Topological Methods for Differential Equations

Degree Theory, Conley Index and Morse Theory

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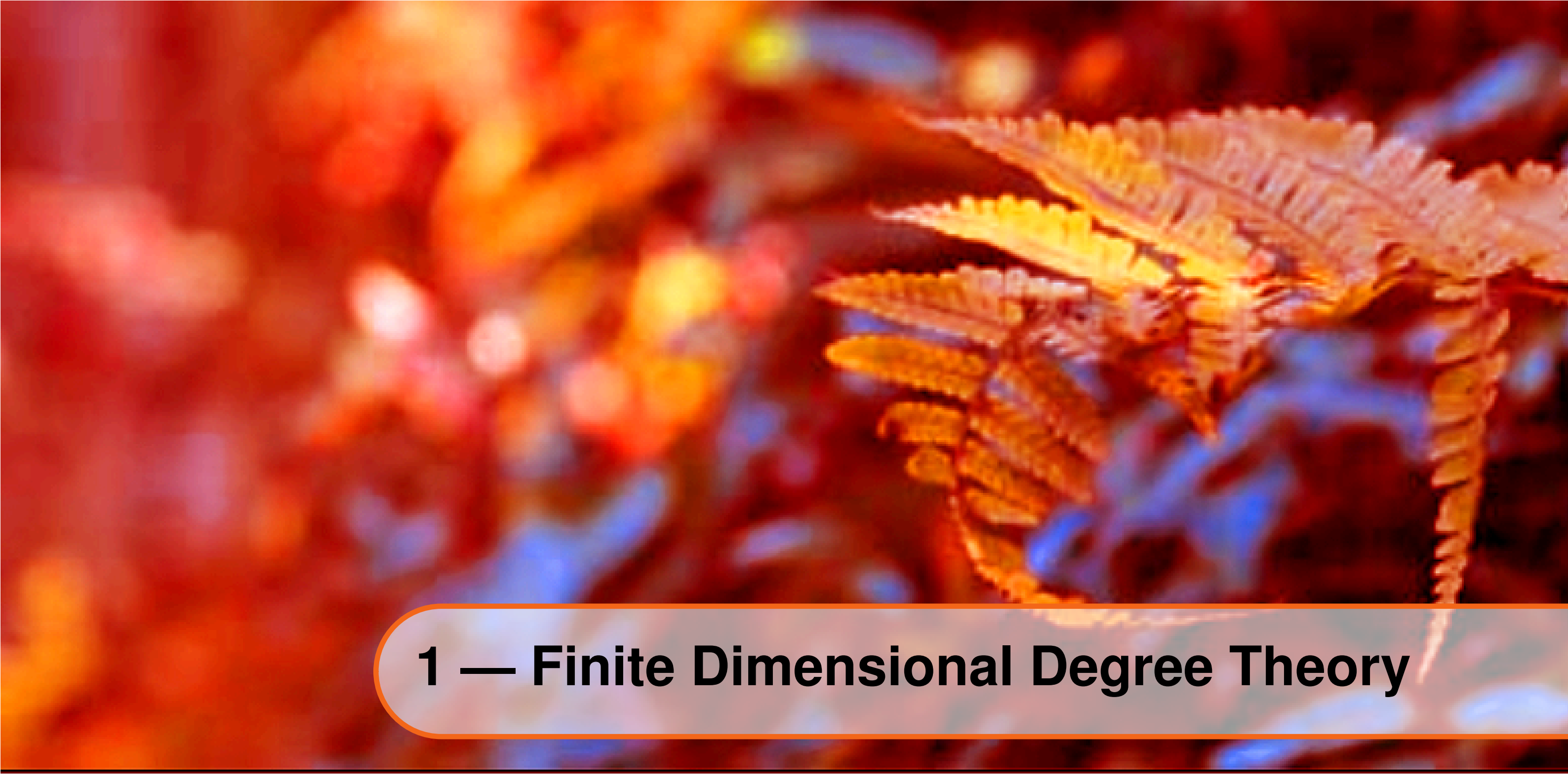
Preface

These notes where composed of lectures given at the VU University in 2006, 2007, 2009, 2013 and 2014. In course various topological principles and techniques are introduced such as finite and infinite degree theory, Morse theory and Conley index theory. The motivation for discussing these topological tools is the application to nonlinear partial differential equations. Throughout theses notes we provide examples and applications to both ordinary differential equations/dynamical systems and partial differential equations. A major part of the course notes is dedicated to Conley index theory and dynamical systems and their relation to degree theory and variational methods such as Morse theory. We chose not to exploit applications to bifurcation theory. Many good texts on this subject can be found in the literature.

The chapters on Conley theory are partly based on lecture notes with W.D. Kalies and K. Mischaikow.



1 — Finite Dimensional Degree Theory

The mapping degree is a topological tool that can be used to find zeroes of functions from $\mathbb{R}^n$ to $\mathbb{R}^n$. Consider the functions $f(x,\lambda) = x^4 - 5x^2 + 4 - \lambda$, and $g(x,\lambda) = x^3 - x - \lambda$. For $\lambda = 0$ both functions have only non-degenerate zeroes. Assign either ± 1 to each root depending on the sign of derivative of the function at a zero and define the mapping degree to be the sum of the signs. For f this number is equal to zero and for g it is equal to 1. By varying the parameter λ, the degree may be computed in most cases, i.e. when the zeroes are all non-degenerate. Notice that for f the answer is always 0 and for g the answer is always 1. In the latter case there is at least one zero, while f does not need to have zeroes at all. In Section 1.2 this idea will be formalized for C^1-mappings $f : \mathbb{R}^n \to \mathbb{R}^n$.

1.1 Notation

Let $\Omega \subset \mathbb{R}^n$ be a bounded,[1] open subset of $\mathbb{R}^n$, which be will referred to as a bounded domain. Its closure is denoted by $\overline{\Omega}$ and the boundary is defined as $\partial\Omega = \overline{\Omega} \backslash \Omega$. The closure $\overline{\Omega}$ is a compact set. Points $x \in \Omega$ are represented in coordinates as follows; $x = (x_1, \cdots, x_n)$. Super-indices will be used to label points in $\mathbb{R}^n$.

The class of functions $f : \overline{\Omega} \subset \mathbb{R}^n \to \mathbb{R}^n$ that are continuous on $\overline{\Omega}$ is denoted by $C^0(\overline{\Omega}; \mathbb{R}^n)$, or $C^0(\overline{\Omega})$ for short. Functions that are continuous on Ω are denoted by $C^0(\Omega; \mathbb{R}^n)$. If $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$ is uniformly continuous, then f can be extended to a continuous function on $\overline{\Omega}$. Therefore $C^0(\overline{\Omega}) \subset C^0(\Omega)$, which is also referred to as the subspace of uniformly continuous functions on Ω. A function f is said to be k-times continuously differentiable on Ω if f and all its derivatives up to

[1] Consider $\mathbb{R}^n$ with the standard Euclidean metric.

order k are continuous on Ω. This class is denoted by $C^k(\Omega;\mathbb{R}^n)$. A function f is k-times continuously differentiable on $\overline{\Omega}$ if f and all derivatives up to order k are uniformly continuous, and thus extend continuously to $\overline{\Omega}$. The class of k-times continuously differentiable on $\overline{\Omega}$ is denoted by $C^k(\overline{\Omega};\mathbb{R}^n)$.

A continuous mapping $f:\mathbb{R}^n \to \mathbb{R}^n$ is said to be proper if $f^{-1}(K) := \{x \in \mathbb{R}^n \mid f(x) \in K\}$ is compact for every compact set $K \subset \mathbb{R}^n$. Proper mappings are closed, i.e. a mapping f is called a *closed mapping* if it maps closed sets $A \subset \overline{\Omega}$ to closed sets $f(A) \subset \mathbb{R}^n$.

1.1 Exercise Show that a proper mapping is a closed mapping. ■

Let $\Omega \subset \mathbb{R}^n$ be an unbounded domain. If we restrict a proper mapping to an unbounded domain $\overline{\Omega}$, then the restriction is a proper mapping on $\overline{\Omega}$. If Ω is a bounded domain, then $f:\overline{\Omega} \subset \mathbb{R}^n \to \mathbb{R}^n$ is a proper mapping since $f^{-1}(K) \subset \overline{\Omega}$ is a closed subset and thus compact. Proper mappings on unbounded domains are a natural extension of continuous mappings on bounded domains.

The length of vectors in $\mathbb{R}^n$ can be measured using the p-norms. For $x = (x_1,\cdots,x_n) \in \mathbb{R}^n$,

$$|x|_p = \Big(\sum_i |x_i|^p\Big)^{1/p}, \quad 1 \le p < \infty, \quad \text{and} \quad |x|_\infty = \max_i\{|x_i|\}.$$

The latter is also referred to as the supremum norm. Since $\mathbb{R}^n$ is finite dimensional all these norms are equivalent.

1.2 Exercise Prove that the p-norms defined above are all equivalent norms on $\mathbb{R}^n$. ■

In the case that no subscript is given, $|\cdot|$ indicates the 2-norm, or Euclidean norm. The 2-norm can be associated to an inner product. For $x,y \in \mathbb{R}^n$, define $\langle x,y\rangle = \sum_i x_i y_i$, and $|x|^2 = \langle x,x\rangle$. The norms given above can also be used to define the notion of distance. For any two points $x,y \in \mathbb{R}^n$ define the distance to be $d_p(x,y) = |x-y|_p$. The distance is also referred to as a metric and $\mathbb{R}^n$ is a complete metric space. The distance between a set Ω and a point x is defined by $d_p(x,\Omega) = \inf_{y\in\Omega} d_p(x,y)$, and more generally, the distance between two sets Ω, and Ω' is then given by $d_p(\Omega',\Omega) = \inf_{x\in\Omega'} d_p(x,\Omega)$. The distance is symmetric in Ω and Ω'. If no subscript is indicated, $d(x,y)$ is the distance associated to the standard Euclidean norm. An open ball in $\mathbb{R}^n$ of radius r and center x is denoted by $B_r(x) = \{y \in \mathbb{R}^n \mid |x-y| < r\}$. If the choice of norm is not indicated in the notation it is usually clear from context.

The linear spaces of C^k-functions can be regarded as a normed linear vector space. For $k=0$ the norm is given by

$$\|f\|_{C^0} = \max_{x\in\overline{\Omega}} |f(x)|_\infty.$$

and for functions $f \in C^1$ the norm is given by $\|f\|_{C^1} = \|f\|_{C^0} + \max_{1\leq i\leq n} \|\partial_{x_i} f\|_{C^0}$, where $\partial_{x_i} f$ denotes the partial derivative with respect to the ith coordinate. The norms for $k \geq 2$ are defined similarly by considering the higher derivatives in the supremum norm. On these normed linear vector spaces the norm can be used to define a distance, or metric as explained above for $\mathbb{R}^n$. Since $\overline{\Omega}$ is compact the spaces $C^k(\overline{\Omega})$, equipped with the above norms, are complete and are therefore Banach spaces. For function $f \in C^k(\overline{\Omega})$ the support is defined to be the closed set

$$\operatorname{supp}(f) = \overline{\{x \in \overline{\Omega} \mid f(x) \neq 0\}}.$$

Functions whose support is contained in Ω are denoted by $C_0^k(\overline{\Omega}) = \{f \in C^k(\overline{\Omega}) \mid \operatorname{supp}(f) \subset \Omega\}$ and form a linear subspace of $C^k(\overline{\Omega})$. As matter of fact $C_0^k(\overline{\Omega})$ is a closed linear subspace and again a Banach space with respect to the norm of $C^k(\overline{\Omega})$.

The *Jacobian* of $f \in C^1(\overline{\Omega})$ at a point $x \in \overline{\Omega}$ is defined as $J_f(x) = \det\big(f'(x)\big)$, where $f'(x)$ is the $n \times n$ matrix of first order partial derivatives, i.e. if $f = (f_1, \cdots, f_n)$, then

$$f'(x) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{pmatrix}.$$

A value $p = f(x)$ is called a *regular value* of f if $J_f(x) \neq 0$ for all $x \in f^{-1}(p) := \{y \in \overline{\Omega} \mid f(y) = p\}$, and p is called a *critical value*, or *singular value* if $J_f(x) = 0$ for some $x \in f^{-1}(p)$. The points $x \in f^{-1}(p)$ for which $J_f(x) \neq 0$ are called *regular points*, and those for which $J_f(x) = 0$ are called *critical points*, or *singular points*. The set of all critical points of f, i.e. all points $x \in \overline{\Omega}$ for which $J_f(x) = 0$, is denoted by $\operatorname{Crit}_f(\overline{\Omega})$, or Crit_f for short.

■ **1.3 Remark** The notions of regular and singular values can also be defined for functions $f : \mathbb{R}^n \to \mathbb{R}^m$, $n, m \geq 1$. In that case $f'(x)$ replaces the role of the Jacobian, i.e. p is regular if $f'(x)$ is of maximal rank for all $x \in f^{-1}(p)$ and singular if $f'(x)$ is not of maximal rank for some $x \in f^{-1}(p)$. A regular point is a point for which $f'(x)$ is of maximal rank and a singular point is a point for which $f'(x)$ is not of maximal rank. In the special case of functions $f : \mathbb{R}^n \to \mathbb{R}$, the critical points are those points for which $f'(x) = 0$. ■

1.2 The C^1-mapping degree

The definition of the C^1-mapping degree is carried out in two steps. The first step is to define the degree in the generic case — regular values —, and the second step entails the extension to singular values using the homotopy invariance of the degree. In Section 1.3 a direct definition of the C^1-mapping degree is given via an integral representation that does not require a distinction between regular and singular values. Because both approaches have specific advantages the two equivalent definitions are explained here.

1.2.a Regular values

Let $f : \overline{\Omega} \subset \mathbb{R}^n \to \mathbb{R}^n$ be a differentiable mapping, i.e. $f \in C^1(\overline{\Omega})$, and let $p \in \mathbb{R}^n$ with $p \notin f(\partial\Omega)$. Since $\overline{\Omega}$ is compact, the pre-image $f^{-1}(p) = \{x \in \overline{\Omega} \mid f(x) = p\}$ is a compact subset of Ω. Indeed, $f^{-1}(p) \subset \overline{\Omega}$ is a bounded set by definition. Let $x^k \in f^{-1}(p)$, with $x^k \to x$, then, by continuity, $f(x^k) \to f(x)$ as $k \to \infty$. Since $f(x^k) = p$ it follows that $f(x) = p$, which proves that $f^{-1}(p)$ is closed and therefore compact. The condition $p \notin f(\partial\Omega)$ implies that $f^{-1}(p) \subset \Omega$.

1.4 Lemma Let $p \notin f(\partial\Omega)$ be a regular value. Then, $f^{-1}(p) \subset \Omega$ consists of finitely many points.

Proof. Since $f^{-1}(p)$ is compact every infinite subset has at least one limit point in $f^{-1}(p)$.[2] Suppose $f^{-1}(p)$ is infinite and let $x^0 \in f^{-1}(p)$ be a limit point. The latter implies that for every $k > 0$ there exists a point $x^k \in f^{-1}(p)$ such that $0 < |x^k - x^0| < 1/k$.

Because f is differentiable the Taylor expansion of f around x^0 gives $f(x^0 + \zeta) = f(x^0) + f'(x^0)\zeta + R_{x^0}(\zeta)$, with $\zeta \in \mathbb{R}^n$. The remainder term $R_{x^0}(\zeta)$ can be estimated as follows: $f(x^0+\zeta) - f(x^0) = \int_0^1 f'(x^0 + t\zeta)\zeta dt$ and

$$R_{x^0}(\zeta) = f(x^0+\zeta) - f(z) - f'(x^0)\zeta = \int_0^1 \Big(f'(x^0+t\zeta) - f'(x^0)\Big)\zeta dt.$$

The derivative f' is uniformly continuous on $\overline{\Omega}$. Thus, for every $\epsilon > 0$ there exists a $\delta > 0$ such that $t|\zeta| < \delta$ implies that $|f'(x^0+t\zeta) - f'(x^0)| < \epsilon$. For the remainder $R_{x^0}(\zeta)$ we obtain

$$|R_{x^0}(\zeta)| \le \epsilon|\zeta|, \quad \text{as } |\zeta| < \delta, \tag{1.2.1}$$

i.e. $R_{x^0}(\zeta) = o(|\zeta|)$ as $|\zeta| \to 0$. Let $\zeta = x^k - x^0$, then $f(x^k) = f(x^0) + f'(x^0)(x^k - x^0) + o(|x^k - x^0|)$ as $|x^k - x^0| \to 0$. By assumption $f(x^k) = f(x^0) = p$, which implies that $f'(x^0)(x^k - x^0) = o(|x^k - x^0|)$ as $|x^k - x^0| \to 0$.

Since p is regular, $f'(x^0)$ is invertible and consequently there exists a positive constant $c > 0$ such that $|f'(x^0)\xi| \ge c|\xi|$, $c > 0$ for all $\xi \in \mathbb{R}^n$. Choose $\epsilon = c/2$. Then, for $k > \delta^{-1}$, Equation (1.2.1) implies that $|f'(x^0)(x^k - x^0)| \le \frac{1}{2}c|x^k - x^0|$ for all $|x^k - x^0| < 1/k$. Combining the latter with the Taylor expansion for f yields

$$0 < c|x^k - x^0| \le |f'(x^0)(x^k - x^0)| \le \tfrac{1}{2}c|x^k - x^0|,$$

[2] In a metric space the notions of compactness, sequential compactness and limit point compactness are equivalent. A set S is sequentially compact if every sequence has a convergent subsequence whose limit is in S. A set S is limit point point compact if every infinite subset has a limit point in S.

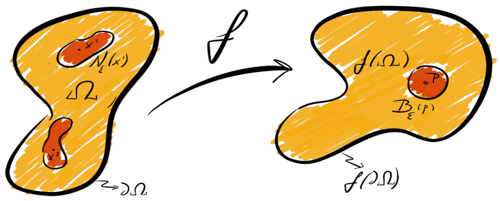


Figure 1.1: The pre-image of small neighborhoods $B_\epsilon(p)$ is the finite union of small neighborhoods $U_{x^j} \subset \Omega$ diffeomorphic to $B_\epsilon(p)$.

which is a contradiction and therefore $f^{-1}(p)$ is a finite set. ■

The fact that $f^{-1}(p)$ consists of finitely many non-degenerate points x allows the following definition of the mapping degree.

1.5 Definition For a regular value $p \notin f(\partial\Omega)$ the C^1-*mapping degree* is defined by

$$\deg(f,\Omega,p) := \sum_{x \in f^{-1}(p)} \operatorname{sign}\Big(J_f(x)\Big). \tag{1.2.2}$$

Note that the mapping degree takes values in $\mathbb{Z}$. The condition $p \notin f(\partial\Omega)$ is essential to have a meaningful topological invariant. If $p \in f(\partial\Omega)$, then the definition is sensitive to perturbations in p.

1.6 Exercise Explain via an example that when $p \in f(\partial\Omega)$, the above definition of degree is not stable under small perturbations in p. ■

Whether p is a regular value of a given function f or not may not be straightforward to decide. Sard's Theorem A.2 states that a value p is regular with 'probability' equal to 1. This fact can be used to extend the definition of degree to arbitrary values p, cf. Sect. 1.4.

The following lemmas show that the C^1-mapping degree in Definition 1.5 is stable with respect to small perturbations in the data f and p.

1.7 Lemma Let $p \notin f(\partial\Omega)$ be a regular value. Then, there exists an $\epsilon > 0$ such that all points $p' \in B_\epsilon(p)$ are regular and satisfy $p' \notin f(\partial\Omega)$, and $\deg(f,\Omega,p') = \deg(f,\Omega,p)$.

