2 \in C_2(P) \). From the calculation of \( H_2(P) \) we know the form of \( b_1 \). It is given by equation (4.27) when \( C_2 \) is taken to be equation (4.26). Thus \( B_1(P) = \mathbb{Z}^{10} \), the direct sum of 10 copies of the abelian group \( \mathbb{Z} \). Next we consider \( z_1 \in Z_1(P) \). Then \( z_1 \in C_1(P) \) and \( \partial z_1 = 0 \). There are 15 one-simplices in \( P \) so that:
\[ z_1 = d_1[v_0, v_1] + d_2[v_0, v_2] + d_3[v_0, v_3] + d_4[v_0, v_4] + d_5[v_0, v_5] + d_6[v_1, v_2] + d_7[v_1, v_3] + d_8[v_1, v_4] + d_9[v_1, v_5] + d_{10}[v_2, v_3] + d_{11}[v_2, v_4] + d_{12}[v_2, v_5] \]
\[ + d_{13}[v_3, v_4] + d_{14}[v_3, v_5] + d_{15}[v_4, v_5] \] (4.29)
and
\[ 0 = \partial z_1 = -[v_0][d_1 + d_2 + d_3 + d_4 + d_5] + [v_1](d_1 - d_6 - d_7 - d_8 - d_9) + [v_2](d_2 + d_6 - d_{10} - d_{11} - d_{12}) + [v_3](d_3 + d_7 + d_{10} - d_{13} - d_{14}) + [v_4](d_4 + d_8 + d_{11} + d_{13} - d_{15}) + [v_5](d_5 + d_9 + d_{12} + d_{14} + d_{15}) \] (4.30)

There are thus six constraint equations that the fifteen integer co-efficients \( d_1, d_2, \ldots, d_{15} \) must satisfy. However only 5 of them are linearly independent. Thus 10 of the 15 co-efficients in (4.29) can be freely chosen, so that
\[ Z_1(P) = \mathbb{Z}^{10} \]
Finally we turn to the determination of $H_1(P)$. Since $B_1(P)$, $Z_1(P)$ are both $\mathbb{Z}^1$ we examine if $h_1 \in H_1(P)$ also imply $h_1 \in B_1(P)$ that would, of course, mean that $H_1(P) = \{0\}$. Comparing (4.25) with (4.29) we see that for this to be the case we must have:

\[
\begin{align*}
  d_1 &= a_1 + a_2 & d_5 &= -a_3 - a_4 & d_9 &= -a_7 - a_8 & d_{13} &= a_6 + a_{10} \\
  d_2 &= a_3 - a_1 & d_6 &= a_1 + a_8 & d_{10} &= a_9 + a_{10} & d_{14} &= a_3 + a_9 \\
  d_3 &= a_3 - a_2 & d_7 &= a_2 + a_6 & d_{11} &= a_5 - a_{10} & d_{15} &= a_4 + a_7 \\
  d_4 &= a_4 - a_5 & d_8 &= a_7 - a_6 & d_{12} &= a_8 - a_9 
\end{align*}
\]

(4.31)

It is straightforward to check that these equations are consistent with the constraint equations implied by equation (4.30) for $d_1, d_2, \ldots, d_{15}$. Since $a_1, a_2, \ldots, a_{10}$ are non-zero positive or negative integers, equation (4.31) implies that whenever $d_1, d_2, \ldots, d_{15}$ are even every such element of $H_1(P)$ belongs to $B_1(P)$. When $d_1, d_2, \ldots, d_{15}$ are not all even, such a cycle need not be a boundary. An example of such a cycle is obtained by dividing the set of equation (4.27) by 2 and setting $a_1 = a_2 = \ldots = a_{10} = +1$.

But if we consider any arbitrary cycle $z_1$ then $2z_1$ will be a cycle with even coefficients and thus belong to $B_1(P)$. Thus

\[ H_1(P) = \mathbb{Z}/2, \text{ the group of integers modulo } 2 \]

**Calculation of $H_0(P)$**. Since $H_0(P) = Z_0(P)/B_0(P)$ we proceed to determine $Z_0(P)$ and $B_0(P)$.

We have already noted several times that $Z_0(P) = C_0(P)$. Since $P$ contains 6 zero-simplexes, $Z_0(P) = \mathbb{Z}^6$. While if $b_0 \in B_0(P)$, then $b_0 \in C_0(P)$ and $b_0 = \partial C_1$ where $C_1 \in C_1(P)$. Equation (4.26) gives the general structure of $b_0$. We note that there are only five independent coefficients in (4.26). Thus $B_0(P) = \mathbb{Z}^5$ and $H_0(P) = \mathbb{Z}$.