Proof. Since $f(\partial\Omega)$ is compact, $d(p,f(\partial\Omega)) > 0$ and we choose $0 < \epsilon < d(p,\partial\Omega)$. This implies that $p' \notin f(\partial\Omega)$ for all $p' \in B_\epsilon(p)$. By the Inverse Function Theorem A.1 there exist open neighborhoods W_x, with $\Omega \supset W_x \ni x$ and $V_x \ni p$ such that $f\colon W_x \to V_x$ is a local diffeomorphism. Choose $\epsilon > 0$ small enough such that $B_\epsilon(p) \subset \bigcap_{x\in f^{-1}(p)} V_x$.[3] This yields neighborhoods $U_x \subset W_x \subset \Omega$, with $x \in f^{-1}(p)$, which are diffeomorphic to $B_\epsilon(p)$ and for which all $p' \in B_\epsilon(p)$ are regular.

[3]We use the fact that $f^{-1}(p)$ is finite and therefore $\bigcap_{x\in f^{-1}(p)} V_x$ is an open neighborhood of p.

By construction $f^{-1}(p') \subset \bigcup_{x\in f^{-1}(p)} U_x$ and $x' \in U_x$ is the unique solution of $f(x') = p'$ in U_x. Since the Jacobian is a continuous function, $\mathrm{sign}\big(J_f(x')\big)$ is constant on U_x. This implies that $\sum_{x'\in f^{-1}(p)} \mathrm{sign}\big(J_f(x')\big) = \sum_{x\in f^{-1}(p)} \mathrm{sign}\big(J_f(x)\big)$, which proves the lemma. ∎

1.8 Lemma Let $p \notin f(\partial\Omega)$ be a regular value. Then, there exists an $\epsilon > 0$ such that for all functions $g \in C^1(\overline{\Omega})$ with $\|f - g\|_{C^1} < \epsilon$, p is a regular value for g, $p \notin g(\partial\Omega)$ and $\deg(g,\Omega,p) = \deg(f,\Omega,p)$.

Proof. Let $0 < \epsilon \le \frac{1}{2} d(p, f(\partial\Omega))$, then for all $g \in C^1(\overline{\Omega})$ with $\|f - g\|_{C^1} < \epsilon$ we have

$$
\begin{aligned}
|p - g(x)| &\ge \Big| |p - f(x)| - |f(x) - g(x)| \Big| \ge d(p, f(\partial\Omega)) - |f(x) - g(x)| \\
&\ge d(p, f(\partial\Omega)) - \epsilon \ge \tfrac{1}{2} d(p, f(\partial\Omega)) > 0,
\end{aligned}
$$

which implies that $p \notin g(\partial\Omega)$.

Consider the equation $g(y) = p$. For a solution $y \in g^{-1}(p)$ we have

$$
f(y) = p + [f(y) - g(y)] = p',
$$

with $p' \in B_\epsilon(p)$. By Lemma 1.7 we can choose $\epsilon > 0$ small enough such that $g^{-1}(p) \subset \bigcup_{x\in f^{-1}(p)} U_x$. Since p is regular we have that $|f'(x)\xi| \ge c|\xi|$, with $c > 0$, for all $x \in f^{-1}(p)$ and for all $\xi \in \mathbb{R}^n$. By the continuity of f' there exists a $\delta > 0$ such that $|f'(y)\xi| \ge \frac{1}{2} c|\xi|$ for all $y \in \bigcup_{x\in f^{-1}(p)} B_\delta(x)$. Choose $\epsilon > 0$ sufficiently small such that $U_x \subset B_\delta(x) \subset \Omega$ for all $x \in f^{-1}(p)$. Then, $|f'(y)\xi| \ge \frac{1}{2} c|\xi|$ for all $y \in \bigcup_{x\in f^{-1}(p)} U_x$. Again by choosing $\epsilon > 0$ smaller if necessary,

$$
|g'(y)\xi| = \ge \Big| |f'(y)\xi| - |[f'(y) - g'(y)]\xi| \Big| \ge \tfrac{1}{4} c|\xi|, \tag{1.2.3}
$$

for all $y \in \bigcup_{x\in f^{-1}(p)} U_x$ and for all $\xi \in \mathbb{R}^n$, which shows that p is a regular value for g. It remains to show that there is a 1-1 correspondence between the sets $f^{-1}(p)$ and $g^{-1}(p)$.

Rewrite the equation $g(y) = p$ as $g(y) - g(x) = p - g(x) = h$, $x \in f^{-1}(p)$ and define $R_x(\zeta) := g(x + \zeta) - g(x) - g'(x)\zeta$, where $\zeta = y - x$. The equation for ζ becomes $g'(x)\zeta + R_x(\zeta) = h$, which translates to the fixed point equation

$$
T(\zeta) := \big[g'(x)\big]^{-1}\big(h - R_x(\zeta)\big) = \zeta, \quad \text{with } |\zeta| < \delta. \tag{1.2.4}
$$

To objective is to show that Equation (1.2.4) has a unique solution in $B_\delta(0)$. Observe that $R_x(\zeta) - R_x(\zeta') = g(x + \zeta) - g(x + \zeta') - g'(x)(\zeta - \zeta')$ and

$$
g(x + \zeta) - g(x + \zeta') = \int_0^1 \Big(g'(x + t\zeta + (1 - t)\zeta') \Big)(\zeta - \zeta') dt. \tag{1.2.5}
$$

For $R_x(\zeta) - R_x(\zeta')$ this yields

$$R_x(\zeta) - R_x(\zeta') = \int_0^1 \Big(g'(x + t\zeta + (1-t)\zeta') - g'(x)\Big)(\zeta - \zeta')dt.$$

and the triangle inequality gives

$$\begin{aligned}|g'(x + t\zeta + (1-t)\zeta') - g'(x)| &\le |g'(x + t\zeta + (1-t)\zeta') - f'(x + t\zeta + (1-t)\zeta')| \\ &\quad + |f'(x + t\zeta + (1-t)\zeta') - f'(x)| + |f'(x) - g'(x)| \\ &\le |f'(x + t\zeta + (1-t)\zeta') - f'(x)| + 2\epsilon.\end{aligned}$$

Combining the estimates we obtain

$$\begin{aligned}|T(\zeta) - T(\zeta')| &\le c_0\Big(|f'(x + t\zeta + (1-t)\zeta') - f'(x)| + 2\epsilon\Big)|\zeta - \zeta'|; \\ |T(\zeta) - \widetilde{h}| &\le c_0\Big(|f'(x + t\zeta) - f'(x)| + 2\epsilon\Big)|\zeta|, \quad \widetilde{h} = \big[g'(x)\big]^{-1}h,\end{aligned}$$

where we used the uniform bound $\|\big[g'(x)\big]^{-1}\| \le c_0$, for all $x \in f^{-1}(p)$, implied by Equation (1.2.3). From the second inequality we can derive the following norm inequality

$$|T(\zeta)| \le c_0\Big(|f'(x + t\zeta) - f'(x)| + 2\epsilon\Big)|\zeta| + c_0|h|.$$

By the definition of $h = p - g(x) = f(x) - g(x)$ and therefore $|h| < \epsilon$. By the uniform continuity of f' there exists a $\delta' > 0$ such that $|f'(x + t\zeta + (1-t)\zeta') - f'(x)| < \frac{1}{4c_0}$ for all $t|\zeta| + (1-t)|\zeta'| < \delta'$. Let $\delta'' = \min\{\delta, \delta'\}$ and choose $\epsilon < \min\{\frac{1}{4c_0}, \frac{\delta''}{4c_0}\}$ small enough such that $U_x \subset B_{\delta''}(x) \subset \Omega$ for all $x \in f^{-1}(p)$. This implies that

$$|T(\zeta) - T(\zeta')| < \tfrac{3}{4}|\zeta - \zeta'|,$$

and

$$|T(\zeta)| < \big(\tfrac{1}{4} + \min\{\tfrac{1}{2}, \tfrac{\delta''}{2}\}\big)\delta'' + \min\{\tfrac{1}{4}, \tfrac{\delta''}{4}\} \le \delta'', \quad \text{for } |\zeta| < \delta'',$$

which makes T a *contraction mapping* on $B_{\delta''}(0)$. By the Contraction Mapping Theorem A.3 $g(y) = p$ has a unique solution in $U_x \subset B_{\delta''}(x)$ for every $x \in f^{-1}(p)$. By the uniform bound on the $g'(x)$ in (1.2.3) we derive that $\operatorname{sign}\big(J_g(y)\big)$ is constant on the sets U_x and therefore $\sum_{y \in g^{-1}(p)} \operatorname{sign}\big(J_g(y)\big) = \sum_{x \in f^{-1}(p)} \operatorname{sign}\big(J_f(x)\big)$, which proves the lemma. ∎

Definition 1.5 of degree was used in the prelude to this chapter and gives a convenient way of computing the mapping degree for regular values p. The condition $p \notin f(\partial\Omega)$ is an isolation condition and makes $\overline{\Omega}$ a set that strictly contains solutions of $f(x) = p$ on Ω, i.e. $\overline{\Omega}$ isolates the solution set $f^{-1}(p)$. This isolation requirement in the definition of degree equips the mapping degree with various robustness properties, see Lemmas 1.7 and 1.8.

From Definition 1.5 a number of crucial properties can be derived. For the identity map $f = \mathrm{id}$ the degree is easily computed, i.e. if $p \in \Omega$, then

$$\deg(\mathrm{id}, \Omega, p) = 1, \tag{1.2.6}$$

and for $p \notin \overline{\Omega}$, $\deg(\mathrm{id}, \Omega, p) = 0$. Another important property that follows from the definition is that the equations $f(x) = p$ and $f(x) - p = 0$ have the same solution set and $J_f = J_{f-p}$. Therefore,

$$\deg(f, \Omega, p) = \deg(f - p, \Omega, 0). \tag{1.2.7}$$

If $\Omega^1, \Omega^2 \subset \Omega$ are two disjoint, open subsets, such that $p \notin f\big(\overline{\Omega} \backslash (\Omega^1 \cup \Omega^2)\big)$, then

$$\deg(f, \Omega, p) = \deg(f, \Omega^1, p) + \deg(f, \Omega^2, p). \tag{1.2.8}$$

Indeed, since $p \notin f\big(\overline{\Omega} \backslash (\Omega^1 \cup \Omega^2)\big)$, then $f^{-1}(p) \subset \Omega^1 \cup \Omega^2$. From Definition 1.5 and the fact that $\Omega^1 \cap \Omega^2 = \varnothing$, Equation (1.2.8) follows, cf. [25]. The three properties in (1.2.6) - (1.2.8) occur in the axioms of Degree Theory, cf. Theorem 1.20. The homotopy axiom, which is still missing, is less obvious and will be discussed in the next subsection.

1.9 Example Consider the mapping $f : \mathbb{D}^2 \subset \mathbb{R}^2 \to \mathbb{R}^2$ defined by $f(x_1, x_2) = (2x_1^2 - 1, 2x_1 x_2)$. This mapping gives a 2-fold covering of the disc $\mathbb{D}^2 := \{x \in \mathbb{R}^2 \mid |x| < 1\}$. The boundary $\partial\mathbb{D}^2 = S^1$ winds around the origin twice under the image of the map f. For the value $(0,0)$, the pre-image consists of the points $x^1 = (-\frac{1}{2}\sqrt{2}, 0)$ and $x^2 = (\frac{1}{2}\sqrt{2}, 0)$, and

$$f'(x^1) = \begin{pmatrix} -2\sqrt{2} & 0 \\ 0 & -\sqrt{2} \end{pmatrix}, \quad f'(x^2) = \begin{pmatrix} 2\sqrt{2} & 0 \\ 0 & \sqrt{2} \end{pmatrix}.$$

Therefore $(0,0)$ is a regular value for f, and since $J_f(x^1) = J_f(x^2) = +1$, the degree is given by $\deg(f, \mathbb{D}^2, 0) = 2$. More details about the relation between the mapping degree and winding numbers are discussed in Sect. 2.2.b.

For regular values p the degree is a count of the elements in $f^{-1}(p)$ with orientation, i.e. a point $x \in f^{-1}(p)$ is counted with either $+1$ or -1 when f is locally orientation preserving or orientation reversing respectively. The degree counts how many times the image $f(\overline{\Omega})$ covers p counted with multiplicity. This is a purely local but stable property for regular values.

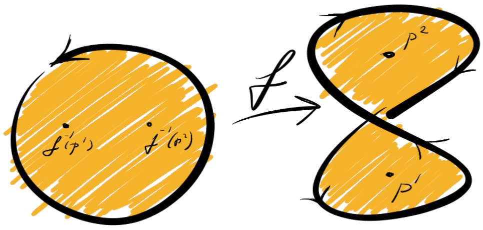


Figure 1.2: Two different orientations with respect to the points p^1 and p^2.

1.10 Example Consider the mapping $f(x_1,x_2) = (2x_1x_2, x_1)$ on $\Omega = \mathbb{D}^2$, and the image points $p^1 = (0,-1/2)$, and $p^2 = (0,1/2)$. Then, as in Example 1.9, $\deg(f,\mathbb{D}^2,p^1) = -\deg(f,\mathbb{D}^2,p^2) = 1$. The positive degree corresponds to a counter clockwise rotation around p^1, and the negative degree corresponds to a clockwise rotation around p^2, see Figure 1.2 and Sect. 2.2.b.

■ **1.11 Remark** A coarser version of degree is the so-called mod-2 degree and is defined as follows; $\deg_2(f,\Omega,p) = \#\big(f^{-1}(p)\big) \mod 2$. This version of the mapping degree contains less information than the degree defined in Definition 1.5, but is important if one considers mappings between non-orientable spaces, cf. [23] ■

1.2.b Homotopy invariance

A crucial property of the C^1-mapping degree is the homotopy invariance with respect to f. Lemma 1.8 shows that the degree remains unchanged under small perturbations in f. Under a large perturbation given by a (continuous) path $t \mapsto f_t$, the value p may not be regular along the path f_t for all t. In order to conclude invariance of the degree under such perturbations in f we need to investigate the behavior of the degree when p is not necessarily regular along the entire path $t \mapsto f_t$.

A continuous path of functions $t \mapsto f_t$, with $t \in [0,1]$, in the homotopy principle below is a continuous mapping $[0,1] \to C^1(\overline{\Omega})$.

1.12 Proposition Let $t \mapsto f_t$, $t \in [0,1]$ be a continuous path in $C^1(\overline{\Omega})$, with $p \notin f_t(\partial\Omega)$ for all $t \in [0,1]$. Suppose p is a regular value for both f_0 and f_1, then $\deg(f_0,\Omega,p) = \deg(f_1,\Omega,p)$.

Before proving this proposition we establish a special version of the homotopy principle.

1.13 Lemma Let $f \in C^1(\overline{\Omega})$ and let p be a regular value such that the line segment $\{tp\}_{\lambda\in[0,1]}$ satisfies $tp \notin f(\partial\Omega)$ for all $\lambda \in [0,1]$. If $f^{-1}(0) = \varnothing$, then $\deg(f,\Omega,p) = \deg(f,\Omega,0) = 0$.

Proof. By (1.2.7) we have that $\deg(f,\Omega,p) = \deg(f-p,\Omega,0)$. Consider the equation $f(x)-(1-s^2)p = 0$ and define $F(s,x) = f(x)-(1-s^2)p$. By Theorem A.4 we may assume that f is smooth, i.e. there exists a smooth perturbation that is C^1-close to f. Lemma 1.8 implies that we can choose the perturbation sufficiently C^1-close to f such that p is still a regular value for the perturbation and the mapping degree is the same. We denote the perturbation again by f. Consequently, $F\colon \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}^n$ is a smooth function. Sard's Theorem A.7 yields the existence of a sequence of regular values $\bar{\epsilon}^k$ for F with $\bar{\epsilon}^k \to 0$ in $\mathbb{R}^n$.

From the hypothesis that $tp \notin f(\partial\Omega)$, for $t \in [0,1]$, and the fact that $f^{-1}(0) = \varnothing$, it follows that $0 \notin F\big(\partial([-1,1]\times\overline{\Omega})\big)$ and $F^{-1}(0)$ is a compact subset of $(-1,1)\times\Omega$. As before, the isolating property implies that $F^{-1}(\bar{\epsilon}^k)$ is a also a compact subset of $(-1,1)\times\Omega$ for $k \geq N$, for some $N > 0$. Since $\bar{\epsilon}^k$ is a regular value for F for every k, the solution set of $F^{-1}(\bar{\epsilon}^k)$ is a smooth, compact manifold (without boundary) of dimension 1 embedded in $\mathbb{R}^{n+1}$. The finitely many connected components of $F^{-1}(\bar{\epsilon}^k)$ are embedded circles $\gamma \subset (-1,1)\times\Omega$. Since p is a regular value for f

The continuity of the mapping degree in Lemma 1.7 with respect to p implies that if $|\bar{\epsilon}^k|$ is sufficiently small, i.e. $k \geq N$, $N > 0$ sufficiently large, then both p and $p+\bar{\epsilon}$ are regular values for f and the mapping degree remains unchanged: $\deg(f,\Omega,p) = \deg(f,\Omega,p+\bar{\epsilon})$.

The transverse intersection of $F^{-1}(\bar{\epsilon})$ with the hyperplane $\{s=0\}$ is the set $f^{-1}(p+\bar{\epsilon})$. Due to the $t \mapsto -t$ symmetry every circle $\gamma \in F^{-1}(\bar{\epsilon})$ that intersects $\{s=0\}$ will necessarily intersect $\{s=0\}$ transversely in exactly two points and thus $f^{-1}(p+\bar{\epsilon})$ consists of an even number of points.

The circles γ in $F^{-1}(\bar{\epsilon})$ are orientable. Let $(0,x^-)$ and $(0,x^+)$ be the intersection points of γ with $\{t=0\}$ and let $\mathbf{t}$ be an oriented tangent vector field along γ with $\mathbf{t}(0,x^\pm) = \pm\mathbf{e}$, where $\mathbf{e} = (1,0,\cdots,0)$ is the unit normal to $\{t=0\}$. We choose an orientation for $\mathbb{R}^{n+1}$ and consider the oriented bases $[\{-\mathbf{e},\boldsymbol{\xi}^-\}] = [\{\mathbf{e},\boldsymbol{\xi}^+\}]$ at $(0,x^-)$ and $(0,x^+)$ respectively. Since $F|_\gamma = \bar{\epsilon}^k$, the derivative F' maps $\{\pm\mathbf{e},\boldsymbol{\xi}^\pm\}$ to an orientated basis of $\mathbb{R}^n$:

$$F'(0,x^\pm)(\pm\mathbf{e}) = 0, \quad [F'(0,x^-)\boldsymbol{\xi}^-] = [F'(0,x^+)\boldsymbol{\xi}^+].$$

The bases $\boldsymbol{\xi}^-$ and $\boldsymbol{\xi}^+$ have opposite orientations and thus the relation $F'(0,x^\pm) = (0 \;\; f'(x^\pm))$ implies that $\operatorname{sign}\Big(J_f(x^-)\Big) = -\operatorname{sign}\Big(J_f(x^+)\Big)$. Conseqeuntly,

$$\deg(f,\Omega,p+\bar{\epsilon}) = \sum_{x\in f^{-1}(p+\bar{\epsilon})} \operatorname{sign}\Big(J_f(x)\Big) = 0,$$

which proves the lemma. ∎

A path $t \mapsto f_t$ is said to be differentiable if it the mapping $F(t,x) := f_t(x)$, $(t,x) \in [0,1]\times\overline{\Omega}$ is a C^1-mapping.

1.14 Lemma Let $t \mapsto f_t$, $t \in [0,1]$ be a differentiable path in $C^1(\overline{\Omega})$, with $p \notin f_t(\partial\Omega)$ for all $t \in [0,1]$. Suppose p is a regular value for both f_0 and f_1, then $\deg(f_0,\Omega,p) = \deg(f_1,\Omega,p)$.