Thus

\[
\begin{align*}
  H_k(P) &= \{0\}, k > 2 \\
  H_2(P) &= \{0\} \\
  H_1(P) &= \mathbb{Z}/2 \\
  H_0(P) &= \mathbb{Z}
\end{align*}
\]

(4.32)

We note that $H_1(P)$ does not contain any freely generated abelian group but only a cyclic group of order 2. This group reflects the twisted nature of the space $P$. 
4.4 RELATIVE HOMOLOGY GROUPS

From the examples it should be apparent that the determination of the homology groups $H_p(K)$ of a polyhedron $K$ starting from the definition of $H_p$, although theoretically feasible is a laborious process. Very often a good deal of the labour can be avoided by considering a subpolyhedron $L \subset K$ and relating $H_p(K)$ to $H_p(L)$. This becomes very fruitful if the notion of the relative homology group $H_p(K, L)$ is introduced. The homology groups $H_p(K)$, $H_p(L)$, and $H_p(K; L)$ then form what is called an exact sequence. This is an algebraic structure which, in certain cases, can actually determine the groups $H_p(K)$ themselves. Our aim now is to establish this important result. First we have to define the relative homology groups $H_p(K; L)$. The intuitive idea is that anything in $K$ belonging to the subpolyhedron $L$ is regarded as belonging to the identity element of $H_p(K; L)$. More precisely let $K$ be a complex and $L \subset K$ be a subcomplex. Let the corresponding chain groups be $C_p(K)$ and $C_p(L)$ respectively then we have:

Definition

The $p$-dimensional chain group of $K$ modulo $L$ or the relative $p$-chain group with integer co-efficients is the quotient group

$$C_p(K; L) = C_p(K) / C_p(L), \quad p > 0$$

Thus each member of $C_p(K, L)$ is a coset

$$c_p + C_p(L), \quad \text{where } c_p \in C_p(K). \quad (4.33)$$

Definition

For $p > 1$, the relative boundary operator $\tilde{\delta}_p$ is the map:

$$\tilde{\delta}_p : C_p(K; L) \rightarrow C_{p-1}(K; L)$$

defined by:

$$\tilde{\delta}_p(c_p + C_p(L)) = \delta_p c_p + C_{p-1}(L)$$

where $c_p + C_p(L) \in C_p(K; L)$ and $\delta_p c_p$ denotes the usual boundary of the $p$-chain $C_p$. It is easy to check that the relative boundary operator is a homomorphism. \quad (4.34)
Definition

The group of relative \( p \)-dimensional cycles on \( K \) modulo \( L \), denoted by \( Z_p(K;L) \) is the kernel of the relative boundary operator. For \( p = 0 \), \( Z_0(K;L) = C_0(K;L) \) \hspace{1cm} (4.35)

Definition

The group of relative \( p \)-dimensional boundaries on \( K \) modulo \( L \), denoted by \( B_p(K;L) \) is the image of \( C_{p+1}(K;L) \) under the relative boundary homomorphism.

The relative homology group can now be defined. We have:

\[ H_p(K;L) = Z_p(K;L)/B_p(K;L), \quad p > 0 \]

The members of \( H_p(K;L) \) are \( z_p + C_p(L) \). Note that it is required that \( \partial z_p \) be a \( (p-1) \)-dimensional chain on \( L \), not that \( z_p \) be an actual cycle. \hspace{1cm} (4.37)

An example illustrating the definitions might be useful.

Example

Let \( K \) be the 2-skeleton of \( \sigma^2 = \{v^0, v^1, v^2\} \), i.e. \( K \) contains all the simplexes of \( \sigma^2 \) and its faces. Let \( L \) be the subcomplex \( \{v^0, v^1\} \) and \( \{v^0, v^1\} \). We will determine \( H_p(K;L) \), \( \forall p \)

A. \( H_0(K;L) \). Since \( H_0(K;L) = Z_0(K;L)/B_0(K;L) \) we must determine \( Z_0(K;L) \) and \( B_0(K;L) \).

We start with \( Z_0(K;L) \). If \( z_0 \in Z_0(K;L) \) then by def. \( z_0 \in C_0(K;L) \) since \( C_0(K;L) = Z_0(K;L) \). \( z_0 \in C_0(K;L) \) means:

\[ z_0 = a_0[v^2] + C_0(L) \]

where \( C_0(L) = \{ b_0[v^1] + g_0[v^0] \} \), \( b_0, g_0 \) integer. Thus \( Z_0(K;L) = \mathbb{Z} \).