Proof. Consider the smooth cut-off function $\omega(s) \in [0,1]$ such that $\omega(s) = 0$ for $s \le -1/2$, $\omega(s) = 1$ for $s \ge 1/2$ and $\omega'(s) > 0$ for $s \in (-1/2,1/2)$. Define

$$G(s,x) = \begin{pmatrix} f_{\omega(s)}(x) - p \\ 4s^2 + 1 \end{pmatrix}$$

which is a C^1-mapping on $[-1,1] \times \overline{\Omega}$. The point $P = (0,2)$ is a regular value of G. Note that the solution set of $G(s,x) = P$ are the points

$$(s,x) \in G^{-1}(P) = \Big(\{-1/2\} \times f_0^{-1}(p)\Big) \cup \Big(\{1/2\} \times f_1^{-1}(p)\Big).$$

The Jacobian for $(s,x) \in G^{-1}(P)$ is given by

$$\operatorname{sign}\Big(J_G(s,x)\Big) = \operatorname{sign}\Big(J_{f_{\omega(s)}}(x)\Big) \cdot \operatorname{sign}(8s), \quad s \in \{-1/2, 1/2\},$$

it follows that

$$\begin{aligned} \deg\big(G,(-1,1) \times \Omega, P\big) &= \deg(f_1 - p,\Omega,0) - \deg(f_0 - p,\Omega,0), \\ &= \deg(f_1,\Omega,p) - \deg(f_0,\Omega,p). \end{aligned}$$

We are now in a position to apply Lemma 1.13. Observe now that $\lambda P = (0,2t)$, $G^{-1}(tP) \in (-1,1) \times \Omega$ for all $t \in [0,1]$ and $G^{-1}((0,0)) = \varnothing$. By Lemma 1.13, $\deg\big(G,(-1,1) \times \Omega, P\big) = 0$, which proves that $\deg(f_1,\Omega,p) = \deg(f_0,\Omega,p)$. ∎

Proof of Proposition 1.12: Reparametrize the path $t \mapsto f_t$ as follows: $t \mapsto h_t$, where $h_t = f_0$ for $0 \le t \le 1/3$, $h_t = f_{3t-1}$ for $1/3 \le t \le 2/3$ and $h_t = f_1$ for $2/3 \le t \le 1$. Define the continuous mapping $H\colon [0,1] \times \overline{\Omega} \to \mathbb{R}^{n+1}$ as $H(t,x) := h_t$. By Theorem A.4 and Remark A.6 there exists a smooth perturbation $\widetilde{H}$ which can be chosen arbitrary close to H in C^0 and such that $\widetilde{f}_0 := \widetilde{H}(0,\cdot)$ and $\widetilde{f}_1 := \widetilde{H}(1,\cdot)$ are arbitrary C^1-close to f_0 and f_1 respectively. From Lemma 1.14 we then derive that $\deg(\widetilde{f}_0,\Omega,p) = \deg(\widetilde{f}_1,\Omega,p)$. From Lemma 1.8 we conclude that

$$\deg(f_0,\Omega,p) = \deg(\widetilde{f}_0,\Omega,p) = \deg(\widetilde{f}_1,\Omega,p) = \deg(f_1,\Omega,p),$$

which completes the proof. ∎

▪ **1.15 Remark** An alternative proof of the homotopy principle can be achieved using the integral characterization of the degree, cf. Lemma 1.38 and Proposition 1.39. ▪

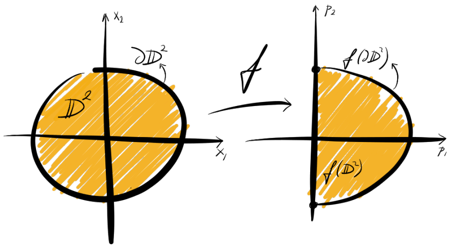


Figure 1.3: The mapping $f(x,y) = (x^2,y)$ maps the disc to a semi-disc: *'folded pancake'*. The semi-circle in the right half plane represents $f(\partial\mathbb{D}^2)$, which is a strict subset of the boundary of the image $\partial f(\mathbb{D}^2)$.

Let $D \subset \mathbb{R}^n \backslash f(\partial\Omega)$ be a connected component,[4] then the degree $\deg(f,\Omega,p)$ is independent of regular values $p \in D$, which is a direct consequence of the homotopy principle.

1.16 Proposition For every path $t \mapsto p_t \in D$, $t \in [0,1]$, with p_0 and p_1 regular values, it holds that $\deg(f,\Omega,p_0) = \deg(f,\Omega,p_1)$.

Proof. From Equation (1.2.7) it follows that $\deg(f,\Omega,p_0) = \deg(f - p_0,\Omega,0)$, and $\deg(f,\Omega,p_1) = \deg(f - p_1,\Omega,0)$. It holds that $p_t \in D$ if and only if $p_t \notin f(\partial\Omega)$. The homotopy $f_t = f - p_t$ therefore satisfies the requirements of Proposition 1.12, and

$$\deg(f,\Omega,p_0) = \deg(f - p_0,\Omega,0) = \deg(f - p_1,\Omega,0) = \deg(f,\Omega,p_1),$$

which proves the statement. ∎

1.17 Example Consider the mapping $f(x,y) = (x^2,y)$ on the the standard 2-dics $\overline{\mathbb{D}}^2$ in the plane. The image of $\overline{\mathbb{D}}^2$ under f is the *'folded pancake'* $f(\overline{\mathbb{D}}^2) = \{p = (p_1,p_2) \in \mathbb{R}^2 \mid p_1 + p_2^2 = 1,\ p_1 \geq 0\}$. The image of the boundary $S^1 = \partial\mathbb{D}^2$ is homeomorphic to a semi-circle and $\mathbb{R}^2\backslash f(\mathbb{D}^2)$ is connected. Note that $f(\partial\mathbb{D}^2) \neq \partial f(\overline{\mathbb{D}}^2)$, see Fig. 1.3. By the homotopy invariance the degree can be evaluated by choosing a regular value $p \in \mathbb{R}^2\backslash f(\partial\mathbb{D}^2)$. Since $\overline{\mathbb{D}}^2$ is compact, then so is the image $f(\mathbb{D}^2)$. We can therefore choose a regular value $p^1 \in \mathbb{R}^2\backslash f(\partial\mathbb{D}^2)$ which does not lie in $f(\overline{\mathbb{D}}^2)$. This implies that $\deg(f,\mathbb{D}^2,p) = 0$. If we choose $p^2 = (1/4,0)$, then $f^{-1}(p^2) = \{(\pm 1/2,0)\}$, which gives a positive and a negative determinant. The sum is zero which confirms the previous calculation.

If we choose a path $t \mapsto p_t$ connecting the regular values p^1 and p^2 and which lies in $\mathbb{R}^2\backslash f(\partial\mathbb{D}^2)$, then p_t crosses the boundary $\partial f(\overline{\mathbb{D}}^2)$ in the vertical. However, $p_t \notin f(\partial\mathbb{D}^2)$ for all $t \in [0,1]$ and therefore $f^{-1}(p_t) \in \mathbb{D}^2$ for all $t \in [0,1]$. The values in $f(\overline{\mathbb{D}}^2)$ on the vertical are necessarily singular. This also shows that the boundary of the image should not be considered as a restriction on p. In the next subsection we show that the degree is defined for all p in $\mathbb{R}^2\backslash f(\partial\mathbb{D}^2)$.

[4]Open subsets of $\mathbb{R}^n$ are connected if and only if they are path-connected.

1.18 Proposition Suppose that $\mathbb{R}^n \backslash f(\partial\Omega)$ is connected, then for every regular value $p \in \mathbb{R}^n \backslash f(\partial\Omega)$ it holds that $\deg(f, \Omega, p) = 0$.

Proof. The image $f(\overline{\Omega})$ is compact and therefore $f(\overline{\Omega}) \subset B_r(0)$ for some $r > 0$. Points $p \in \mathbb{R}^n \setminus B_r(0)$ are regular by default since $f^{-1}(p) = \varnothing$. Since $f(\partial\Omega) \subset B_r(0)$ we have that $\mathbb{R}^n \setminus f(\partial\Omega) \supset \mathbb{R}^n \setminus f(\overline{\Omega})$ and thus the set of regular values in $\mathbb{R}^n \setminus f(\partial\Omega)$ is non-empty. Since $\mathbb{R}^n \setminus f(\partial\Omega)$ is connected, Proposition 1.16 implies that the degree constant on $\mathbb{R}^n \setminus f(\partial\Omega)$. For $p \in \mathbb{R}^n \setminus f(\overline{\Omega})$ we have $\deg(f, \Omega, p) = 0$, and thus $\deg(f, \Omega, p) = 0$ for all $p \in \mathbb{R}^n \setminus f(\partial\Omega)$, which completes the proof. ∎

1.2.c The degree for arbitrary values

The homotopy invariance established in the previous subsection can be used to extend the definition of the C^1-mapping degree to arbitrary values $p \in \mathbb{R}^n \backslash f(\partial\Omega)$.

Sard's Theorem A.7 states that the *set of singular values* $S_f = f(\mathrm{Crit}_f)$ has Lebesgue measure zero and therefore has empty interior. Since $S_f \subset \mathbb{R}^n$ is closed the complement S_f^c is open and $\overline{S_f^c} = \mathrm{int}(S_f)^c = \varnothing^c = \mathbb{R}^n$, which shows that S_f^c, the set of regular values, is *open and dense*.

Let $p \in D \subset \mathbb{R}^n \setminus f(\partial\Omega)$, where D is a connected component. Then, since D is open, there exists a small neighborhood $B_\epsilon(p) \subset D$. Because S_f^c is dense there exists a sequence $\{p^k\} \subset S_f^c$ with $p^k \to p$ and $p^k \in B_\epsilon(p)$ for all $k \geq N$ for some $N > 0$. By Proposition 1.16, $\deg(f, \Omega, p^k)$ is constant for all $k \geq N$ and thus the limit is well-defined and independent of the chosen sequence. Indeed, if $\{p'^k\}$ is a different sequence of regular values converging to p, then there exists an N' such that $p'^k \in B_\epsilon(p)$ for all $k \geq N'$. By Proposition 1.16 $\deg(f, \Omega, p^k) = \deg(f, \Omega, p'^k)$ for all $k \geq \max\{N, N'\}$ which shows the independence of the chosen sequence and thus justifies the following extension of the C^1-mapping degree.

1.19 Definition Let D a connected component of $\mathbb{R}^n \backslash f(\partial\Omega)$ and let $p \in D$. Then define the C^1-mapping degree by

$$\deg(f, \Omega, p) := \lim_{k \to \infty} \deg(f, \Omega, p^k),$$

where $p^k \to p$ and $p^k \in \mathbb{R}^n \setminus f(\partial\Omega)$ are regular values.

For triples (f, Ω, p), with $f \in C^1(\overline{\Omega})$, $\Omega \subset \mathbb{R}^n$ a bounded domain and $p \in \mathbb{R}^n \setminus f(\partial\Omega)$, the C^1-mapping degree is well-defined. Such triples are called *admissible triples*. If $p, p' \in D$, then there exist sequences of regular values $p^k \to p$ and $p'^k \to p'$, and for $k \geq N$ for some $N > 0$, $p^k, p'^k \in D$. By Proposition 1.16, $\deg(f, \Omega, p^k) = \deg(f, \Omega, p'^k)$, which proves, via Definition 1.19, that $\deg(f, \Omega, p) = \deg(f, \Omega, p')$ for all $p, p' \in D$ and which justifies the notation $\deg(f, \Omega, p) = \deg(f, \Omega, D)$.

The properties of the *generic*[5] C^1-mapping degree listed in Equations (1.2.6) - (1.2.8) and Proposition 1.12 also hold for the general C^1-mapping degree and are the fundamental axioms that define a degree theory, see Section 2.1.

1.20 Theorem — Degree Theory, cf. [21]. The degree function $\deg(f,\Omega,p)$ in Definition 1.19 satisfies the following axioms:
(A1) if $p \in \Omega$, then $\deg(\mathrm{id},\Omega,p) = 1$;
(A2) for $\Omega^1,\Omega^2 \subset \Omega$, disjoint open subsets of Ω and $p \notin f\big(\overline{\Omega}\backslash(\Omega_1 \cup \Omega_2)\big)$, it holds that $\deg(f,\Omega,p) = \deg(f,\Omega^1,p) + \deg(f,\Omega^2,p)$;
(A3) for every continuous path $t \mapsto f_t$, $f_t \in C^1(\overline{\Omega})$, with $p \notin f_t(\partial\Omega)$, it holds that $\deg(f_t,\Omega,p)$ is independent of $t \in [0,1]$;
(A4) $\deg(f,\Omega,p) = \deg(f-p,\Omega,0)$.
The application $(f,\Omega,p) \mapsto \deg(f,\Omega,p)$ is called a C^1-degree theory.

Proof. Axiom (A1) follows from Equation (1.2.6). As for Axiom (A2) we argue as follows. By assumption, $f^{-1}(p) \subset \Omega^1 \cup \Omega^2$ and therefore $f^{-1}(p') \subset \Omega^1 \cup \Omega^2$ for every regular value p' sufficiently close to p. Let $p^k \to p$, then, by Definition 1.19, for $k \geq N$ for some $N > 0$,

$$\begin{aligned}\deg(f,\Omega,p) &= \deg(f,\Omega,p') = \deg(f,\Omega^1,p^k) + \deg(f,\Omega^2,p^k)\\ &= \deg(f,\Omega^1,p) + \deg(f,\Omega^2,p),\end{aligned}$$

which verifies Axiom (A2).

Let $p^k \to p$ be a sequence of regular values for both f_0 and f_1. Such sequences exist since every p^k is regular for both f_0 and f_1 when p^k is close enough to p. By assumption $d(p, f_t(\partial\Omega) \geq \delta > 0$ and therefore we can choose $k \geq N$, for some $N > 0$ such that $p^k \in B_{\delta/2}(p)$, which gives

$$|p^k - f_t(x)| \geq |p - f_t(x)| - |p^k - p| > \delta/2 > 0,$$

for all $x \in \partial\Omega$. Consequently, for all $k \geq N$, $p^k \notin f_t(\partial\Omega)$. Proposition 1.12 and Definition 1.19 this implies

$$\deg(f_0,\Omega,p) = \deg(f_0,\Omega,p^k) = \deg(f_1,\Omega,p^k) = \deg(f_1,\Omega,p), \quad \forall\, k \geq N.$$

By considering the homotopy $t \mapsto f_{t_0t}$ it follows that $\deg(f_0,\Omega,p) = \deg(f_{t_0},\Omega,p)$, for every $t_0 \in [0,1]$, which proves Axiom (A3).

Finally, let $p^k \to p$ be a sequence of regular values. Then, by Equation (1.2.8), $\deg(f,\Omega,p) = \deg(f,\Omega,p^k) = \deg(f-p^k,\Omega,0)$ for all $k \geq N$ for some $N > 0$. Consider the homotopy $f_t = (1-t)(f-p) + t(f-p^k) = f-(1-t)p - tp^k$. Since p'

[5] The word 'generic' is used to indicate that the choice of regular values is from an open en dense subset of $\mathbb{R}^n$.

is close to p, the line-segment $\{(1-t)p + tp^k\}_{t\in[0,1]}$ does not intersect $f(\partial\Omega)$, and therefore $0 \notin f_t(\partial\Omega)$. From Axiom (A3) it then follows that

$$\deg(f,\Omega,p) = \deg(f,\Omega,p^k) = \deg(f - p^k,\Omega,0) = \deg(f-p,\Omega,0),$$

which proves Axiom (A4). ∎

1.3 Integral representations

The expression for the C^1-mapping degree for regular values points to an obvious integral definition of the degree which allows for a formulation of of the C^1-degree without distinguishing between regular and singular values. The integral formulation is is also useful for establishing various properties analytically such as the homotopy invariance.

1.3.a Regular integrals

Let $\omega : \mathbb{R}^n \to \mathbb{R}$ be a continuous function with $\mathrm{supp}(\omega) \subset B_\epsilon(p)$ and let $p \notin f(\partial\Omega)$ be a regular value for f. Choose $\epsilon > 0$ small enough such that $B_\epsilon(p) \subset \mathbb{R}^n \backslash f(\partial\Omega)$ and is a coordinate neighborhood of p with respect to the change of coordinates $y = f(x)$, near $y = p$, see Figure 1.1. The weight function ω can be normalized via

$$\int_{\mathbb{R}^n} \omega(x)dx = 1.$$

A function ω that satisfies the above conditions is called a *weight function*, or *test function*. In calculations it is convenient to use the notation of differential forms on $\mathbb{R}^n$. Let $dx = dx_1 \wedge \cdots \wedge dx_n$ denote the standard n-form or the Lebesgue measure on $\mathbb{R}^n$ depending on the context. In Sect. 1.3.b we give a short introduction to differential forms and differential forms notation and operations such as the wedge product. Consider the differential n-forms

$$\boldsymbol{\omega} = \omega(y)dy, \quad \text{and} \quad f^*\boldsymbol{\omega} = \omega(f(x))J_f(x)dx.$$

The latter is called the pullback under f, where $y = f(x)$. The n-form dx provides $\mathbb{R}^n$ with a standard orientation. With this notation most of the calculations simplify considerably. The space of compactly supported continuous n-forms on $\mathbb{R}^n$ is denoted by $\Gamma_c^{n,0}(\mathbb{R}^n)$, cf. [19].

1.21 Proposition Let $p \notin f(\partial\Omega)$ be a regular value and ω a weight function as defined above. Then C^1-mapping degree is retrieved by the integral

$$\int_\Omega f^*\boldsymbol{\omega} = \deg(f,\Omega,p). \tag{1.3.9}$$

Proof. By Lemma 1.4 $f^{-1}(p)$ is a finite set contained in Ω. Since J_f is non-zero at points $x \in f^{-1}(p)$, the Inverse Function Theorem A.1 yields that f maps neighborhoods U_x of points in $x \in f^{-1}(p)$ diffeomorphically onto $B_\epsilon(p)$, see Figure 1.1. Thus f is a local change of coordinates on a neighborhood of every point $x \in f^{-1}(p)$. The integral $\int_\Omega f^*\boldsymbol{\omega}$ splits in k local integrals

$$\begin{aligned} \int_\Omega f^*\boldsymbol{\omega} &= \sum_j \int_{U_{x^j}} f^*\boldsymbol{\omega} = \sum_j \operatorname{sign}\Big(J_f(x^j)\Big) \int_{B_\epsilon(p)} \boldsymbol{\omega} \\ &= \sum_j \operatorname{sign}\Big(J_f(x^j)\Big) = \deg(f,\Omega,p), \end{aligned}$$

which proves that both $\int_\Omega f^*\boldsymbol{\omega}$ is independent of ω and represents the C^1-mapping degree defined in Definition 1.5, where we used that locally f is a coordinate transformation $y = f(x)$ and $\operatorname{sign}\Big(J_f(x^i)\Big) \int \omega(f(x))J_f(x)dx = \int \omega(y)dy$. ∎

> **1.22 Exercise** Prove the change of coordinates formula for the integral: $\operatorname{sign}\Big(J_f(x^i)\Big) \int \omega(f(x))J_f(x)dx = \int \omega(y)dy$. ▪

▪ **1.23 Remark** If in Proposition 4.56 we choose weight functions ω with the property that $\int_\Omega \boldsymbol{\omega} \neq 0$, then

$$\deg(f,\Omega,p) \cdot \int_{\mathbb{R}^n} \boldsymbol{\omega} = \int_\Omega f^*\boldsymbol{\omega}.$$

See also Remark 1.35. ▪

1.3.b The Poincaré Lemma

Before extending the integral representation of the mapping degree we first need to establish some basic fact about differential forms on $\mathbb{R}^n$. Define the linear functions $dx_i \colon \mathbb{R}^n \to \mathbb{R}$, $i = 1, \cdots, n$, by $dx_i(\xi) = \xi_i$, with $\xi = (\xi_1, \cdots, \xi_n) \in \mathbb{R}^n$. In order to define arbitrary anti-symmetric multi-linear functions on $\mathbb{R}^n$ we introduce the wedge product of linear functions dx_i. A *increasing multi-index* $I = \{i^1, \cdots, i^p\}$ of length $|I| = p$ is characterized by the restriction $1 \le i^1 < \cdots < i^p \le n$, $1 \le p \le n$. Define the multi-linear function $dx_I := dx_{i^1} \wedge \cdots \wedge dx_{i^p}$ by:

$$dx_I(\xi) := \sum_{\sigma \in S_p} (-1)^{|\sigma|} \xi^{\sigma(I)}, \tag{1.3.10}$$

where S_p is the symmetric group of permutation on p elements, $|\sigma|$ is the order of the permutation and $\xi^I = \xi_{i^1} \cdots \xi_{i^p}$. The set of increasing multi-indices of length p is denoted by $\mathscr{M}^p$. By construction dx_I is a multi-linear function on $\mathbb{R}^n$.

1.24 Example For $p = 2$ we have the expansion $(dx_i \wedge dx_j)(\xi, \eta) = \xi_i \eta_j - \xi_j \eta_i$ and $dx_i \wedge dx_j = 0$ when $i = j$, and for $p = 3$, $(dx_i \wedge dx_j \wedge dx_k)(\xi, \eta, \zeta) = \xi_i \eta_j \zeta_k - \xi_i \eta_k \zeta_j - \xi_j \eta_i \zeta_k - \xi_k \eta_j \zeta_i + \xi_j \eta_k \zeta_i + \xi_k \eta_i \zeta_j$.

1.25 Exercise Use Equation (1.3.10) and Example 1.24 to show that n-form $dx_1 \wedge \cdots \wedge dx_n$ satisfies: $(dx_1 \wedge \cdots \wedge dx_n)(\xi^1, \cdots, \xi^n) = \det(\xi^1, \cdots, \xi^n)$. ■

We the above notation we can now define arbitrary differential p-forms on $\mathbb{R}^n$. Let $1 \leq p \leq n$, then a C^k p-form on $\mathbb{R}^n$ is defined by

$$\boldsymbol{\mu} := \sum_{I \in \mathscr{M}^p} \mu^I(x) dx_I, \tag{1.3.11}$$

where $\mu^I \in C^k(\mathbb{R}^n; \mathbb{R})$. The linear vector space of C^k p-forms on $\mathbb{R}^n$ is denoted by $\Gamma^{p,k}(\mathbb{R}^n)$ and the smooth p-forms are denoted buy $\Gamma^p(\mathbb{R}^n)$. Compactly supported C^k and smooth p-forms are obtained by considering compactly supported coefficient functions μ^I and are denoted by $\Gamma_c^{p,k}(\mathbb{R}^n)$ and $\Gamma_c^p(\mathbb{R}^n)$ respectively. We can also restrict p-forms to a subset $\Omega \subset \mathbb{R}^n$ by considering coefficient functions $\mu^I \in C^k(\Omega; \mathbb{R})$. This leads to the vector spaces $\Gamma^{p,k}(\Omega)$, $\Gamma^p(\Omega)$. If $\Omega \subset \mathbb{R}^n$ is an open set it makes sense to also consider the vector spaces of compactly supported p-forms $\Gamma_c^{p,k}(\Omega)$ and $\Gamma_c^p(\Omega)$. The support of a p-form is defined as the closure of the set

$$\{x \in \mathbb{R}^n \mid \mu^I(x) \neq 0, \text{ for some } I \in \mathscr{M}^p\}$$

and is denoted by $\operatorname{supp}(\boldsymbol{\mu})$.

A few properties of differential p-forms follow from the definition in (1.3.10) and (1.3.11). From Example 1.24 we have that $dx_i \wedge dx_i = 0$ and $dx_i \wedge dx_j = -dx_j \wedge dx_i$. The wedge product can also be defined as a product of p-forms. The wedge rules for dx_i suffice in this book. For more details on differential p-forms see [19].

Another important operation on differential forms is the exterior derivative. Let $\boldsymbol{\mu} \in \Gamma^{p,k}(\mathbb{R}^n)$, $k \geq 1$, then

$$d\boldsymbol{\mu} := \sum_{I \in \mathscr{M}^p} \frac{\partial \mu^I(x)}{\partial x_i} dx_i \wedge dx_I = \sum_{I \in \mathscr{M}^p} \frac{\partial \mu^I(x)}{\partial x_i} dx_i \wedge dx_{i^1} \wedge \cdots \wedge dx_{i^p}, \tag{1.3.12}$$

which is a differential $(p+1)$-form in $\Gamma^{p+1,k-1}(\mathbb{R}^n)$. Whether the exterior derivative is well-defined is determined by the coefficient function μ^I. The exterior derivative is also defined as an operator on $\Gamma^{p,k}(\Omega)$ and $\Gamma_c^{p,k}(\Omega)$. A p-form $\boldsymbol{\mu}$ is *closed* if $d\boldsymbol{\mu} = 0$ and $\boldsymbol{\mu}$ is *exact* if there exists a $(p-1)$-form $\boldsymbol{\theta}$ such that $\boldsymbol{\mu} = d\boldsymbol{\theta}$.

The classical Poincaré Lemma states that a smooth, closed p-form on a contractible, open subset $\Omega \subset \mathbb{R}^n$ is exact. Here we give a extension of the Poincaré Lemma for C^k, compactly supported n-forms, cf. [19].

1.26 Proposition — Poincaré Lemma. Let $D \subset \mathbb{R}^n$ be a connected, open subset and let $\boldsymbol{\mu}$ be a C^k, compactly supported n-form on $\mathbb{R}^n$ with $\int_{\mathbb{R}^n} \boldsymbol{\mu} = 0$ and $\operatorname{supp}(\boldsymbol{\mu}) \subset D$. Then there exists a C^{k+1}, compactly supported $(n-1)$-form $\boldsymbol{\theta}$ on $\mathbb{R}^n$, with $\operatorname{supp}(\boldsymbol{\theta}) \subset D$ such that $\boldsymbol{\mu} = d\boldsymbol{\theta}$.

■ **1.27 Remark** The extension of the Poincaré Lemma for C^k, compactly supported p-forms also applies to the case $1 \le p < n$, i.e. if $\boldsymbol{\mu}$ be a C^k, compactly supported p-form on $\mathbb{R}^n$, then there exists a C^{k+1}, compactly supported $(p-1)$-form $\boldsymbol{\theta}$ on $\mathbb{R}^n$ such that $\boldsymbol{\mu} = d\boldsymbol{\theta}$. For a proof see [19]. ■

In order to prove the general version of the Poincaré Lemma we start with the special case of supports contained in an n-dimensional cube $Q^n = [a,b]^n = [a,b] \times \cdots \times [a,b]$.

1.28 Lemma Let $\boldsymbol{\mu}$ be a C^k, compactly supported n-form on $\mathbb{R}^n$ with $\int_{\mathbb{R}^n} \boldsymbol{\mu} = 0$ and $\operatorname{supp}(\boldsymbol{\mu}) \subset \operatorname{int}(Q^n)$. Then there exists a C^{k+1}, compactly supported $(n-1)$-form $\boldsymbol{\theta}$ on $\mathbb{R}^n$, with $\operatorname{supp}(\boldsymbol{\theta}) \subset \operatorname{int}(Q^n)$ such that $\boldsymbol{\mu} = d\boldsymbol{\theta}$.

Proof. A C^k, compactly supported n-form $\boldsymbol{\mu}$ is given by the expression $\boldsymbol{\mu} = \mu(x)dx_1 \wedge \cdots \wedge dx_n = \mu(x)dx$, where $\mu \in C_c^k(\mathbb{R}^n)$. In order to establish the exactness condition $\boldsymbol{\mu} = d\boldsymbol{\theta}$ we represent $\boldsymbol{\theta}$ by

$$\boldsymbol{\theta} = \sum_{i=1}^{n} (-1)^{i-1} \theta^i(x) dx_1 \wedge \cdots \wedge dx_{i-1} \wedge dx_{i+1} \wedge \cdots \wedge dx_n,$$

where $\theta^i \in C^{k+1}(\mathbb{R}^n)$. The exactness condition now translates into finding a vector field $\Theta(x) = (\theta^1(x), \cdots, \theta^n(x))$ such that $\mu = \operatorname{div} \Theta$. Indeed,

$$\begin{aligned} d\boldsymbol{\theta} &= \sum_{i=1}^{n} (-1)^{i-1} \frac{\partial \theta^i}{\partial x_i}(x) dx_i \wedge dx_1 \wedge \cdots \wedge dx_{i-1} \wedge dx_{i+1} \wedge \cdots \wedge dx_n \\ &= \sum_{i=1}^{n} \frac{\partial \theta^i}{\partial x_i}(x) dx_1 \wedge \cdots \wedge dx_n = \operatorname{div} \Theta(x)\, dx. \end{aligned}$$

For $n = 1$ we take $\theta^1(x) = \int_{-\infty}^{x} \mu(s)ds$. By assumption $\operatorname{supp}(\mu) \subset \operatorname{int}(Q^1) = [a,b]$ and $\int_{\mathbb{R}} \mu(s)ds = \int_a^b \mu(s)ds = 0$ and therefore $\theta^1(x) = 0$ for all $x \le a$ and $x \ge b$. This proves that $\operatorname{supp}(\theta) \subset \operatorname{int}(Q^1)$ and $\frac{d}{dx}\theta^1(x) = \mu(x)$. If $\mu \in C_c^k(\mathbb{R})$, then $\theta^1 \in C_c^{k+1}(\mathbb{R})$.

Suppose the above statement is true in dimension $n-1$. Write $x = (y, x_n)$, with $y = (x_1, \ldots, x_{n-1})$ and consider the $(n-1)$-form $\boldsymbol{\alpha} = \alpha(y)dx_1 \wedge \cdots \wedge dx_{n-1}$, where $\alpha(y) = \int_{\mathbb{R}} \mu(y, x_n)dx_n$. By the assumptions on $\boldsymbol{\mu}$ we have that $\mu \in C_c^k(\mathbb{R}^n)$, $\operatorname{supp}(\mu) \subset \operatorname{int}(Q^n)$ and $\int_{\mathbb{R}^n} \mu(x)dx = 0$, and therefore $\alpha \in C_c^k(\mathbb{R}^{n-1})$, $\operatorname{supp}(\alpha) \subset \operatorname{int}(Q^{n-1})$ and $\int_{\mathbb{R}^{n-1}} \alpha(y)dy = \int_{\mathbb{R}^n} \mu(x)dx = 0$. By the induction hypothesis α is of divergence form, i.e. $\alpha = \text{div}\ \xi$, for some vector field ξ, with $\xi^i \in C_c^{k+1}(\mathbb{R}^{n-1})$ and $\text{supp}(\xi^i) \subset \text{int}(Q^{n-1})$. Let $\tau \in C^\infty(\mathbb{R})$ with $\text{supp}(\tau) \subset \text{int}(Q^1) = [a,b]$ and $\int_{\mathbb{R}} \tau(s)ds = 0$, define the function

$$\theta^n(y, x_n) = \int_{-\infty}^{x_n} \Big(\mu(y,s) - \tau(s)\alpha(y)\Big)ds.$$

By construction $\text{supp}(\theta^n) \subset \text{int}(Q^n)$ and $\frac{\partial \theta^n}{\partial x_n} = \mu(x) - \tau(x_n)\alpha(y)$. Now let

$$\Theta(x) = \big(\tau(x_n)\xi(y), \theta^n(y, x_n)\big),$$

then

$$\begin{aligned} \text{div}\ \Theta(x) &= \tau(x_n)\text{div}\ \xi(y) + \frac{\partial \theta^n}{\partial x_n}(x) \\ &= \tau(x_n)\alpha(y) + \mu(x) - \tau(x_n)\alpha(y) \\ &= \mu(x), \end{aligned}$$

and $\text{supp}(\theta^i) \subset \text{int}(Q^n)$. ■

Proof of Proposition 1.26. Define the centered cubes $Q^n_{\epsilon_x}(x) = [x - \epsilon_x, x + \epsilon_x]^n$. Since D is open there exists an $\epsilon_x > 0$ for every $x \in D$ such that $Q^n_{\epsilon_x}(x) \subset D$. Consider the open covering $\text{supp}(\mu) \subset \bigcup_{x \in \text{supp}(\mu)} \text{int}(Q^n_{\epsilon_x}(x))$. By the compactness of $\text{supp}(\mu)$ there exists a finite sub-covering $\text{supp}(\mu) \subset \bigcup_j \text{int}(Q^n_j)$, $j = 1, \cdots, N$, where $Q^n_j = Q^n_{\epsilon_{x^j}}(x^j)$ for a choice of points $x^j \in \text{supp}(\mu)$. By construction

$$\text{supp}(\mu) \subset \bigcup_j \text{int}(Q^n_j) \subset \bigcup_j Q^n_j \subset D.$$

Let $\{\eta^j\}$ be a partition of unity subordinate to $\{Q^n_j\}$ and define the n-forms $\boldsymbol{\mu}^j = \eta^j \boldsymbol{\mu}$. Because $\sum_j \eta^j = 1$ we have that

$$\sum_j \boldsymbol{\mu}^j = \boldsymbol{\mu}, \quad \text{supp}(\boldsymbol{\mu}^j) \subset Q^n_j.$$

Although $\int_{\mathbb{R}^n} \boldsymbol{\mu} = 0$, the individual integrals $c^j = \int_{\mathbb{R}^n} \boldsymbol{\mu}^j$ need not be zero. If $c^j = 0$, then by Lemma 1.28 there exists a C^{k+1}, compactly supported $(n+1)$-form $\boldsymbol{\theta}^j$ with $\text{supp}(\boldsymbol{\theta}^j) \subset Q^n_j$, such that

$$\boldsymbol{\mu}^j = d\boldsymbol{\theta}^j. \tag{1.3.13}$$

For the remaining cases that $c^j \neq 0$ we use the (path) connectedness of D.

Let $\boldsymbol{\mu}^0$ be a C^k, compactly supported n-form with $\text{supp}(\boldsymbol{\mu}^0) \subset \text{int}(Q^n_0) \subset Q^n_0 \subset D$ and $\int_{\mathbb{R}^n} \boldsymbol{\mu}^0 = 1$. For every cube Q^n_j choose a point $x^j \in \text{int}(Q^n_j)$ and consider a continuous path $t \to \gamma_t$, $t \in [0,1]$, with $\gamma_0 \in Q^n_0$ and $\gamma_1 = x^j$. The compactness of $\{\gamma_t\}_{t \in [0,1]}$ allows a finite, open covering with cubes K^n_i, $i = 0, \cdots, M$, such that

$$\{\gamma_t\}_{t \in [0,1]} \subset \bigcup_i \text{int}(K^n_i), \quad \text{int}(K^n_i) \cap \text{int}(K^n_{i+1}) \neq \varnothing, \quad K^n_0 = Q^n_0, \ K^n_M = Q^n_j,$$

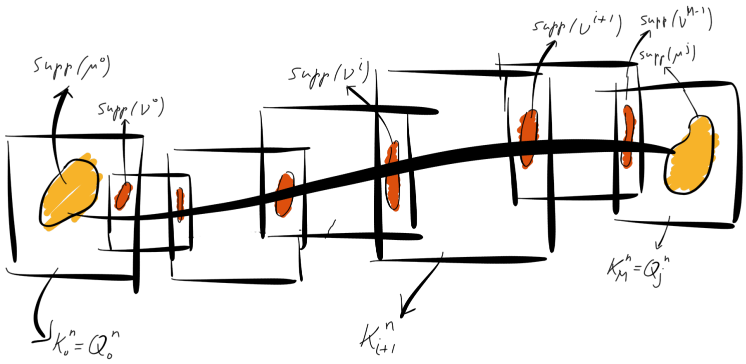


Figure 1.4: The covering of the connecting path $t \mapsto \gamma_t$ with cubes K_j^n between Q_0^n and Q_j^n. In the overlaps the supports of the n-forms $\boldsymbol{\nu}^i$ are chosen.

where $K_i^n = [a^i, b^i]^n \subset D$, cf. Fig. 1.4. Choose n-forms $\boldsymbol{\nu}^i$, $i = 0, \cdots, M-1$, such that

$$\operatorname{supp}(\boldsymbol{\nu}^i) \subset \operatorname{int}(K_i^n) \cap \operatorname{int}(K_{i+1}^n), \quad \text{and} \quad \int_{\mathbb{R}^n} \boldsymbol{\nu}^i = 1.$$

By construction $\int_{\mathbb{R}^n} (\boldsymbol{\nu}^i - \boldsymbol{\nu}^{i+1}) = 0$ and $\operatorname{supp}(\boldsymbol{\nu}^i - \boldsymbol{\nu}^{i+1}) \subset \operatorname{int}(K_{i+1}^n)$, $i = 0, \cdots, M-2$. Lemma 1.28 now implies that

$$\boldsymbol{\nu}^i - \boldsymbol{\nu}^{i+1} = d\boldsymbol{\lambda}^{i+1}, \quad i = 0, \cdots, M-2, \tag{1.3.14}$$

where $\boldsymbol{\lambda}^{i+1}$ are C^{k+1}, compactly supported $(n-1)$-forms with $\operatorname{supp}(\boldsymbol{\lambda}^{i+1}) \subset K_{i+1}^n$. For K_0^n we have that $\int_{\mathbb{R}^n} (\boldsymbol{\mu}^0 - \boldsymbol{\nu}^0) = 0$ and $\operatorname{supp}(\boldsymbol{\mu}^0 - \boldsymbol{\nu}^0) \subset \operatorname{int}(K_0^n)$ and therefore

$$\boldsymbol{\mu}^0 - \boldsymbol{\nu}^0 = d\boldsymbol{\lambda}^0, \tag{1.3.15}$$

where $\boldsymbol{\lambda}^0$ is a C^{k+1}, compactly supported $(n-1)$-form with $\operatorname{supp}(\boldsymbol{\lambda}^0) \subset K_0^n$. For K_M^n we have that $\int_{\mathbb{R}^n} (c^j \boldsymbol{\nu}^{M-1} - \boldsymbol{\mu}^j) = 0$, where $c^j = \int_{\mathbb{R}^n} \boldsymbol{\mu}^j$, and $\operatorname{supp}(c^j \boldsymbol{\nu}^{M-1} - \boldsymbol{\mu}^j) \subset \operatorname{int}(K_M^n)$ and consequently

$$c^j \boldsymbol{\nu}^{M-1} - \boldsymbol{\mu}^j = d\boldsymbol{\lambda}^M, \tag{1.3.16}$$

where $\boldsymbol{\lambda}^M$ is a C^{k+1}, compactly supported $(n-1)$-form with $\operatorname{supp}(\boldsymbol{\lambda}^M) \subset K_M^n$. Combining (1.3.b)-(1.3.16) we obtain

$$d(c^j \boldsymbol{\lambda}^0 + \cdots + c^j \boldsymbol{\lambda}^{M-1} + \boldsymbol{\lambda}^M) = c^j \boldsymbol{\mu}^0 - \boldsymbol{\mu}^j,$$

and if we set $\boldsymbol{\theta}^j = -c^j \boldsymbol{\lambda}^0 - \cdots - c^j \boldsymbol{\lambda}^{M-1} - \boldsymbol{\lambda}^M$, then

$$d\boldsymbol{\theta}^j = \boldsymbol{\mu}^j - c^j \boldsymbol{\mu}^0, \tag{1.3.17}$$

and $\boldsymbol{\theta}^j$ is a C^{k+1}, compactly supported $(n+1)$-form $\boldsymbol{\theta}^j$ with $\operatorname{supp}(\boldsymbol{\theta}^j) \subset D$. Equation (1.3.17) retrieves Equation (1.3.13) if $c^j = 0$. Using the fact that $\sum_j \boldsymbol{\mu}^j = \boldsymbol{\mu}$ and $\sum_j c^j = 0$ we obtain

$$\sum_j d\boldsymbol{\theta}^j = \sum_j \boldsymbol{\mu}^j - \sum_j c^j \boldsymbol{\mu}^0 = \sum_j \boldsymbol{\mu}^j - \boldsymbol{\mu}^0 \sum_j c^j = \boldsymbol{\mu},$$

which establishes the exactness of $\boldsymbol{\mu}$ and $\boldsymbol{\theta} := \sum_j d\boldsymbol{\theta}^j$ is a C^{k+1}, compactly supported $(n-1)$-form with $\operatorname{supp}(\boldsymbol{\theta}) \subset D$. ∎

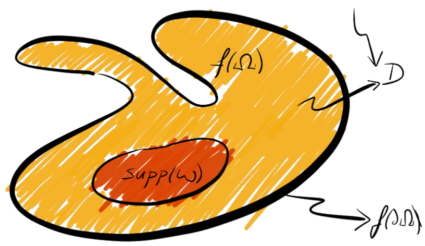


Figure 1.5: The support of the weight function ω in a connected component D of $\mathbb{R}^n \setminus f(\partial\Omega)$.