Next consider \( B_0(K;L) \). For \( b_0 \) to be an element of \( B_0(K;L) \) we must have:

\[ b_0 \in C_0(K;L) \text{ and } b_0 = \delta c_1, c_1 \in C_1(K;L) \]

\[ b_0 \in C_0(K;L) \Rightarrow b_0 = h_0[v^2] + C_0(L) \]
THE HOMOLOGY GROUPS

For $b_0 = \bar{\partial}C_1$ we note that

$$ C_1 \in C_1(K;L) \Rightarrow C_1 = a_1[v^1, v^2] + b_1[v^0, v^2] + C_1(L) $$

where $C_1(L) = \{h[v^0, v^1], h_1 = \text{integer}\}$

$$ \bar{\partial}G = (a_1 + b_1)[v^2] - a_1[v_1] - b_1[v^0] + C_1(L) $$

$$ = (a_1 + b_1)[v^2] + C_1(L) $$

Thus any element of $C_0(K;L)$ is the relative boundary of some element $C_1 of C_1(K;L)$.

Thus

$$ B_0(K;L) = \mathbb{Z} \quad \text{and} \quad H_0(K;L) = \{0\} $$

B. $H_1(K;L) = z_1(K;L)/B_1(K;L)$

$$ z_1 \in Z_1(K;L) \Rightarrow z_1 \in C_1(K;L) \quad \text{and} \quad \bar{\partial}z_1 = 0 $$

$$ z_1 \in C_1(K;L) \Rightarrow z_1 = a_1[v^1, v^2] + b_1[v^0, v^2] + C_1(L) $$

$$ 0 = \bar{\partial}z_1 = (a_1 + b_1)[v^2] + C_1(L) = 0 $$

Therefore

$$ a_1 + b_1 = 0, \quad a_1 = -b_1 = g, $$

say

$$ z_1 = g[[v^1, v^2] - [v^0, v^2]] + C_1(L) $$

Therefore $Z_1(K, L) = \mathbb{Z}$

$$ b_1 \in B_1(K;L) \Rightarrow b_1 \in C_1(K;L) \quad \text{and} \quad b_1 = \bar{\partial}C_2 $$

But

$$ C_2 = \{0\}, $$

therefore

$$ b_1 = \{0\}, B_1(K;L) = \{0\} $$

Thus

$$ H_1(K;L) = \mathbb{Z} $$

C. $H_k(K;L)$, $k > 1$, since $C_k(K;L) = \{0\}, k > 1 \quad H_k(K;L) = \{0\}, k > 1$

The relative homology groups $H_k(K;L)$ by their construction, are insensitive to $L$. This suggests the following interesting possibility. If we excise or cut out the interior $L_0$ of $L$ will $H_k(K - L_0, L - L_0)$ be isomorphic to $H_k(K, L)$? The answer is yes and we have:
Theorem (excision theorem)

Let $K$ be a complex containing a closed subcomplex $L$. If $L_0$ is an open subcomplex of $L$ such that $\overline{L}_0$ the closure of $L_0$ is contained in the interior of $L$ then

\[ H_p(K; L) = H_p(K - L_0, L - L_0), \forall p \]  \hspace{1cm} (4.38)

This theorem is extremely useful in calculations as we shall see shortly. We next turn to the relationships that exist between the groups $H_p(K)$; $H_p(L)$ and $H_p(K; L)$.

A. **Relation between $H_p(L)$ and $H_p(K)$**. Since $L$ is a subcomplex of $K$ we can relate $L$ to $K$ by means of the inclusion homomorphism on the corresponding chain groups

\[ i : C_p(L) \rightarrow C_p(K) \] \hspace{1cm} (4.39)

defined as:

\[ i[C_p] = c_p, \ c_p \in C_p(L) \subseteq C_p(K) \]

In turn this map induces the group homomorphism:

\[ i^* : H_p(L) \rightarrow H_p(K) \]

B. **Relation between $H_p(K)$ and $H_p(K; L)$**. This relationship is established by considering the homomorphism: \( j : C_p(K) \rightarrow C_p(K; L) \) defined by:

\[ j[C_p] = c_p + C_p(L), \ c_p \in C_p(K) \]

Then $j$ induces a homomorphism: $j^* : H_p(K) \rightarrow H_p(K; L)$ as we now demonstrate. Observe that by definition: $[\partial C_p] = \partial c_p + C_{p-1}(L)$. On the other hand $\partial[j(C_p)] = \partial[c_p + C_p(L)] = \partial c_p + C_{p-1}(L)$. Where $\partial$ represents the relative boundary operator. From these two expressions we see that

\[ j \partial = \partial j \]

Now suppose $z_p \in H_p(K)$. From the result just established we then have:

\[ \partial j(z_p) = j(\partial z_p) \]

But $\partial z_p = 0$, since $z_p \in H_p(K)$, thus $\partial j(z_p) = 0$ which means that $j(z_p) \in Z_p(K; L)$. Thus using $j$ each class $[z_p] \in H_p(K)$ can be mapped into the class $[j(z_p)] \in H_p(K; L)$. It is straightforward to check that such a mapping is a homomorphism. We write:

\[ j^* : H_p(K) \rightarrow H_p(K; L), \forall p \] \hspace{1cm} (4.40)

C. **Relation between $H_p(K; L)$ and $H_{p-1}(L)$**. We now come to the most interesting of the inter-relationships between homology groups. This
relationship is interesting because it relates two groups which differ in their dimensional index \( p \), namely \( H_p(K, L) \) and \( H_{p-1}(L) \). To see how this comes about consider \( z_p \in H_p(K; L) \). By definition this implies that \( z_p \in C_p(K; L) \) and \( \partial z_p = 0 \). i.e.

\[
\partial z_p + C_{p-1}(L) = 0
\]

\( \partial z_p \) is some element \( C_{p-1} \), in \( C_{p-1}(L) \). Furthermore, \( C_{p-1} \) is not just a chain but a cycle, since \( \partial C_{p-1} = \partial^2 z_p \) and \( \partial^2 z_p = 0 \), by Theorem (4.12). Thus \( C_{p-1} \) determines a unique member of \( H_{p-1}(L) \). We write this correspondence (which can be shown to be a group homomorphism) as:

\[
\partial^*: H_p(K; L) \rightarrow H_{p-1}(L)
\]

(4.41)

given by: \( \partial^*([z_p + C_p(L)]) = [\partial z_p] \), where \( [z_p + C_p(L)] \in H_p(K; L) \), and \( [\partial z_p] \), as we just saw, is an element of \( H_{p-1}(L) \).

All of the inter-relationships, which we have described, can be fitted together into a sequence of groups and homomorphisms called the homology sequence of the pair \( (K; L) \). This is defined as:

### 4.5 EXACT SEQUENCES

**Definition**

The homology sequence of the complex \( K \) with subcomplex \( L \) is the sequence of groups and homomorphisms:

\[
\ldots \rightarrow H_p(L) \xrightarrow{i^*} H_p(K) \xrightarrow{j^*} H_p(K; L) \xrightarrow{\partial^*} H_{p-1}(L) \xrightarrow{i^*} \ldots
\]

(4.42)

The homology sequence has the following important property:

**Theorem**

The homology sequence of the complex \( K \) with subcomplex \( L \) is exact, that is, the image of each homomorphism in the sequence is equal to the kernel of the next homomorphism.

(4.43)

The proof of this Theorem is straightforward and consists of establishing the following six results:

1. Image \( i^* \subseteq \) Kernel \( j^* \)
2. Kernel \( j^* \subseteq \) Image \( i^* \)
3. Image \( j^* \subseteq \) Kernel \( \partial^* \)
4. Kernel $\partial^* \subseteq \text{Image } j^*$
5. Image $\partial^* \subseteq \text{Kernel } i^*$
6. Kernel $i^* \subseteq \text{Image } \partial^*$

We sketch the proof of (5) and (6) and leave the other results for the reader to establish. Recall that the map $\partial^*: H_p(K, L) \to H_{p-1}(L)$ sent an element $z_p + C_p(L)$ of $H_p(K, L)$ into the element $\partial z_p$ of $C_{p-1}(L)$. By definition this is a boundary element and hence in the kernel of the inclusion homomorphism $i^*$. This establishes (5). Next suppose $b_{p-1}$ is an element of the kernel of the inclusion homomorphism. Then, by definition, $b_{p-1} = \partial z_p$, $z_p \in H_p(L)$ which, in turn, can be written as an element $z_p + C_p(L)$ of $H_p(K, L)$ as we saw earlier since $\partial(z_p + C_p(L)) = 0$. This establishes (6). Establishing Theorem (4.43) is useful because there are many theorems that compare the groups of an exact sequence. We, state, as an example the simplest of these theorems.