1.3.c A general representation

The integral characterization of the C^1-degree in the generic case motivates a representation of the C^1-mapping degree in general, i.e. regardless whether p is a regular value or not. In order for the integral representation in Equation (1.3.9) to serve as a definition of the C^1-mapping degree for arbitrary values $p \in \mathbb{R}^n \setminus f(\partial\Omega)$, the independence on ω needs to be established.

Let ω be a continuous weight function on $\mathbb{R}^n$ with the properties

$$\operatorname{supp}(\omega) \subset D \subset \mathbb{R}^n \backslash f(\partial\Omega), \quad \text{and} \quad \int_{\mathbb{R}^n} \boldsymbol{\omega} = 1,$$

where D is the connected component of $\mathbb{R}^n \backslash f(\partial\Omega)$ containing p. The first property allows for a larger class of weight functions since $\operatorname{supp}(\omega)$ is not necessarily a local coordinate neighborhood of p, see Fig. 1.5. For $\boldsymbol{\omega} \in \Gamma_c^{n,0}(D)$, with $\int_{\mathbb{R}^n} \boldsymbol{\omega} = 1$, define the integral

$$\mathscr{I}(f,\Omega,D) := \int_\Omega f^*\boldsymbol{\omega}. \tag{1.3.18}$$

The notation is justified by Lemma 1.29 which show that the integral does not depend on $\boldsymbol{\omega}$, but does depend on the connected component D containing $\operatorname{supp}(\omega)$. In Lemma 1.32 we establish that $\mathscr{I}$ is integer valued. For regular $p \in D$ and $\operatorname{supp}(\omega) \subset B_\epsilon(p)$, with $B_\epsilon(p)$ a local coordinate neighborhood, the integral representation in Equation (1.3.9) is retrieved.

1.29 Lemma Let $D \subset \mathbb{R}^n \setminus f(\partial\Omega)$ be a connected component and let $\boldsymbol{\omega}, \boldsymbol{\omega}' \in \Gamma_c^{n,0}(D)$ be two compactly supported n-forms on D, with $\int_{\mathbb{R}^n} \boldsymbol{\omega} = \int_{\mathbb{R}^n} \boldsymbol{\omega}' = 1$. Then

$$\int_\Omega f^*\boldsymbol{\omega} = \int_\Omega f^*\boldsymbol{\omega}'.$$

Proof. Let $\boldsymbol{\mu} := \boldsymbol{\omega}' - \boldsymbol{\omega}$, then $\int_{\mathbb{R}^n} \boldsymbol{\mu} = 0$ and $\operatorname{supp}(\boldsymbol{\mu}) \subset D$. By the Poincaré Lemma in Proposition 1.26 we have the existence of a C^1, compactly supported $(n-1)$-form $\boldsymbol{\theta}$, with $\operatorname{supp}(\boldsymbol{\theta}) \subset D$ such that

$$\boldsymbol{\mu} = \boldsymbol{\omega}' - \boldsymbol{\omega} = d\boldsymbol{\theta}.$$

Now choose a open set $\Omega' \subset \overline{\Omega'} \subset \Omega$ with smooth boundary such that $\mathrm{supp}(f^*\boldsymbol{\mu}) \subset \Omega'$. By Stokes' Theorem

$$\begin{aligned}\int_\Omega f^*\boldsymbol{\omega}' - \int_\Omega f^*\boldsymbol{\omega} &= \int_\Omega f^*(\boldsymbol{\omega}' - \boldsymbol{\omega}) = \int_\Omega f^*\boldsymbol{\mu} \\ &= \int_\Omega f^* d\boldsymbol{\theta} = \int_{\Omega'} f^* d\boldsymbol{\theta} \\ &= \int_{\Omega'} d(f^*\boldsymbol{\theta}) = \int_{\partial\Omega'} f^*\boldsymbol{\theta} = 0,\end{aligned}$$

since $\mathrm{supp}(f^*\boldsymbol{\theta}) \subset \Omega' \subset \Omega$. This proves the lemma. ∎

1.30 Exercise Check, using differential forms calculus, that $f^*d\boldsymbol{\theta} = d(f^*\boldsymbol{\theta})$ (Hint: show this first for C^2-functions). ∎

∎ **1.31 Remark** The condition that $\mathrm{supp}(\boldsymbol{\omega})$ is contained in a connected component D of $\mathbb{R}^n \setminus f(\partial\Omega)$ is crucial. If we allow any n-form on $\mathbb{R}^n$ (connected), then the Poincaré Lemma is applicable and by Stokes' Theorem, under the assumption that $\partial\Omega$ is smooth, we obtain

$$\int_\Omega f^*\boldsymbol{\omega}' - \int_\Omega f^*\boldsymbol{\omega} = \oint_{\partial\Omega \cap f^{-1}(K)} f^*\boldsymbol{\theta},$$

where $K = \mathrm{supp}(\boldsymbol{\omega}) \cup \mathrm{supp}(\boldsymbol{\omega}')$. The latter integral need not be zero. This explains why the condition $K \subset \mathbb{R}^n \setminus f(\partial\Omega)$ is important. Indeed, $K \subset \mathbb{R}^n \setminus f(\partial\Omega)$ implies that $f^{-1}(K) \subset \Omega$. However, the assumption $\mathrm{supp}(\boldsymbol{\omega}) \subset \mathbb{R}^n \setminus f(\partial\Omega)$, without restricting $\mathrm{supp}(\boldsymbol{\omega})$ to a connected component, is not enough since the Poincaré Lemma is not applicable. For example, let $f = \mathrm{id}$ and let $\Omega = B_1(0)$. Then, $\mathbb{R}^n \setminus \partial\Omega = D^1 \cup D^2$, where $D^1 = B_1(0)$ and $D^2 = \mathbb{R}^n \setminus \overline{B_1(0)}$. Consider the n-form $\boldsymbol{\omega} = \boldsymbol{\omega}^1 + \boldsymbol{\omega}^2$, with $\mathrm{supp}(\boldsymbol{\omega}^1) \subset D^1$ and $\mathrm{supp}(\boldsymbol{\omega}^2) \subset D^2$ and $\int_{\mathbb{R}^n} \boldsymbol{\omega}^i = 1/2$, for $i = 1,2$. Since $f^{-1}(D^2) = \varnothing$ we have that $\int_\Omega f^*\boldsymbol{\omega} = \int_{B_1(0)} \boldsymbol{\omega}^1 = 1/2$. On the other hand, if we consider $\boldsymbol{\omega}' = 2\boldsymbol{\omega}^1$, we obtain $\int_\Omega f^*\boldsymbol{\omega} = 2\int_{B_1(0)} \boldsymbol{\omega}^1 = 1$, which shows that $\int_\Omega f^*\boldsymbol{\omega}$ is not necessarily independent of $\boldsymbol{\omega}$ when $\mathrm{supp}(\boldsymbol{\omega})$ is not contained in a connected component of $\mathbb{R}^n \setminus f(\partial\Omega)$. ∎

1.32 Lemma Let $D \subset \mathbb{R}^n \setminus f(\partial\Omega)$ be a connected component and let $\boldsymbol{\omega} \in \Gamma_c^{n,0}(D)$, with $\int_{\mathbb{R}^n} \boldsymbol{\omega} = 1$ and $\mathrm{supp}(\boldsymbol{\omega}) \subset D$. Then

$$\int_\Omega f^*\boldsymbol{\omega} = \deg(f, \Omega, p) \in \mathbb{Z},$$

for every regular value $p \in D$.

Proof. By Sard's Theorem A.7 there exists a sequence $p^k \to p \in D$ with the property that $p^k \in D$ for $k \geq N$ for some $N > 0$. Choose a coordinate neighborhood $B_\epsilon(p^k) \subset$

D for some p^k, $k \geq N$. Let ω' be an n-form with $\text{supp}(\omega') = B_\epsilon(p^k)$ and $\int_{\mathbb{R}^n} \omega' = 1$. From Proposition 4.56 it follows that $\int_\Omega f^*\omega' = \deg(f,\Omega,p)$ and from Lemma 1.29

$$\int_\Omega f^*\omega = \int_\Omega f^*\omega' = \deg(f,\Omega,p),$$

which proves the lemma. ∎

It is is clear from the previous considerations that the degree is independent of $p \in D$ and coincides with the definition of degree in the regular case; Definition 1.5. The advantage of the integral representation is that a lot of properties of the degree can be obtained via fairly simple proofs.

This leads to the following alternative definition of the mapping degree for arbitrary values $p \in D$.

1.33 Definition Let $p \in D \subset \mathbb{R}^n \backslash f(\partial\Omega)$ and $\omega \in \Gamma_c^n(D)$, with $\int_{\mathbb{R}^n} \omega = 1$. Define C^1-mapping degree by

$$\deg(f,\Omega,p) := \mathscr{I}(f,\Omega,D) = \int_\Omega f^*\omega.$$

1.34 Exercise Let $p \in D \subset \mathbb{R}^n \backslash f(\partial\Omega)$ and let $\omega \in \Gamma_c^n(D)$, with $\int_{\mathbb{R}^n} \omega \neq 0$, i.e. ω not exact. Prove that $\deg(f,\Omega,p) = \int_\Omega f^*\omega / \int_{\mathbb{R}^n} \omega$. ▪

▪ **1.35 Remark** In Appendix E.1.b we introduced C^k, compactly supported de Rham cohomology. By the Poincaré Lemma all compactly supported cohomology of connected, open subsets of $\mathbb{R}^n$ vanishes up to order $p < n$ and is isomorphic to $\mathbb{R}$ for $p = n$. Let $f \in C^1(\overline{\Omega})$, then $f\colon \Omega \to \mathbb{R}^n \setminus f(\partial\Omega)$ yields a homomorphism f^* in compactly supported cohomology defined by $[\omega] \mapsto [f^*\omega]$.

By restricting to n-forms supported in a connected component $D \subset \mathbb{R}^n \setminus f(\partial\Omega)$ the analysis in this section yields the commuting diagram

$$\begin{tikzcd}
H_c^n(D) \arrow[r, "f^*"] \arrow[d, "\cong"'] & H_c^n(\Omega) \arrow[d, "\int_\Omega"] \\
\mathbb{R} \arrow[r, "\deg(f,\Omega,p)"] & \mathbb{R}
\end{tikzcd}$$

which is expressed in the relation $\deg(f,\Omega,p)\int_{\mathbb{R}^n} \omega = \int_\Omega f^*\omega$. ∎

1.36 Exercise Let $D \subset \mathbb{R}^n$ be connected, open subset. Use the Poincaré Lemma to prove that $\int_{\mathbb{R}^n} \colon H_c^n(D) \to \mathbb{R}$, given by $[\omega] \mapsto \int_{\mathbb{R}^n} \omega$, is well-defined and is an isomorphism. ▪

1.3.d Homotopy invariance

The treatment of the mapping degree in Sect. 1.3.c shows that $\deg(f, \Omega, p)$ is independent of $p \in D$, with $D \subset \mathbb{R}^n \backslash f(\Omega)$, a connected component. Consequently, $\deg(f, \Omega, p_t)$ is a constant function of t for every curve $t \mapsto p_t$ in D; homotopy invariance of the degree under homotopies in p.

The integral representation of the mapping degree can be used as an alternative to establish homotopy invariance of the degree with respect to f. The general homotopy invariance of the degree will be proved in several steps. The key ingredient is the continuity of the integral representation with respect to f.

1.37 Lemma The function $f \mapsto \int_\Omega f^*\boldsymbol{\omega} = \int_\Omega \omega(f(x))J_f(x)dx$ is continuous with respect to the C^1-topology.

Proof. By the continuity of $\omega(x)$, the condition $\|f - g\|_{C^1} < \delta$, implies that $|\omega(f(x)) - \omega(g(x))| < \epsilon$ uniformly for $x \in \overline{\Omega}$. Similarly, since $J_f(x)$ is a polynomial term in the partial derivatives $\frac{\partial f_i}{\partial x_j}$, the condition $\|f - g\|_{C^1} < \delta$ implies that $|J_f(x) - J_g(x)| < \epsilon$, uniformly in $x \in \overline{\Omega}$. These continuity properties yield the continuity of the integral $\int_\Omega f^*\boldsymbol{\omega}$ with respect to f. ■

1.38 Lemma Let $t \mapsto f_t$ and $t \mapsto \boldsymbol{\omega}_t$, $t \in [0,1]$ be a continuous paths in and assume that $\mathrm{supp}(\boldsymbol{\omega}_t) \cap f_t(\partial\Omega) = \varnothing$ for all $t \in [0,1]$, then $\int_\Omega f_t^*\boldsymbol{\omega}_t = \mathrm{const}$.

Proof. By assumption, for each $t \in [0,1]$ the integral represents a degree, i.e. $\int_\Omega f_t^*\boldsymbol{\omega}_t = \deg(f_t, \Omega, p_t)$ for some $p_t \in \mathrm{supp}(\boldsymbol{\omega})$. Therefore the integral is integer valued. On the other hand by Lemma 1.37 the integral is a continuous function of t and thus constant. ■

1.39 Proposition Let $t \mapsto f_t$ and $t \mapsto p_t$, $t \in [0,1]$ be a continuous paths and assume that $p_t \notin f_t(\partial\Omega)$ for all $t \in [0,1]$. Then, $\deg(f_t, \Omega, p_t)$ is a continuous function of t and is therefore constant along (f_t, Ω, p_t).

Proof. Choose an $\epsilon > 0$ small enough such that $B_\epsilon(p_t) \subset \mathbb{R}^n \backslash f_t(\partial\Omega)$. Define a form $\boldsymbol{\omega} = \omega(x)dx$ such that $\mathrm{supp}(\boldsymbol{\omega}) = \overline{B_\epsilon(0)}$ and set $\boldsymbol{\omega}_t = \omega(x - p_t)dx$. Consequently $t \mapsto \boldsymbol{\omega}_t$ is a continuous path with $\mathrm{supp}(\boldsymbol{\omega}_t) \cap f_t(\partial\Omega) = \varnothing$ for all $t \in [0,1]$ and $\int_\Omega f_t^*\boldsymbol{\omega}_t = \deg(f_t, \Omega, p_t)$. By Lemma 1.38 the integral $\int_\Omega f_t^*\boldsymbol{\omega}_t$ is constant, which proves the lemma. ■

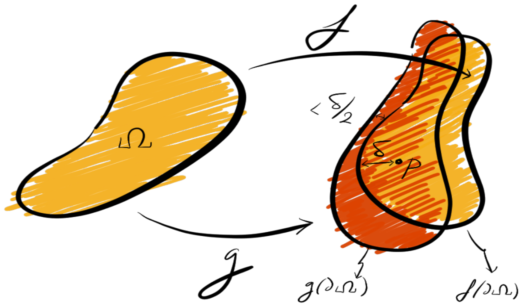


Figure 1.6: For small perturbations g of f, the point p is not contained in $g(\partial\Omega)$ [left]. The same holds for homotopies h_t. The second figure shows $f(\Omega)$ and $h_t(\Omega)$ for $t \in [0,1]$ [right].

1.4 The Brouwer degree

The C^1-mapping degree defined in Section 1.2 uses the fact that f is differentiable. The homotopy invariance of the C^1-degree can be used to extend the degree to the class of continuous functions on $\mathbb{R}^n$, which is essentially the approach due to Nagumo.[24] At the core of the definition of the C^0-mapping degree, or Brouwer degree is the fact that C^1-functions can be approximated by C^0-functions.

1.4.a Definition of the Brouwer degree

Using approximations of f via smooth mappings and homotopy invariance leads to the definition of the C^0-degree, or Brouwer degree.

1.40 Definition Let $f \in C^0(\overline{\Omega})$ and let $p \notin f(\partial\Omega)$. Then, for any sequence $f^k \in C^1(\overline{\Omega})$ converging to f in C^0, define

$$\deg(f,\Omega,p) := \lim_{k\to\infty} \deg(f^k,\Omega,p),$$

as the Brouwer degree of the triple (f,Ω,p).

The properties of the C^1-mapping degree imply that this definition makes sense, i.e. the limit exists and is independent of the chosen sequence f^k. Approximating sequences exist by virtue of Theorem A.4. Since $p \in D \subset \mathbb{R}^n \backslash f(\partial\Omega)$ it holds that $\delta = d(p, f(\partial\Omega)) > 0$ (compactness of $f(\partial\Omega)$).[6] Let $g,\tilde{g} \in C^1(\overline{\Omega})$ be approximations of f such that $\|g-f\|_{C^0}, \|\tilde{g}-f\|_{C^0} < \delta/2$. Consider the homotopy $h_t(x) = (1-t)g(x) + t\tilde{g}(x), \quad t \in [0,1]$. The choices of g and $\tilde{g}$ give

$$\begin{aligned} \|h_t - f\|_{C^0} &\leq (1-t)\|g-f\|_{C^0} + t\|\tilde{g}-f\|_{C^0} \\ &< (1-t)\,\delta/2 + t\,\delta/2 = \delta/2, \end{aligned}$$

and for $x \in \partial\Omega$ it holds that

$$|h_t(x) - p| \geq |f(x) - p| - |h_t(x) - f(x)| \geq \delta/2.$$

[6] The continuous image of a compact set is compact, which implies that $\delta = d(p, f(\partial\Omega)) > 0$.

Therefore, $p \notin h_t(\partial\Omega)$ for all $t \in [0,1]$ and the degree $\deg(h_t, \Omega, p)$ is constant in t by the homotopy invariance of the degree (e.g. Proposition 1.39). We conclude that $\deg(g, \Omega, p) = \deg(\tilde{g}, \Omega, p)$. For any approximating sequence f^k it holds that $\|f^k - f\|_{C^0} < \delta/2$, for k large enough. Therefore, we may assume in the above definition, that $p \notin f^k(\partial\Omega)$. These observations prove that the limit in Definition 1.40 exists and is independent of the chosen sequence f^k.

■ **1.41 Remark** In approximating C^0-fucntions via C^1-functions it is not necessary to assume that p is a regular value for the sequence f^k. Approximations can always be chosen such that this is the case, which can be useful sometimes. ■

1.42 Exercise Let $p \in \mathbb{R}^n \backslash f(\partial\Omega)$. Show that one can always approximate f with C^1-maps f^k with the additional property that p is regular value for all f^k. ■

1.43 Proposition The Brouwer degree $\deg(f, \Omega, p)$ is continuous in $f \in C^0(\overline{\Omega})$.

Proof. Let $g \in C^0(\overline{\Omega})$ be any continuous mapping such that $\|g - f\|_{C^0} < \delta/4$. Then $\deg(g, \Omega, p)$ well-defined, since, for $x \in \partial\Omega$, it holds that $|g(x) - p| \geq |f(x) - p| - |g(x) - f(x)| \geq 3\delta/4$ and thus $\operatorname{dist}(p, g(\partial\Omega)) > 3\delta/4$ which implies that $p \notin g(\partial\Omega)$.