**Theorem**

Suppose that an exact sequence has a section of four groups:

\[
\{0\} \to A \xrightarrow{g} B \xrightarrow{h} \{0\}
\]

where $\{0\}$ denotes the trivial group. Then $g$ is an isomorphism from $A$ onto $B$.

\[\text{(4.44)}\]

**Proof**

The image $f(\{0\}) = \{0\}$ contains only the identity element of $A$. Exactness of the sequence then means that $g$ has kernel $\{0\}$, so that $g$ is one-to-one. The kernel of $h$ is all of $B$ and this, again by exactness, must be the image $g(A)$. Thus $g$ is an isomorphism.

A few examples illustrating how the exact homology sequence combined with the excision Theorem (4.38) is used to determine homology groups will now be given.

**Example 1.** As our first example we prove, using the exact homology sequence and the excision theorem, that $H_p(S^1) = \{0\}$ for $p > 1$. Introduce, $A^1 = S^1 - [n]$, where $[n]$ represents the north pole of the unit circle $S^1$ and $B^1 = S^1 - [s]$, where $[s]$ represents the south pole of $S^1$. It is clear that: $A^1 \cap B^1 = S^1$, and $X^1 = A^1 \cap B^1 = R_1 \cup R_2$, where $R_1 \cap R_2 = \phi$. Figure 4.4 explains the geometrical ideas involved.

Consider now the following exact homology sequences for $p > 1$

\[
\to H_p(A^1) \to H_p(S^1) \to H_p(S^1, A^1) \to H_{p-1}(A^1) \to H_{p-1}(S^1) \to \ldots
\]

\[\text{(4.45)}\]
and
\[ H_p(X^1) \rightarrow H_p(B^1) \rightarrow H_p(B^1, X^1) \rightarrow H_{p-1}(X^1) \rightarrow H_{p-1}(B^1) \rightarrow \ldots \] (4.46)

Observe now that \( A^1 \) and \( B^1 \) are contractible spaces hence:
\[ H_p(A^1) = H_p(B^1) = \{0\} \text{ for } p > 0 \] (4.47)
so that (4.45) and (4.46) become for \( p > 1 \)
\[ \{0\} \rightarrow H_p(S^1) \rightarrow H_p(S^1, A) \rightarrow \{0\} \]
and
\[ \{0\} \rightarrow H_p(B^1, X^1) \rightarrow H_{p-1}(X^1) \rightarrow \{0\} \]
Theorem (4.44) applies and it immediately follows that
\[ H_p(S^1) = H_p(S^1, A^1) \]
and
\[ H_p(B^1, X^1) = H_{p-1}(X^1) \] (4.48)

Next note by removing a point say, the south pole \([s]\) from \( S^1 \) and \( A^1 \), \( S^1 \) becomes \( B^1 \) and \( A^1 \) becomes \( X^1 \). Thus from the excision theorem (4.38) we have:
\[ H_p(S^1, A^1) = H_p(S^1 - [s], A^1 - [s]) \]
\[ = H_p(B^1, X^1) \]
Hence (4.48) implies:
\[ H_p(S^1) = H_{p-1}(X^1), \ p > 1 \] (4.49)

Finally we note that \( X^1 \) can be retracted to two points \([e]\) and \([w]\) say. Hence by Theorem (4.1a)
\[ H_{p-1}(X^1) = H_{p-1}([e] \cup [w]) \] (4.50)
The space consisting of the two points \([e]\) and \([w]\) has dimension zero, by definition and thus by definition, \( C_p([e] \cup [w]) = \{0\} \) for \( p > 0 \) so that \( H_{p-1}([e] \cup [w]) = \{0\} \) for \( p > 1 \) which establishes the result we were after.
Example 2. As our second example we prove that if $K$ is a simplicial complex then

$$H_p(K) = H_{p+1}(\Sigma(K)),$$

for $p > 0$

where $\Sigma(K)$ is the suspension of $K$. In general for any compact space $X$, the suspension of $X$, written $\Sigma(X)$ is homeomorphic to the topological space $(X \times [-1,1])/\sim$, where $\sim$ is the equivalence relation in $X \times [1,-1]$ which identifies all points in $(X \times -1)$ and all points in $(X \times 1)$. Figure 4.5 explains what is involved.

$$\begin{array}{c}
\xymatrix{\Sigma(X) \ar[r] & \Sigma(X) \\
-1 \ar[u] \ar[r] & X \ar[u]}
\end{array}$$