Let $f^k \in C^1(\overline{\Omega})$ and $g^k(\overline{\Omega}) \in C^1(\overline{\Omega})$ be sequences that converge to f and g respectively. Choose k large enough such that $\|f^k - f\|_{C^0} < \delta/4$, and $\|g^k - g\|_{C^0} < \delta/4$. Since $\|g^k - g\|_{C^0} < \delta/4 < 3\delta/8$ we have that $\deg(g, \Omega, p) = \deg(g^k, \Omega, p)$, and similarly $\deg(f, \Omega, p) = \deg(f^k, \Omega, p)$, since $\|f^k - f\|_{C^0} < \delta/4 < \delta/2$.

On the other hand

$$\|g^k - f\|_{C^0} \leq \|g - f\|_{C^0} + \|g^k - g\|_{C^0} < \delta/2,$$

which implies that $\deg(f, \Omega, p) = \deg(g^k, \Omega, p)$, and therefore $\deg(f, \Omega, p) = \deg(g, \Omega, p)$, establishing the continuity of deg with respect to f. ■

Using the continuity of the degree in f the invariance under continuous homotopies can be derived.

1.44 Proposition For any continuous path $t \mapsto f_t$ in $C^0(\overline{\Omega})$, with $f_0 = f$ and $p \notin f_t(\partial\Omega)$, $t \in [0,1]$, it holds that $\deg(f_t, \Omega, p) = \deg(f, \Omega, p)$ for all $t \in [0,1]$.

Proof. By definition $t \mapsto f_t$ is continuous in $C^0(\overline{\Omega})$ and therefore by Proposition 1.43, $\deg(f_t, \Omega, p)$ depends continuously on $t \in [0,1]$. Since the degree is integer valued it has to be constant along the homotopy f_t. ■

1.45 Proposition The Brouwer degree satisfies the translation property, i.e. for any $q \in \mathbb{R}^n$ it holds that $\deg(f - q, \Omega, p - q) = f(f, \Omega, p)$.

Proof. Choose a sufficiently small perturbation $g \in C^1(\overline{\Omega})$ of f, then Axiom (A4) implies that

$$\begin{aligned} \deg(g - q, \Omega, p - q) &= \deg(g - q - (p - q), \Omega, 0) \\ &= \deg(g - p, \Omega, 0) = \deg(g, \Omega, p). \end{aligned}$$

By definition $\deg(f - q, \Omega, p - q) = \deg(g - q, \Omega, p - q)$ and $\deg(f, \Omega, p) = \deg(g, \Omega, p)$, which proves the lemma. ∎

∎ **1.46 Remark** If $t \mapsto p_t$ is a continuous path such that $p_t \notin f_t(\partial\Omega)$, then the translation property of the degree, Proposition 1.45, shows, since $f_t - p_t$ is a homotopy, that

$$\deg(f_t, \Omega, p_t) = \deg(f_t - p_t, \Omega, 0) = \deg(f - p, \Omega, 0) = \deg(f, \Omega, p).$$

We conclude that the Brouwer degree is an invariant for cobordant triples $(f, \Omega, p) \sim (g, \Omega, q)$, or $(f, \Omega, D) \sim (g, \Omega, D')$. In the Section 2.1 the more general version will be given allowing variations in Ω in the context of Axioms for degree theory. ∎

1.4.b The index of isolated zeroes

It the case that a mapping has only isolated zeroes, and thus finitely many, Property (A3) gives the degree as a sum of the local degrees. More precisely, let $x^i \in \Omega$ be the zeroes of f and let $\Omega^i \subset \Omega$ be sufficiently small neighborhoods of x^i, such that x^i the only solution of $f(x) = p$ in Ω^i for all i. Then $\deg(f, \Omega, p) = \sum_i \deg(f, \Omega^i, p)$ and we define

$$\iota(f, x^i, p) := \deg(f, \Omega^i, p),$$

which is called the index of an isolated zero of f. The index for isolated zero does not depend on the domain Ω^i. Indeed, if Ω^i and $\widetilde{\Omega}^i$ are both neighborhoods of x^i for which x^i is the only zero of $f(x) = p$, then we can define a cobordism between (f, Ω^i, p) and $(f, \widetilde{\Omega}^i, p)$ as follows. Let $\Omega = \bigcup_{t \in [0,1]} \Omega_t$ with

$$\Omega_t = \begin{cases} \Omega^i & \text{for } t < \frac{1}{2}, \\ \Omega^i \cap \widetilde{\Omega}^i & \text{for } t = \frac{1}{2}, \\ \widetilde{\Omega}^i & \text{for } t > \frac{1}{2}, \end{cases}$$

and $f_t = F(\cdot,t) = f$, $p_t = p$. By Theorem **??** $\deg(f,\Omega^i,p) = \deg(f,\widetilde{\Omega}^i,p)$. The expression for the degree becomes

$$\deg(f,\Omega,p) = \sum_{x\in f^{-1}(p)} \iota(f,x,p). \tag{1.4.19}$$

It is not hard to find mappings with isolated zeroes of arbitrary integer index.

1.47 Exercise Show that if $x \in f^{-1}(p)$ is a non-degenerate zero of f, then $\iota(f,x,p) = (-1)^\beta$, where $\beta = \#\{\text{negative real eigenvalues}\}$ (counted with multiplicity). ▪

1.4.c Linear vector spaces

Let V be a real linear vectorspace of dimension n. A continuous mapping $f : \overline{\Omega} \subset V \to V$ is differentiable on $\overline{\Omega}$ if $\widetilde{f} : \overline{D} \subset \mathbb{R}^n \to \mathbb{R}^n$ is differentiable on $\overline{D}$, where $\widetilde{f} = q \circ f \circ q^{-1}$, $q : V \to \mathbb{R}^n$ is a linear isomorphism (linear chart), and $D = q(\Omega)$. A value $p \in V$ is called regular for f if and only if $\widetilde{p} = q(p) \in \mathbb{R}^n$ is regular for $\widetilde{f}$. We now define the C^1-mapping degree by

$$\deg(f,\Omega,p;V) := \deg(\widetilde{f},D,\widetilde{p}), \tag{1.4.20}$$

for $p \notin f(\partial\Omega)$. The definition is independent of the chosen isomorphism q since the determinants in the expression of the C^1-mapping degree do not depend on the particular isomorphism, and the zeroes of $\widetilde{f}$ and $\widehat{f}$ are in 1-1 correspondence.[7]

1.48 Exercise Show that the degree $\deg(f,\Omega,p;V)$ is well-defined for all $p \notin f(\partial\Omega)$. ▪

1.5 Elementary applications of the mapping degree

In this section we will discuss additional applications of the Brouwer degree.

1.5.a The degree for holomorphic functions

The Brouwer degree can also be used in complex function theory. A complex function $f : \mathbb{C} \to \mathbb{C}$ can be regarded as a mapping from $\mathbb{R}^2$ to $\mathbb{R}^2$ via the following correspondence. Set $z = x_1 + ix_2$ and $f(z) = u(x_1,x_2) + iv(x_1,x_2)$, then $f : \mathbb{R}^2 \to \mathbb{R}^2$ is defined by

$$(x_1,x_2) \mapsto f(x_1,x_2) = \big(u(x_1,x_2), v(x_1,x_2)\big).$$

A complex mapping f is holomorphic on a bounded open set $\Omega \subset \mathbb{C}$, if the Cauchy-Riemann equations are satisfied, i.e. $\overline{\partial} f = 0$, which is equivalent to

$$\frac{\partial u}{\partial x_1} = \frac{\partial v}{\partial x_2}, \quad \frac{\partial v}{\partial x_1} = -\frac{\partial u}{\partial x_2},$$

[7] Here $\widehat{f}$ is the transformed mapping under a different isomorphism.

for all $z = x_1 + ix_2 \in \Omega$. The Brouwer degree for the triple (f, Ω, z), with $z \in \mathbb{C} \backslash f(\partial\Omega)$, is defined as the degree of the mapping $f = (u, v)$ on $\mathbb{R}^2$.

From complex function theory it follows that zeroes of holomorphic functions are isolated, or the function is identically equal to zero. This leads to a following result about the mapping degree for holomorphic functions.

1.49 Proposition Let $f : \overline{\Omega} \subset \mathbb{C} \to \mathbb{C}$ be a holomorphic function, not identically equal to zero, and $f(z^0) = 0$, for some $z^0 \in \Omega$. Then there exists an $\epsilon > 0$, and a ball $B_\epsilon(z^0) \subset \Omega$ such that $f(z) \neq 0$, for all $z \in B_\epsilon(z^0) \backslash \{z^0\}$, and

$$\deg(f, B_\epsilon(z^0), 0) = m \geq 1,$$

where m is the order of z^0, i.e. $f(z) = (z - z^0)^m g(z)$, $z \in B_\epsilon(z^0)$, and $|g(z)| \geq a > 0$, for all $z \in B_\epsilon(z^0)$.

Proof. Since f is not identically equal to zero, z^0 is an isolated zero of f, and there there exists a ball $B_\epsilon(z^0) \subset \Omega$ on which f is non-zero, except at z^0. Also, by analyticity, it follows that z^0 is a finite order zero; $f(z) = (z - z^0)^m g(z)$, $m \geq 1$, and $|g(z)| \geq a > 0$ in $B_\epsilon(z^0)$. These consideration make that the degree $\deg(f, B_\epsilon(z^0), 0)$ is well-defined, since $|f(z)| = \epsilon^m a > 0$, for $z \in \partial B_\epsilon(z^0)$. In the case $m = 1$ the degree can be easily computed from the definition. In general, for a homolomorphic function, $J_f(z) = \frac{1}{2}\|\nabla f\|^2$. Since 0 is a regular value, $J_f(z^0)$ can be computed as follows: $f(z) = (z - z^0)g(z)$, and thus $f'(z) = g(z) + (z - z^0)g'(z)$. Therefore

$$J_f(z^0) = |g(z^0)|^2 = a^2 > 0,$$

and $\deg(f, B_\epsilon(z^0), 0) = 1$.

Consider the holomorphic function $p(z) = (z - z^0)^m g(z^0)$, and the homotopy $f_\lambda(z) = \lambda f(z) + (1 - \lambda)p(z)$, $\lambda \in [0, 1]$, which is a homotopy of holomorphic functions. Choose $\epsilon > 0$ small enough such that $|g(z) - g(z^0)| < \frac{1}{2}|g(z^0)|$, for all $z \in B_\epsilon(z^0)$. In order to use the homotopy property of the degree it needs to be verified that $0 \notin \partial f(B_\epsilon(z^0))$, for all $\lambda \in [0, 1]$. Let $|z - z^0| = \epsilon$, then

$$\begin{aligned}
|f_\lambda(z)| &= \big|\lambda(z - z^0)^m g(z) + (1 - \lambda)(z - z^0)^m g(z^0)\big| \\
&= \epsilon^m \big|\lambda g(z) + (1 - \lambda) g(z^0)\big| \\
&= \epsilon^m \big|g(z^0) + \lambda(g(z) - g(z^0))\big| \\
&\geq \epsilon^m \big|g(z^0)\big| - \lambda\big|g(z) - g(z^0)\big| \\
&\geq \frac{1}{2}\epsilon^m |g(z^0)|.
\end{aligned}$$

If we choose $\delta = \frac{1}{4}\epsilon^m |g(z^0)|$, then $f_\lambda(z) = \delta$ has no solutions on $\partial B_\epsilon(z^0)$, for all $\lambda \in [0, 1]$. Consequently,

$$\deg(f, B_\epsilon(z^0), \delta) = \deg(p, B_\epsilon(z^0), \delta).$$

It remains to evaluate $\deg(p, B_\epsilon(z^0), \delta)$. The associated equation is

$$p(z) = (z - z^0)^m g(z^0) = \delta = \frac{1}{4}\epsilon^m |g(z^0)|.$$

This implies that zeroes lie on $|z - z^0| = \epsilon 4^{-\frac{1}{m}}$. For the arguments it holds that

$$m \arg(z - z^0) + \arg(g(z^0)) = 2\pi n, \quad n \in \mathbb{Z}.$$

It follows immediately that the above equation has exactly m non-degenerate solutions, and therefore, $\deg(p, B_\epsilon(z^0), \delta) = m$.

At the boundary $\partial B_\epsilon(z^0)$, $|(z - z^0)|^m |g(z)| = \epsilon^m a$. Consider the path $\xi_\lambda = \frac{1}{2}\lambda\epsilon^m a$, then $f(z) \neq \xi_\lambda$ on $\partial B_\epsilon(z^0)$, for all $\lambda \in [0,1]$. Consequently, $d(f, B_\epsilon(z^0), \xi_\lambda)$ is constant all $\lambda \in [0,1]$, and

$$\deg(f, B_\epsilon(z^0), 0) = \deg(f, B_\epsilon(z^0), \delta) = \deg(p, B_\delta(z^0), \delta) = m,$$

which completes the proof. ∎

A direct consequence of the above proposition is a result on mapping degree holomorphic functions in general.

1.50 Theorem Let $f : \overline{\Omega} \subset \mathbb{C} \to \mathbb{C}$ be a holomorphic function. Assume that $0 \notin f(\partial\Omega)$. Then,

$$\deg(f, \Omega, 0) \geq 0.$$

Proof. By analyticity f has only isolated zeroes $z^i \in \Omega$. Let $B_\epsilon(z^i) \subset \Omega$ be sufficiently small neighborhoods containing exactly one zero each. The excision and summation properties of the degree then give

$$\deg(f, \Omega, 0) = \deg(f, \cup_i B_\epsilon(z^i), 0) = \sum_i \deg(f, B_\epsilon(z^i), 0) = \sum_i m_i,$$

where the numbers $m_i \geq 1$ are the orders of the zeroes z^i. Since $\sum_i m_i \geq 0$ this yields the desired result. ∎

Another consequence of Proposition 1.49 is the Fundamental Theorem of Algebra.

1.51 Theorem Every polynomial $p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_0$, with real coefficients a_i, has exactly n complex roots counted with multiplicity.

Proof. Write $p(z) = z^n + r(z)$, then $|p(z)| = |z^n + r(z)| \geq \big||z^n| - |r(z)|\big|$. On the circle $|z| = R > 0$, for R sufficiently large, we have $|r(z)| \leq CR^{n-1}$, and thus

$$|p(z)| = |z^n + r(z)| \geq \big||z|^n - |r(z)|\big| \geq R^n - CR^{n-1} > 0,$$

which proves that all zeroes are contained in the ball $B_R(0)$, and $\deg(f, B_R(0), 0)$ is well-defined. The same holds for the homotopy $p_\lambda(z) = z^n + \lambda r(z)$, $\lambda \in [0,1]$. The homotopy invariance of the Brouwer degree combined with Proposition 1.49 then yields

$$\deg(p, B_R(0), 0) = \deg(z^n, B_R(0), 0) = n > 0,$$

implying that $p(z) = 0$ has at least one solution z_1 in $B_R(0)$. Now repeat the argument for the polynomial $p_1(z) = \frac{p(z)}{z - z_1}$. This again produces a zero z_2. This process terminates after n steps, proving the desired result. ■

1.5.b Periodic orbits in planar systems of differential equations

Consider $\mathbb{R}^2$ with standard polar coordinates $(r,\theta) \in \mathbb{R}^+ \times \mathbb{R}/2\pi\mathbb{Z}$ and $x = r\cos(\theta)$, $y = r\sin(\theta)$.

Consider the explicit system of differential equations given by $\dot{r} = r(r-1)$ and $\dot{\theta} = \theta_0$. This system can be solved explicitly and consist of an unstable fixed point at $r = 0$ and a stable periodic orbit that $r = 1$. All other non-trivial orbits converge to the stable periodic orbit.

Consider the perturbed system

$$\begin{cases} \dot{r} = r(r-1) + \eta(r,\theta), \\ \dot{\theta} = \theta_0 + \xi(r,\theta), \end{cases}$$

where $\eta(0,\theta) = 0$ and η is bounded, and $\xi(r,\theta) > -\theta_0$. The solution of this system defines a flow and is denoted by $\varphi(t,\cdot)$ which describes the evolution of the system in time. We now ask the question whether this system still has a periodic solution. By the assumptions η, $r = 0$ is still an unstable fixed point. The assumption on ξ reveals that $\dot{\theta} > 0$ which implies that a Poincaré section can be defined. The set $L = \{(x,0) \mid x > 0\}$ is a Poincaré section, i.e. the flow lines of the above systems intersect L transversely and for every point $(x,0) \in L$ there exists a time $\tau = \tau(x,0)$ such that the flow returns to L. Indeed, $\dot{\theta} > 0$. The function $\tau\colon L \to \mathbb{R}^+$ is continuous (even smooth) and is called the first-return time. With the Poincaré section L comes a first-return map $f\colon L \to L$, which is defined as follows: Let $p = (x,0) \in L$, then

$$f(p) = \varphi(\tau(p), p).$$

1.52 Exercise Show that the first-return time $\tau\colon L \to \mathbb{R}^+$ is a continuous function. ■

1.53 Exercise Show that the first-return map $f\colon L \to L$ is a continuous function. ▪

A periodic orbit for φ corresponds to a fixed of f, i.e. a point $p \in L$ such that $f(p) = p$. Define $g(p) = f(p) - p$ and $g\colon L \to L$ is a continuous map. Since $\dot{r} > 0$ for r sufficiently large we conclude that g is a proper map. Indeed, after a revolution r has strictly increased provided $|p|$ is sufficiently large.

Consider the homotopy $g_\lambda(p) = f_\lambda(p) - p$, where f_λ is defined by the equations

$$\begin{cases} \dot{r} = r(r-1) + \lambda\eta(r,\theta), \\ \dot{\theta} = \theta_0 + \lambda\xi(r,\theta), \end{cases}$$

with $\lambda \in [0,1]$. By the previous considerations g_λ is proper for all $\lambda \in [0,1]$ and $g_0^{-1}(0) = \{r = 1\}$ and the Brouwer degree is given by $\deg(g_0, L, 0) = 1$. By the homotopy invariance of the degree we now conclude that $\deg(g, L, 0) = \deg(g_1, L, 0) = \deg(g_0, L, 0) = 1$, which proves the existence of a zero and thus a periodic solution for φ. links with intersection forms linking in terms of homology/cohomology classes Poincare-Hopf also w.r.t. Hairy Ball Thm simplify pull-back self-linking number via the ribbon

1.6 Problems

1.54 Problem By identifying $\mathbb{C}$ and $\mathbb{R}^2$ the application $z \mapsto z^n$ can be identified with a smooth mapping f on $\mathbb{R}^2$. Show that $\iota(f,0) = n$. Find a class of mappings on $\mathbb{R}^2$ for 0 is an isolated zero and $\iota(f,0) = -n$.

1.55 Problem Let $f \in C^1(\overline{\Omega})$, with $\overline{\Omega} \subset \mathbb{R}^n$ a bounded domain and f is one-to-one. Prove that $\deg(f, \Omega, p) = \pm 1$.

1.56 Problem Let $f : \overline{B_1(0)} \to \mathbb{R}^n$ and $f(x) \neq \mu x$ for $\mu \geq 0$ and for all $x \in \partial B_1(0)$. Show that $f(x) = 0$ has a non-trivial solution in $B_1(0)$.

1.57 Problem Let $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ be a polynomial with $a_n \neq 0$.

(i) Show that for fixed coefficients $a_0, \cdots, a_n$ there exists an $r > 0$ such that $f^{-1}(0) \in (-r, r)$.

(ii) Prove for n odd that $\deg(f, (-r,r), 0) = 1$.

(iii) Prove that n even that $\deg(f, (-r,r), 0) = 0$.

(Hint: use the integral representation of the degree with $\omega(x) = 1 - x^2$ on $(-1,1)$ and zero outside).

1.58 Problem Prove Proposition 1.18.

Let N and M be smooth, orientable n-dimensional manifolds and let $f : \overline{\Omega} \subset N \to M$ be continuous C^1-mapping on a compact subset $\overline{\Omega} \subset N$. A value $p \in M$

is regular if $f'(x)\colon T_xN \to T_pM$ has maximal rank for all $x \in f^{-1}(p)$, i.e. $f'(x)$ is invertible.

1.59 Problem (i) Show that the compactness of $\overline{\Omega}$ implies that $f^{-1}(p)$ is a compact set.

(ii) For a regular value $p \in M$ such that $p \notin f(\partial\Omega)$. Define the local C^1-mapping degree by

$$\deg(f,\Omega,p) = \sum_{x \in f^{-1}(p)} \operatorname{sign}\Big(J_f(x)\Big), \tag{1.6.21}$$

where $J_f(x) = \det(f'(x))$. Show that (1.6.21) does not depend on the choice of regular value. If N is closed then $\deg(f) := \deg(f,N,p)$ is well-defined for any regular value $p \in M$.

(iii) Let $f_t\colon N \to M$ be a homotopy such that $p \notin f_t(\partial\Omega)$. Show that the mapping degree is invariant under homotopy.

The construction of the C^1-mapping degree via integration can be carried for compactly supported n-forms $\boldsymbol{\omega}$ on M. Let $D \subset M \backslash f(\partial\Omega)$ be a connected component, then for every $p \in D$ and every $\boldsymbol{\omega} \in \Gamma_c^n(D)$ with $\int_M \boldsymbol{\omega} = 1$, define

$$I(f,\Omega,p) = \int_\Omega f^*\boldsymbol{\omega}.$$

1.60 Problem Use the definition of I to show the homotopy invariance if I with respect to p and f.

1.61 Problem Show that $I(f,\Omega,p) = \deg(f,\Omega,p)$ given in (1.6.21).

1.62 Problem Describe an extension of the local mapping degree to continuous mappings $f\colon N \to M$.

2 — Axiomatic Degree Theory

2.1 Properties and axioms for the Brouwer degree

In this section a number of useful properties of the mapping degree will be given. In principle these properties can be proved using the definitions of the C^1-mapping degree and the Brouwer degree. Another approach is to single out the most fundamental properties and show that these determine the Brouwer degree uniquely, and that all properties can be derived from these properties, or axioms. The axiomatic approach to degree theory can be found in Amann and Weiss.[2] For a extensive exposition on the axiomatic degree theory we refer to Llyod.[21]

2.1 Definition Consider triples (f,Ω,p), where $\Omega \subset \mathbb{R}^n$ are bounded open sets, $f \in C^0(\overline{\Omega})$, and $\mathbb{R}^n \ni p \notin f(\partial\Omega)$. Such triple are called admissible triples. To each admissible triple (f,Ω,p) assign an integer $\deg(f,\Omega,p)$, which satisfies the following four axioms;

(A1) if $p \in \Omega$, then $\deg(\mathrm{Id},\Omega,p) = 1$;

(A2) for $\Omega^1,\Omega^2 \subset \Omega$, disjoint open subsets of Ω, and $p \notin f\big(\overline{\Omega}\backslash(\Omega_1 \cup \Omega_2)\big)$, it holds that $\deg(f,\Omega,p) = \deg(f,\Omega^1,p) + \deg(f,\Omega^2,p)$;

(A3) for any continuous path $t \mapsto f_t$, $f_t \in C^0(\overline{\Omega})$ and $p \notin f_t(\partial\Omega)$, it holds that $\deg(f_t,\Omega,p)$ is independent of $t \in [0,1]$;

(A4) $\deg(f,\Omega,p) = \deg(f-p,\Omega,0)$.

The application $(f,\Omega,p) \mapsto \deg(f,\Omega,p)$ is called a degree theory.

2.2 Theorem — Existence. The Brouwer degree $\deg(f,\Omega,p)$ for admissible triples (f,Ω,p) satisfies the Axioms (A1)-(A4), i.e. the Brouwer degree is a degree theory.

Proof. In order to verify Axiom (A1) consider the equation $x = p$. Clearly, there exists a unique solution and $J_{\mathrm{Id}}(x) = \mathrm{Id}$, which proves (A1). Axiom (A3) follows from Proposition 1.44 and Remark 1.46.

Since f is continuous and $\overline{\Omega}\backslash(\Omega^1 \cup \Omega^2)$ is compact, we have that

$$|f(x) - p| \geq \delta > 0, \quad \forall x \in \overline{\Omega}\backslash(\Omega^1 \cup \Omega^2).$$

As a consequence the open ball $B_\delta(p) \subset f(\overline{\Omega}\backslash(\Omega^1 \cup \Omega^2))$. If $g \in C^1(\overline{\Omega})$, with $\|f - g\|_{C^0} < \delta/2$, then $|g(x) - p| \geq \delta/2$ and thus $B_{\delta/2}(p) \subset \overline{\Omega}\backslash(\Omega^1 \cup \Omega^2)$. We now have $\deg(f, \Omega, p) = \deg(g, \Omega, p)$.

Choose a regular value $p' \in B_{\delta/2}(p)$ such that $\deg(g, \Omega, p) = \deg(g, \Omega, p')$. By 1.2.a, $\deg(g, \Omega, p') = \deg(g, \Omega^1, p') + \deg(g, \Omega^2, p')$, and $\deg(g, \Omega^i, p') = \deg(g, \Omega^i, p)$, $i = 1, 2$. By the choices of δ and g, $\deg(f, \Omega^i, p) = \deg(g, \Omega^i, p)$, which proves the theorem. ∎

■ **2.3 Remark** For the Brouwer degree for maps from $\mathbb{R}^n$ to $\mathbb{R}^n$, Axioms (A3) and (A4) are equivalent to the following two alternative axiom:

(A3′) for any continuous paths $t \mapsto f_t$, $f_t \in C^0(\overline{\Omega})$ and $t \mapsto p_t$, with $p_t \notin f_t(\partial\Omega)$, i.e. (f_t, Ω, p_t) is admissible for all t, it holds that $\deg(f_t, \Omega, p_t)$ is independent of $t \in [0, 1]$;

If the degree is considered for mappings between manifolds Axiom (A4) need not be well-defined. ■

2.4 Exercise Show that the Axioms (A3) and (A4) combined are equivalent to Axiom (A3′). ■

2.1.a Properties of degree theories

The above theorem shows that there exists a degree theory satisfying Axioms (A1)-(A4); the Brouwer degree. The remainder of this section is a list of properties that are derived from Axioms (A1)-(A4) and a proof that the Brouwer degree is the only degree satisfying (A1)-(A4).

The axiomatic approach can also be used for degree theories in other contexts such as infinite dimensional spaces as we will discuss in the next chapter.

2.5 Theorem — Validity of the degree. If $p \notin f(\overline{\Omega})$, then $\deg(f, \Omega, p) = 0$. Conversely, if $\deg(f, \Omega, p) \neq 0$, then there exists a $x \in \Omega$, such that $f(x) = p$.

Proof. By chosing $\Omega_1 = \Omega$ and $\Omega_2 = \varnothing$ it follows from Axiom (A2) that $\deg(f, \varnothing, p) = 0$. Now take $\Omega_1 = \Omega_2 = \varnothing$ in Axiom (A2), then $\deg(f, \Omega, p) = 2 \cdot \deg(f, \varnothing, p) = 0$.

Suppose that there exists no $x \in \Omega$, such that $f(x) = p$, i.e. $f^{-1}(p) = \varnothing$. Since $p \notin f(\partial\Omega)$, it follows that $p \notin f(\overline{\Omega})$, and thus $\deg(f, \Omega, p) = 0$, a contradiction. ∎

2.6 Property (Continuity of the degree) The degree $\deg(f,\Omega,p)$ is continuous in f, i.e. there exists a $\delta = \delta(p,f) > 0$, such that for all g satisfying $\|f-g\|_{C^0} < \delta$, it holds that $p \notin g(\partial\Omega)$, and $\deg(g,\Omega,p) = \deg(f,\Omega,p)$.

Proof. See Proposition 1.43. ∎

2.7 Property (Dependence on path components) The degree only depends on the path components $D \subset \mathbb{R}^n \backslash f(\partial\Omega)$, i.e. for any two points $p,q \in D \subset \mathbb{R}^n \backslash f(\partial\Omega)$ it holds that $\deg(f,\Omega,p) = \deg(f,\Omega,q)$. For any path component $D \subset \mathbb{R}^n \backslash f(\partial\Omega)$ this justifies the notation $\deg(f,\Omega,D)$.

Proof. Let p and q be connected by a path $t \mapsto p_t$ in D, then by Axiom (A4) the degree $\deg(f,\Omega,p_t)$ is constant in $t \in [0,1]$. ∎

2.8 Property (Translation invariance) The degree is invariant under translation, i.e. for any $q \in \mathbb{R}^n$ it holds that $\deg(f-q,\Omega,p-q) = \deg(f,\Omega,p)$.

Proof. The degree $d(f-tq,\Omega,p-tq)$ is well-defined for all $t \in [0,1]$. Indeed, since $p - tq \notin f(\partial\Omega) - tq$, it follows from Axiom (A3) that $\deg(f-tq,\Omega,p-tq) = \deg(f,\Omega,p)$, for all $t \in [0,1]$. ∎

2.9 Property (Excision) Let $\Lambda \subset \Omega$ be a closed subset in Ω, and $p \notin f(\Lambda)$. Then, $\deg(f,\Omega,p) = \deg(f,\Omega\backslash\Lambda,p)$.

Proof. In Axiom (A2) set $\Omega_1 = \Omega\backslash\Lambda$ and $\Omega_2 = \varnothing$, then $\deg(f,\Omega,p) = \deg(f,\Omega\backslash\Lambda,p) + \deg(f,\varnothing,p) = \deg(f,\Omega\backslash\Lambda,p)$. ∎

2.10 Property (Additivity) Suppose that $\Omega^i \subset \Omega$, $i = 1,\cdots,k$, are disjoint open subsets of Ω, and $p \notin f\big(\overline{\Omega}\backslash(\cup_i\Omega_i)\big)$, then $\deg(f,\Omega,p) = \sum_i \deg(f,\Omega^i,p)$.

Proof. The property holds trivially for $k=1$. Now assume it holds for $k-1$, then by Axiom (A2)

$$\deg(f,\Omega,p) = \deg\left(f,\bigcup_{i=1}^{k-1}\Omega^i,p\right) + \deg(f,\Omega^k,p) = \sum_{i=1}^{k}\deg(f,\Omega^i,p),$$

by the induction hypothesis. ∎

2.11 Exercise Show that the above statement holds true for countable collections of disjoint open subsets Ω_i of Ω.

Let $\Omega \subset \mathbb{R}^n \times [0,1]$ be a bounded and relatively open subset of $\mathbb{R}^n \times [0,1]$ and let $F : \overline{\Omega} \to \mathbb{R}^n$ a continuous function on $\overline{\Omega}$, with $f_t = F(\cdot,t)$, such that

(i) $f_0 = f$, and $f_1 = g$;

(ii) $\Omega_0 = \Omega_f$, and $\Omega_1 = \Omega_g$;

(iii) there exists a continuous path $t \mapsto p_t$, $p_0 = p$ and $p_1 = q$, such that (f_t,Ω_t,p_t) is admissible for all $t \in [0,1]$;

then $(f,\Omega_f,p) \sim (g,\Omega_g,q)$ are homotopic, or corbordant (notation: $(f,\Omega_f,p) \sim (g,\Omega_g,q)$), and (f_t,Ω_t,p_t) is an admissible homotopy.

2.12 Property (Homotopy invariance) For an admissible homotopy (f_t,Ω_t,p_t), the degree $\deg(f_t,\Omega_t,p_t)$ is constant in $t \in [0,1]$.

Proof. Assume without loss of generality that $p_t = p$ for all t. Choose a small ball $B_\epsilon(p)$, then following the reasoning in the proof of Theorem **??**, $f_t^{-1}(B_\epsilon(p)) \subset C_{t_i}$, $t \in (t_i - \delta_i, t_i + \delta_i)$, for finitely many sets $C_{t_i} \subset \Omega$. By excision, Property 2.9, it follows that $\deg(f_t,\Omega_t,p) = \deg(f_t,C_{t_i},p)$, and since the sets $C_{t_i} \times (t_i - \delta_i, t_i + \delta_i)$ form an open covering of $F^{-1}\big(B_\epsilon(p) \times [0,1]\big)$, the degree $\deg(f_t,\Omega_t,p)$ is constant for all $t \in [0,1]$.

By (A4) $\deg(f_t,\Omega_t,p_t) = \deg(f_t - p_t,\Omega_t,0)$. Therefore, without loss of generality, assume that $p_t = p$ is constant.

Choose $\epsilon > 0$ small enough such that $B_\epsilon(p) \subset \mathbb{R}^n \setminus \cup_{t\in[0,1]} f((\partial\Omega)_t)$. As before, write $\deg(f_t,\Omega_t,p) = \int_{\Omega_t} f_t^*\boldsymbol{\omega} = \int_{(\overline{\Omega})_t} f_t^*\boldsymbol{\omega}$, where $\mathrm{supp}(\boldsymbol{\omega}) = \overline{B}_\epsilon(p)$. By assumption, the set $F^{-1}\big(B_\epsilon(p) \times [0,1]\big) \in \Omega$ is compact. At every $t \in [0,1]$, the sets $f_t^{-1}(B_\epsilon(p))$ can be covered by open cylinders $C_t \times (t-\delta, t+\delta) \subset \Omega$. At each t, by compactness and continuity of F, δ can be small enough such that $f_{t'}^{-1}(B_\epsilon(p)) \subset C_{t'} \times (t'-\delta, t'+\delta)$ for all $t' \in (t-\delta,t+\delta)$. For all $t \in [0,1]$ these sets form an open covering of $F^{-1}\big(B_\epsilon(p) \times [0,1]\big)$, which has a finite subcovering, $C_{t_i} \times (t_i - \delta_i, t_i + \delta_i)$, $i = 1,\cdots,k$. Therefore, for given $t' \in (t_i - \delta_i, t_i + \delta_i)$, $\int_{\Omega_t} f_{t'}^*\boldsymbol{\omega} = \int_{f_{t'}^{-1}(B_\epsilon(p))} f_{t'}^*\boldsymbol{\omega} = \int_{C_{t_i}} f_{t'}^*\boldsymbol{\omega}$, which is continuous in t' by Lemma 1.37 and thus constant in t'. Since the sets $C_{t_i} \times (t_i - \delta_i, t_i + \delta_i)$, $i = 1,\cdots,k$, form an open covering, the degree is the same for all $t \in [0,1]$. ∎

2.13 Property (Orientation) Let $A \in \mathrm{GL}(\mathbb{R}^n)$ and Ω any open neighborhood of $0 \in \mathbb{R}^n$, then $\deg(A,\Omega,0) = \mathrm{sign}\ \det(A)$.

Proof. The group $\mathrm{GL}(\mathbb{R}^n)$ consists of two path components GL^+ and GL^-. If $A \in \mathrm{GL}^+$ choose a path $t \mapsto A_t$, connecting Id and A. Clearly, $(A_t,\Omega,0)$ is admissible for all t, and therefore by Axioms (A1) and (A3) it follows that $\deg(A,\Omega,0) = \deg(A_t,\Omega,0) = \deg(\mathrm{Id},\Omega,0) = 1$, which proves the statement for $A \in \mathrm{GL}^+$.

For $A \in \mathrm{GL}^-$ choose a path $t \mapsto A_t$, connecting $R = \mathrm{diag}(-1, 1, \cdots, 1)$ and A. As before, $(A_t, \Omega, 0)$ is admissible for all t, and thus by Axiom (A3) it follows that $\deg(A, \Omega, 0) = \deg(A_t, \Omega, 0) = \deg(R, \Omega, 0)$. It remains therefore to determine $\deg(R, \Omega, 0)$. Consider the homotopy

$$f_t(x) = \left(|x_1| - \frac{1}{2} + t, x_2, \cdots, x_n\right) : (-1, 1) \times \Omega' \to \mathbb{R}^n,$$

where $\Omega' \subset \mathbb{R}^{n-1}$ is an open neighborhood of $0 \in \mathbb{R}^{n-1}$. It is clear that $(f_t, (-1, 1) \times \Omega', 0)$ is admissible for all $t \in [0, 1]$, and thus by Axiom (A3)

$$\deg(f_t, (-1, 1) \times \Omega', 0) = \deg(f_1, (-1, 1) \times \Omega', 0) = 0,$$

since the equation $f_1(x) = 0$ has no solutions (Property 2.5). At $t = 0$, the value 0 is a regular value and the equation $f_0(x) = 0$ has exactly two non-degenerate solutions $x^- = \left(-\frac{1}{2}, 0, \cdots, 0\right)$ and $x^+ = \left(\frac{1}{2}, 0, \cdots, 0\right)$. Choosing two sufficiently small open neighborhoods Ω^- and Ω^+ of x^- and x^+ respectively, Axiom (A2) yields that

$$\deg(f_0, (-1, 1) \times \Omega', 0) = \deg(f_0, \Omega^-, 0) + \deg(f_0, \Omega^+, 0) = 0.$$

For f_0 it holds that $f_0(x) = \left(-x_1 - \frac{1}{2}, x_2, \cdots, x_n\right)$ on Ω^- and $f_0(x) = \left(x_1 - \frac{1}{2}, x_2, \cdots, x_n\right)$ on Ω^+. Set $p = \left(\frac{1}{2}, 0, \cdots, 0\right)$, then by Property 2.8

$$\deg(f_0, \Omega^+, 0) = \deg(\mathrm{Id} - p, \Omega^+, 0) = \deg(\mathrm{Id}, \Omega^+, p) = 1.$$

The latter follows using Property 2.12, with

$$\Omega = \{(x_1 - t/2, x_2, \cdots, x_n) \mid x \in \Omega^+\},$$

and $p_t = \left(\frac{1-t}{2}, 0, \cdots, 0\right)$, i.e. $\deg(\mathrm{Id}, \Omega^+, p) = \deg(\mathrm{Id}, \Omega_1, 0) = 1$ by Axiom (A1). Similarly,

$$\begin{aligned} \deg(f_0, \Omega^-, 0) &= \deg(R - p, \Omega^-, 0) \\ &= \deg(R, \Omega^-, p) = -\deg(f_0, \Omega^+, 0) = -1. \end{aligned}$$

Using the homotopy property (Property 2.8) as above it follows that $\deg(R, \Omega, 0) = -1$. ∎

2.1.b Characterization and uniqueness of degree theories

2.14 Theorem — Uniqueness. If (f, Ω, p) is an admissible triple, with $f \in C^1(\overline{\Omega})$ and p regular, then

$$\deg(f, \Omega, p) = \sum_{x \in f^{-1}(p)} \mathrm{sign}\Big(J_f(x)\Big).$$

For an admissible triple in general there exists an admissible triple (g, Ω, q) with $g \in C^1(\overline{\Omega})$ and q regular, which homotopy to (f, Ω, p). Moreover, $\deg(f, \Omega, p) =$

$\deg(g,\Omega,q)$.