Figure 4.5

In terms of the figure let us call the ‘northern hemisphere’ of $\Sigma(X)$, $\Gamma^+(X)$ and the ‘southern hemisphere’ of $\Sigma(X)$, $\Gamma^-(X)$. Now consider the following exact sequences:

$$\rightarrow H_p(K) \rightarrow H_p(\Gamma^+(K)) \rightarrow H_p(\Gamma^+(K), K) \rightarrow H_{p-1}(K) \rightarrow H_{p-1}(\Gamma^+(K)) \rightarrow$$

and

$$\rightarrow H_p(\Gamma^-(K)) \rightarrow H_p(\Sigma(K)) \rightarrow H_p(\Sigma(K), \Gamma^-(K)) \rightarrow H_{p-1}(\Gamma^-(K)) \rightarrow H_{p-1}(\Sigma(K)) \rightarrow \cdots$$

(4.51)

It is possible to prove that the spaces $\Gamma^+(K)$, $\Gamma^-(K)$ are contractible so that $H_p(\Gamma(K)) = H_p(\Gamma^-(K)) = \{0\}$, for $p > 0$. Equation (4.51) then becomes for $p > 1$

$$\{0\} \rightarrow H_p(\Gamma^+(K), K) \rightarrow H_{p-1}(K) \rightarrow \{0\}$$

and

$$\{0\} \rightarrow H_p(\Sigma(K)) \rightarrow H_p(\Sigma(K), \Gamma^-(K)) \rightarrow \{0\}$$

These imply

$$H_p(\Gamma^+(K), K) = H_{p-1}(K) = \{0\}$$

(4.52)

and

$$H_p(\Sigma(K)) = H_p(\Sigma(K), \Gamma^-(K)) = \{0\}$$
Finally we note that if the point \(-1\) is removed from \(\Sigma(K)\) it is retractable to \(\Gamma\)(\(K\)), while removing the point \(-1\) from \(\Gamma\)(\(K\)) makes it retractable to \(K\). The excision theorem then gives:

\[
H_p(\Sigma(K), \Gamma\)(K)) = H_p(\Sigma(K) - (-1), \Gamma\)(K) - [-1]) = H_p(\Gamma\)(K), K)
\]

which combined with (4.52) establishes the result we were after.

**Example 3.** As our final example we calculate \(H_p(S^n)\) using exact homology sequences and the excision Theorem (4.38).

Introduce

\[
A^n = S^n - [n], [n] = \text{north polar point} \\
B^n = S^n - [s], [s] = \text{the south polar point}
\]

Then \(A^n \cup B^n = S^n\), and \(X^n = A^n \cap B^n\) is retractable to \(S^{n-1}\) for \(n > 1\). A picture explaining the geometry in the case \(n = 2\) is given in the Fig. 4.6.

![Figure 4.6](image)

Again, we have the exact sequences:

\[
\rightarrow H_p(A^n) \rightarrow H_p(S^n) \rightarrow H_p(S^n, A^n) \rightarrow H_{p-1}(A^n) \rightarrow H_{p-1}(S^n) \rightarrow \ldots
\]

and

\[
\rightarrow H_p(X^n) \rightarrow H_p(B^n) \rightarrow H_p(B^n, X^n) \rightarrow H_{p-1}(X^n) \rightarrow H_{p-1}(B^n) \rightarrow \ldots
\]

From these and the observation that \(H_p(A^n) = H_p(B^n) = \{0\}\) for \(p > 0\) it immediately follows that

\[
H_p(S^n) = H_p(S^n, A^n)
\]

and

\[
H_p(B^n, X^n) = H_{p-1}(X^n), \text{ for } p > 1
\]

Now we use the excision theorem (4.38) to prove that

\[
H_p(S^n, A^n) = H_p(S^n - [s], A^n - [s]) = H_p(B^n, X^n)
\]
and hence

\[ H_p(S^n) = H_{p-1}(X^n) \]

For \( n > 1 \), \( X^n \) is retractable to \( S^{n-1} \), we thus have:

\[ H_p(S^n) = H_{p-1}(S^{n-1}) \] (4.53)

valid for \( n > 1 \), \( p > 1 \). By repeated use of (4.53) and a direct computation of \( H_1(S^1) \) it is possible to show that:

\[ H_p(S^n) = \begin{cases} \{ \mathbb{Z} \text{ for } p = 0, p = n \\ \{0\} \text{ otherwise} \end{cases} \] (4.54)

We end our discussion of the homology groups \( H_p(K) \) by making a few remarks.

### 4.6 Torsion, Kunneth Formula, Euler–Poincaré Formula and Singular Homology

#### Torsion

The homology groups \( H_p(K) \) were obtained from the chain groups \( C_p(K) \). The elements of the chain group \( C_p(K) \) were formal linear combinations of the oriented \( p \)-simplexes of \( K \) multiplied by integer co-efficients. It is possible to generalize the chain groups by introducing as co-efficients of the \( p \)-simplexes elements of an arbitrary abelian group \( G \) rather than the integers i.e. writing \( C_p \in C_p(K) \) as:

\[ C_p = \sum_{i=1}^{l_p} g_i \sigma_i^p, g_i \in G \]

These are the chain groups defined over the abelian group \( G \), written \( C_p(K, G) \) and the corresponding homology groups \( H_p(K; G) \) can be constructed. It is quite remarkable that the groups \( H_p(K, G) \) are not really more general than the groups \( H_p(K) \) we have considered. Indeed it is possible to prove that a knowledge of the groups \( H_p(K) \) and \( G \) is enough to completely determine the groups \( H_p(K; G) \). We do not prove this theorem here but point out the following amusing feature of homology theory. If the integer co-efficients are replaced by rational co-efficients \( Q \) the corresponding homology groups \( H_p(K; Q) \), far from being more general, contain in them less information than the corresponding homology groups \( H_p(K) \) with integer co-efficients.
The reason for this is simple. We recall that the integer co-efficient homology groups \( H_p(K) \) had the general structure: \( H_p(K) = G_p(K) \oplus T_p(K) \), where elements of \( T_p(K) \), the torsion subgroup, were finite order cyclic group. This meant that if \( t_p \in T_p(K) \) was of order \( n \) then \( nt_p = 0 \), the identity element. For chains with rational rather than integer co-efficients, the equation \( nt_p = 0 \) would mean that \( t_p = 0 \). Thus \( H_p(K; \mathbb{Q}) \) does not have any torsion subgroup, while \( H_p(K) \) does.

The Kunneth formula

In the Chapter on the fundamental group we proved that if \( K = X \times Y \), then

\[
\pi_1(X \times Y, x_0 \times y_0) = \pi_1(X, x_0) \oplus \pi_1(Y, y_0)
\]

Is there an analogous formula for the homology groups? There is, but it is more complicated. We write down, the formula for homology groups with rational rather than integer co-efficients so that complications due to the torsion subgroups are not present. The formula, known as the Kunneth formula is:

\[
H_p(X \times Y; Q) = \bigoplus_{k+q = p} H_k(X; Q) \otimes H_q(Y; Q) \tag{4.55}
\]

As an application of the Kunneth formula we determine \( H_p(T^2) \), where \( T^2 = S^1 \times S^1 \) the 2-torus. We know that

\[
H_p(S^1) = \begin{cases} \mathbb{Z}, & p = 0, 1 \\ \{0\}, & \text{otherwise.} \end{cases}
\]

Thus using (4.55) we get

\[
H_0(T^2) = H_0(S^1 \otimes S^1) = H_0(S^1) \otimes H_0(S^1) = \mathbb{Z}
\]

\[
H_1(T^2) = H_0(S^1) \otimes H_1(S^1) \oplus H_1(S^1) \otimes H_0(S^1)
\]

\[
= \mathbb{Z} \oplus \mathbb{Z}
\]

\[
H_2(T^2) = H_0(S^1) \otimes H_2(S^1) \oplus H_1(S^1) \otimes H_1(S^1) \oplus H_2(S^1) \otimes H_0(S^1)
\]

\[
= \mathbb{Z}
\]

and

\[
H_p(T^2) = \{0\}, \text{ for } p > 3.
\]

A more general version of (4.55) with the torsion subgroups taken into account is discussed in reference [2].
The Euler-Poincaré formula

We have commented on the fact that the rank of \( H_p(K) \) is related to the number of \((p+1)\) dimensional holes present in the space. There is a remarkable formula involving \( R_p(K) \) and the number of \( p \)-simplexes of \( K \), called the Euler-Poincaré formula, which we will now prove. First we write down the formula:

\[
\chi(K) = \sum_{p=0}^{n} (-1)^p l_p = \sum_{p=0}^{n} (-1)^p R_p(K) \tag{4.56}
\]

where \( l_p \), \( p = 0, 1, \ldots, n \) denotes the number of \( p \)-simplexes present in \( K \). Observe that since the groups \( H_p(K) \) are topological invariants the sum \( \sum_{p=0}^{n} (-1)^p R_p(K) \) is a topological invariant. Thus \( \chi(K) \) which can be calculated simply from a knowledge of the number of different dimensional simplexes present in \( K \) is a topological invariant of \( K \). \( \chi(K) \) is called the Euler characteristic of \( K \). To prove the formula we note that for each dimension \( p \), \( 0 \leq p \leq n \) we have:

\[
\partial: C_p(K) \rightarrow C_{p-1}(K)
\]

where \( C_{-1} \) is by definition the zero space. From the rank and nullity theorem of linear algebra it follows that:

\[
l_p = \text{dimension of } C_p(K) = \text{dimension (kernel } \partial) + \text{dimension (image } \partial) = \text{dimension of } Z_p(K) + \text{dimension } B_{p-1}(K)
\]

While:

\[
R_p(K) = \text{dimension } H_p(K) = \text{dimension } [Z_p(K)/B_p(K)] = \text{dimension } Z_p(K) - \text{dimension } B_p(K)
\]

Thus:

\[
\chi(K) = \sum_{p=0}^{n} (-1)^p R_p(K)
\]

\[
= \sum_{p=0}^{n} (-1)^p [\text{dimension } Z_p(K) - \text{dimension } B_p(K)]
\]

\[
= \sum_{p=0}^{n} (-1)^p \text{ dimension } Z_p(K) + \sum_{p=0}^{n} (-1)^{p+1} \text{ dimension } B_p(K)
\]
Now observe that:

\[
\sum_{p=0}^{n} (-1)^{p+1} \dim B_p(K) = \sum_{p=0}^{n} (-1)^p \dim B_{p-1}(K)
\]

Since \( \dim B_p = 0 \), for \( p = n \) and \( \dim B_{-1} = 0 \). Thus

\[
\chi(K) = \sum_{p=0}^{n} (-1)^p [\dim Z_p(K) + \dim B_{p-1}(K)]
\]

\[
= \sum_{p=0}^{n} (-1)^p lp
\]

which establishes the formula.

**Singular homology**

As our final remark we briefly sketch how the homology groups for an arbitrary topological space \( X \) can be constructed. We start by introducing the standard \( p \)-simplexes \( \Delta_p \). These are the set of points \( (x_0, \ldots, x_p) \) in euclidean space with the property: \( 0 \leq x_i \leq 1 \) and \( \sum_{i=0}^{p} x_i = 1 \). The singular \( p \)-simplexes \( \lambda^p_\gamma \) in \( X \) can now be defined as continuous mappings: \( \lambda^p_\gamma : \Delta_p \to X \). The term singular is used to indicate that the maps \( \lambda^p_\gamma \) need not be invertible. In terms of the singular \( p \)-simplexes \( \lambda^p_\gamma \) the singular chain group \( S_p(X) \) can be defined as the set of formal sums:

\[
S_p = \sum \lambda^p_\gamma g, \quad g_i \in G, \text{ an abelian group and the addition of two singular chain elements is defined as:}
\]

\[
\sum \lambda^p_\gamma g_i + \sum \lambda^p_\gamma h_i = \sum \lambda^p_\gamma (g_i + h_i)
\]

when \( g_i, h_i \in G \).

To construct the singular homology groups the boundary map \( \partial \) on the singular chain \( S_p(x) \) has to be defined. Suppose, as before, that \( \partial \) is linear so that all we really need to know is the action of \( \partial \) on a singular \( p \)-simplex \( \lambda^p \). This is defined as:

\[
\partial \lambda^p = \sum_{r=0}^{p} (-1)^r \lambda^p \circ F^r
\]

where \( \lambda^p \circ F^r \) denotes the \( r \)th face of the singular \( p \)-simplex \( \lambda^p \) and in turn is defined as follows:

\[
\lambda^p \circ F^r : \Delta_{p-1} \to X
\]
where
\[ F' : \Delta_{p-1} \to \Delta_p \]
denotes the map:
\[ F'(x_0, \ldots, x_{n-1}) = (x_0, \ldots, \hat{x}_i, \ldots, x_n) \]
The notation \( \hat{x}_i \) means that the vertex \( x_i \) has been omitted. It is possible to show that \( \partial^2 = 0 \) and define the singular homology group for the arbitrary topological space \( X \) as:
\[ H_p(X; G) = \frac{\text{Ker} [\partial : S_p \to S_{p-1}]}{\text{Image} [\partial : S_{p+1} \to S_p]} \]
For further information regarding singular homology theory, references 2 and 3 can be consulted.

**REFERENCES**