Proof. For a regular value p is the inverse image $f^{-1}(p) = \{x^j\}$ is a finite set in Ω (see Lemma 1.21). For $\epsilon > 0$ sufficiently small, $f^{-1}(B_\epsilon(p))$ consists of disjoint homeomorphic balls $N_\epsilon(x^j) \subset \Omega$. By the additivity (Property 2.10)

$$\deg(f,\Omega,p) = \sum_j \deg(f,N_\epsilon(x^j),p).$$

If $\epsilon > 0$ is chosen small enough then

$$\deg(f,N_\epsilon(x^j),p) = \deg(f'(x^j),B_{\epsilon'}(0),0) = \operatorname{sign}\Big(J_f(x^j)\Big),$$

which proves the first statement. The latter identity can be proved as follows. By assumption $f'(x^j)$ is invertible for all j. Define the homotopy $f_t = (1-t)f + tp + tf'(x^j)(x - x^j)$, then for $x \in N_\epsilon(x^j)$ it holds that $f_t - p = f'(x^j)(x - x^j) + (1 - t)R(x,x^j)$, and $\|R\| = o(\|x - x^j\|)$, for $\|x - x^j\|$ sufficiently small. This gives the estimate

$$\begin{aligned} \|f_t(x) - p\| &\geq \|f'(x^j)(x - x^j)\| - (1-t)\|R(x,x^j)\| \\ &\geq C\|x - x^j\| - o(\|x - x^j\|), \end{aligned}$$

and thus $\|f_t - p\| > 0$, for all $t \in [0,1]$, provided $\|x - x^j\| = \epsilon'$ is small enough. Using Axiom (A3) it follows that $\deg(f,N_\epsilon(x^j),p) = \deg(f'(x^j)(x - x^j),B_{\epsilon'}(x^j),p)$. Now use the Properties 2.12 and 2.13, as in the previous proof, to show that

$$\deg(f'(x^j)(x - x^j),B_{\epsilon'}(x^j),p) = \deg(f'(x^j),B_{\epsilon'}(0),0) = \operatorname{sign}\Big(J_f(x^j)\Big).$$

In Section 1.4 it was proved that for each admissible triple (f,Ω,p) there exists an admissible triple (g,Ω,q), with $g \in C^1(\overline{\Omega})$ and q regular, so that $(f,\Omega,p) \sim (g,\Omega,q)$. Then, by Axiom (A3), $\deg(f,\Omega,p) = \deg(g,\Omega,q)$. ∎

Theorem 2.14 shows that the Brouwer degree is the unique degree theory satisfying Axioms (A1)-(A4).

(Composition) Let $f \in C^0(\overline{\Omega})$, $g \in C^0(\overline{\Lambda})$, with $f(\Omega) \subset \Lambda$ and Ω and Λ both bounded and open. Let D_i be the path components of $\Lambda \backslash f(\partial\Omega)$. Assume that $p \notin g(\partial\Lambda) \cup g(f(\partial\Omega))$, then

$$\deg(g \circ f,\Omega,p) = \sum_i \deg(g,D_i,p) \cdot \deg(f,\Omega,D_i),$$

which is finite sum.

Identify $\mathbb{R}^n$ with $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$, and let $\Omega_1 \subset \mathbb{R}^{n_1}$, and $\Omega_2 \subset \mathbb{R}^{n_2}$, be open and bounded subsets. (Cartesian product) Let (f,Ω_1,p) and (g,Ω_2,q), with $f \in C^0(\overline{\Omega}_1)$, and $g \in C^0(\overline{\Omega}_2)$, be admissible triples. Then $(f \times g,\Omega_1 \times \Omega_2,p \times q)$ is admissible, and $\deg(f \times g,\Omega_1 \times \Omega_2,p \times q) = \deg(f,\Omega_1,p) \cdot \deg(g,\Omega_2,q)$.

Proof. By Theorem 2.14 it suffices to prove this statement for C^1-functions f and g, and regular values p and q respectively. The product $f \times g$ is also C^1, $p \times q$ a regular value, and $(f \times g)^{-1}(p \times q) = f^{-1}(p) \times g^{-1}(q) = \{\xi^i, \zeta^j\}_{i,j}$. For the degree this yields

$$
\begin{aligned}
\deg((f \times g, \Omega_1 \times \Omega_2, p \times q) &= \sum_{i,j} \operatorname{sign}\Big(J_{f\times g}(\xi^i, \zeta^j)\Big) \\
&= \left[\sum_i \operatorname{sign}\Big(J_f(\xi^i)\Big)\right] \cdot \left[\sum_j \operatorname{sign}\Big(J_g(\zeta^j)\Big)\right],
\end{aligned}
$$

since for $f \times g$ it holds that $J_{f\times g}(\xi^i, \zeta^j) = J_f(\xi^i) \cdot J_g(\zeta^j)$, which completes the proof of Property 2.1.b. ∎

In **??** we explained how the mapping degree is defined for mappings on a vectorspace V. The Brouwer degree extends to mappings on V by using approximation of C^1-functions.

> **2.15 Exercise** Carry out the construction of the Brouwer mapping degree for continuous mappings on a real linear vectorspace V. ▪

2.1.c The mapping degree and homology

In the Sect. 1.4 we introduced the Brouwer degree for continuous mappings via approximation with differentiable functions and showing that the definition is independent of the chosen approximation. In section we follow a more abstract approach towards the mapping degree which does not require approximation but is less intuitive. We prove that these concepts lead to the same notion of degree. In Remark 1.35 we already showed the connection between the C^1-mapping degree and the de Rahm cohomology. A similar approach can be followed for the Brouwer degree.

Let us start with the notion of mapping degree for mappings between closed[1] orientable smooth manifolds of dimension n. Let $f\colon M \to M'$ be a continuous mapping and let $H_n(M;\mathbb{Z})$ and $H_n(M';\mathbb{Z})$ be the singular homologies of M and M' in degree n. For closed orientable n-manifolds we have $H_n(M) \cong \mathbb{Z}$ and $H_n(M') \cong \mathbb{Z}$, eg. [22]. Since the latter groups are infinite cyclic the induced homomorphism in singular homology is given by $f_*(\alpha) = d \cdot \alpha$, $d \in \mathbb{Z}$, where $\alpha \in H_n(M)$ is a generator. We define $\deg(f) := d$. In particular this construction defines the degree for $f\colon S^n \to S^n$, see Sect. 2.2.a.

> **2.16 Exercise** Show that the above definition of mapping degree does not depend on the choice of orientation. ▪

[1] By definition a manifold has no boundary. A manifold is closed if it is a compact space.

Using the mapping degree for mappings between spheres we can now given an alternative definition of the Brouwer degree. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set $n > 0$. As before let $f \colon \overline{\Omega} \subset \mathbb{R}^n \to \mathbb{R}^n$ be a continuous mapping and let $p \in \mathbb{R}^n$ such that $p \notin f(\partial\Omega)$. This implies that $f^{-1}(p) \subset \Omega$ is compact. Via the stereographic projection we have the canonical embedding $\pi^{-1} \colon \overline{\Omega} \to S^n \subset \mathbb{R}^{n+1}$ which makes $\pi^{-1}(\overline{\Omega})$ a subset of S^n. For notational convenience we will write $\overline{\Omega} \subset S^n$ and $f \colon \overline{\Omega} \subset S^n \to S^n$.[2] The inclusion $j \colon (S^n, \varnothing) \subset (S^n, S^n \setminus f^{-1}(p))$ yields the injection (monomorphism) $\mathbb{Z} \cong H_n(S^n) \to H_n(S^n, S^n \setminus f^{-1}(p))$. Since $S^n \setminus \Omega$ is closed and contained in the open set $S^n \setminus f^{-1}(p)$, the excision property of homology yields the isomorphism $H_n(S^n, S^n \setminus f^{-1}(p)) \cong H_n(\Omega, \Omega \setminus f^{-1}(p))$. The mapping f yields the homomorphism $f_* \colon H_n(\Omega, \Omega \setminus f^{-1}(p)) \to H_n(S^n, S^n \setminus \{p\}) \cong \tilde{H}_n(S^n) = H_n(S^n) \cong \mathbb{Z}$. Combining the above homomorphisms we obtain

$$\mathbb{Z} \xrightarrow{j_*} H_n(\Omega, \Omega \setminus f^{-1}(p)) \xrightarrow{f_*} H_n(S^n, S^n \setminus \{p\}) \cong \mathbb{Z}.$$

As before we define $f_*(j_*(1)) = d(f, \Omega, p)$ as the *local* mapping degree. It remains to show that the local degree is in fact the Brouwer degree.

■ **2.17 Remark** From excision is also follows that $H_n(S^n, S^n \setminus \{p\}) \cong H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{p\})$, which implies that we can consider the mapping $f \colon (\Omega, \Omega \setminus f^{-1}(p)) \to (\mathbb{R}^n, \mathbb{R}^n \setminus \{p\})$ in homology

$$\mathbb{Z} \xrightarrow{j_*} H_n(\Omega, \Omega \setminus f^{-1}(p)) \xrightarrow{f_*} H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{p\}) \cong \mathbb{Z}.$$

The above construction is a way to determine a generator in $H_n(\Omega, \Omega \setminus f^{-1}(p))$. ■

The easiest way to show that $d(f, \Omega, p)$ is in fact the Brouwer degree is to use the results from Theorems 2.2 and 2.14. We therefore need to verify the Axioms (A1)-(A4) which define a degree theory, Definition 2.1. For this we follow the approach presented by A. Dold, cf. [9].

2.18 Theorem Under the above hypotheses $d(f, \Omega, p) = \deg(f, \Omega, p)$.

Proof. If $f = \mathrm{id}$, then f_* defines a monomorphism and therefore $f_*(1) = 1 = d(f, \Omega, p)$ which verifies Axiom (A1).

Let $t \mapsto f_t$ be a homotopy as described in Axiom (A3). Let $K \subset S^n$ be a compact set such that $f^{-1}(p) \subset K \subset U \subset \Omega$. Then, from the properties of singular homology (excision and inclusion) we derive the following commuting diagram:

$$\begin{tikzcd}
H_n(S^n)) \arrow[r, "j_*"] \arrow[dr, "j_*"'] & H_n(S^n, S^n \setminus f_t^{-1}(p)) \arrow[r, leftrightarrow, "\cong"] & H_n(\Omega, \Omega \setminus f_t^{-1}(p)) \arrow[r, "(f_t)_*"] & H_n(S^n, S^n \setminus \{p\}) \\
& H_n(S^n, S^n \setminus K) \arrow[u] \arrow[r, leftrightarrow, "\cong"] & H_n(U, U \setminus K). \arrow[u] \arrow[ur, "(f_t)_*"] &
\end{tikzcd}$$

[2] The mapping f should be regarded as the mapping $\tilde{f} = \pi^{-1} \circ f \circ \pi$.

(2.1.1)

Let $U = \Omega$, then we derive that the composition

$$H_n(S^n) \to H_n(S^n, S^n \setminus K) \cong H_n(\Omega, \Omega \setminus K) \to H_n(S^n, S^n \setminus \{p\}) \cong H_n(S^n),$$

also defines the degree $d(f, \Omega, p)$. Since $f_t\colon (\Omega, \Omega \setminus K) \to (S^n, S^n \setminus \{p\})$ defines a homotopy between the mappings f_0 and f_1, the homotopy invariance of singular homology implies that $(f_0)_* = (f_1)_*$. This proves that $d(f_0, \Omega, p) = d(f_1, \Omega, p)$, which complete the verification of Axiom (A3).

The shift $x \mapsto x - p$ yields the mapping $f - p\colon (\Omega, \Omega \setminus f^{-1}(p)) \to (S^n, S^n \setminus \{0\})$ and the commuting diagram

$$\begin{tikzcd}
H_n(\Omega, \Omega \setminus f^{-1}(p)) \arrow[r, "f_*"] \arrow[d, leftrightarrow, "\mathrm{id}_*"'] & H_n(S^n, S^n \setminus \{p\}) \arrow[d, leftrightarrow, "\cong"] \\
H_n(\Omega, \Omega \setminus f^{-1}(p)) \arrow[r, "(f-p)_*"] & H_n(S^n, S^n \setminus \{0\})
\end{tikzcd}$$

which implies that $d(f, \Omega, p) = d(f - p, \Omega, 0)$ and verifies Axiom (A4).

Finally we show that Axiom (A2) is satisfied. Suppose $p \notin f(\overline{\Omega} \setminus (\Omega^1 \cup \Omega^2))$, then by the commuting diagram in (2.1.1), $d(f, \Omega, p) = d(f, \Omega^1 \cup \Omega^2, p)$, where $\Omega^1 \cap \Omega^2 = \varnothing$. It remains to show the additivity. Let $f_i = f|_{\Omega^i}$, $\widetilde{\Omega} = \Omega^1 \cup \Omega^2$ and consider the diagrams

$$\begin{tikzcd}
H_n(S^n) \arrow[r] \arrow[d, "\widetilde{\mathrm{id}}"'] & H_n(S^n, S^n \setminus f^{-1}(p)) \arrow[r, leftrightarrow, "\cong"] \arrow[d, "i_*"'] & H_n(\widetilde{\Omega}, \widetilde{\Omega} \setminus f^{-1}(p)) \arrow[d, leftrightarrow, "\cong"] \\
H_n(S^n) \oplus H_n(S^n) \arrow[r] & \bigoplus_{i=1,2} H_n(S^n, S^n \setminus f_i^{-1}(p)) \arrow[r, leftrightarrow, "\cong"] & \bigoplus_{i=1,2} H_n(\Omega^i, \Omega^i \setminus f_i^{-1}(p))
\end{tikzcd}$$

where $\widetilde{\mathrm{id}}$ is given by $\alpha \mapsto (\alpha, \alpha)$, and i_* is induced by the inclusions $(S^n, S^n \setminus f^{-1}(p)) \subset (S^n, S^n \setminus f_i^{-1}(p))$. Moreover,

$$\begin{tikzcd}
H_n(\widetilde{\Omega}, \widetilde{\Omega} \setminus f^{-1}(p)) \arrow[r, "f_*"] \arrow[d, leftrightarrow, "\cong"'] & H_n(S^n, S^n \setminus \{p\}) \arrow[r, leftrightarrow, "\cong"] & H_n(S^n) \\
\bigoplus_{i=1,2} H_n(\Omega^i, \Omega^i \setminus f_i^{-1}(p)) \arrow[r, "(f_1)_* \oplus (f_2)_*"] & \bigoplus_{i=1,2} H_n(S^n, S^n \setminus \{p\}) \arrow[r, leftrightarrow, "\cong"] \arrow[u, "\mathrm{id}_* + \mathrm{id}_*"] & H_n(S^n) \oplus H_n(S^n) \arrow[u, "\mathrm{id}_* + \mathrm{id}_*"']
\end{tikzcd}$$

where $\mathrm{id}_* + \mathrm{id}_*$ represents the mapping $(\alpha, \beta) \mapsto \alpha + \beta$. If the diagrams commutes then the additivity follows. Indeed, the top line yields $\alpha \mapsto d(f, \widetilde{\Omega}, p)\alpha$ and the bottom line yields $(\alpha,\beta) \mapsto \big(d(f^1,\Omega^1,p)\alpha, d(f^2,\Omega^2,p)\beta\big)$. By the commutativity we have

$$\begin{aligned}\alpha \mapsto (\alpha,\alpha) \mapsto \big(d(f^1,\Omega^1,p)\alpha, d(f^2,\Omega^2,p)\alpha\big) &\mapsto \big(d(f^1,\Omega^1,p) + d(f^2,\Omega^2,p)\big)\alpha \\ &= d(f,\widetilde{\Omega},p)\alpha = d(f,\Omega,p)\alpha,\end{aligned}$$

which verifies (A2). We now show that the diagrams commute. Consider the diagram

$$\begin{tikzcd} H_n(S^n, S^n \setminus f^{-1}(p)) \arrow[r, leftrightarrow, "\cong"] \arrow[d, "(i_i)_*"'] & H_n(\widetilde{\Omega}, \widetilde{\Omega} \setminus f^{-1}(p)) \\ H_n(S^n, S^n \setminus f_i^{-1}(p)) & H_n(\Omega^i, \Omega^i \setminus f_i^{-1}(p)) \arrow[l, "(\pi_i^{-1})_*"'] \arrow[u, "(\iota_i)_*"'] \end{tikzcd}$$

where $\iota_i \colon \Omega^i \to \widetilde{\Omega}$ and $\pi_i^{-1} \colon \Omega^i \to S^n$ are inclusion and the diagram commutes. The second option is the diagram

$$\begin{tikzcd} H_n(S^n, S^n \setminus f^{-1}(p)) \arrow[r, leftrightarrow, "\cong"] \arrow[d, "(i_j)_*"'] & H_n(\widetilde{\Omega}, \widetilde{\Omega} \setminus f^{-1}(p)) \\ H_n(S^n, S^n \setminus f_j^{-1}(p)) & H_n(\Omega^i, \Omega^i \setminus f_i^{-1}(p)) \arrow[l, "(\pi_i^{-1})_*"'] \arrow[u, "(\iota_i)_*"'] \end{tikzcd}$$

and since $\pi_i^{-1} \colon \Omega^i \to S^n \setminus f_j^{-1}(p)$, the induced mapping $(\pi_i^{-1})_* = 0$. Combing these composition makes that the diagrams above commute. ∎

■ **2.19 Remark** If we consider a bounded domain Ω with a smooth boundary $\partial\Omega$ then the commuting diagram in (2.1.1) and the fact that $\partial\Omega$ allows a normal vector field yield that the Brouwer degree is given by the mapping

$$\mathbb{Z} \xrightarrow{j_*} H_n(\overline{\Omega}, \partial\Omega) \xrightarrow{f_*} H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{p\}) \cong \mathbb{Z}.$$

Compare Remark 2.23. ■

2.2 Boundary dependence of the degree

The homotopy invariance of the Brouwer degree can be used to prove that the degree on depends only on the restriction of f to the boundary.

2.20 Proposition Let $\varphi : \partial\Omega \to \mathbb{R}^n \setminus \{p\}$ be a continuous mapping. Then, for any two continuous extensions[a] $f, g \in C^0(\overline{\Omega})$, such that

$$f|_{\partial\Omega} = g|_{\partial\Omega} = \varphi,$$

it holds that $\deg(f,\Omega,p) = \deg(g,\Omega,p)$ and therefore the Brouwer degree only depends on the restriction $f|_{\partial\Omega}$ to $\partial\Omega$.

[a] Continuous extensions exist by virtue of Tietze's Extension Theorem, see Appendix A.2, Theorem A.8

Proof. Consider the homotopy $h_t = (1-t)f + tg$, $t \in [0,1]$. Then, since $f = g = \varphi$ on $\partial\Omega$, it holds that $h_t = \varphi$ on $\partial\Omega$ for all $t \in [0,1]$ and therefore $p \notin h_t(\partial\Omega) = \varphi(\partial\Omega)$ for all $t \in [0,1]$. Consequently, (h_t,Ω,p) is an admissible homotopy and by the homotopy invariance of the degree $\deg(h_t,\Omega,p)$ is independent of $t \in [0,1]$. ■

Proposition 2.20 makes it possible to define a degree theory for continuous mappings on compact sets that occur as boundaries of bounded open sets in $\mathbb{R}^n$. Continuous mappings from $\partial\Omega$ to $\mathbb{R}^n$ cannot be surjective.[3] Therefore assume, without loss of generality, that maps φ act from $\partial\Omega$ to $\mathbb{R}^n \setminus \{p\}$ for some $p \notin \varphi(\partial\Omega)$.

2.21 Definition Let $\varphi : \partial\Omega \to \mathbb{R}^n \setminus \{p\}$ be a continuous mapping. The mapping degree is defined by

$$W_{\partial\Omega}(\varphi,p) := \deg(f,\Omega,p),$$

for any continuous extension[a] $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$, with $f|_{\partial\Omega} = \varphi$. The mapping degree for mappings $\varphi : \partial\Omega \to \mathbb{R}^n \setminus \{p\}$ may be regarded as a generalized notion of the winding number.

[a] A continuous extension of φ to $\overline{\Omega}$ always exists by virtue of Tietze's Extension Theorem A.8.

2.2.a Generalized winding numbers

From the translation property of the degree (see Property 2.8) it follows that $\deg(f,\Omega,p) = \deg(f-p,\Omega,0)$, which implies that $W_{\partial\Omega}(\varphi,p) = W_{\partial\Omega}(\varphi-p,0)$. Define the normalized mapping

$$\psi := \frac{\varphi - p}{|\varphi - p|} : \partial\Omega \to S^{n-1}, \tag{2.2.2}$$

where $S^{n-1} \subset \mathbb{R}^n$ denotes the standard unit sphere in $\mathbb{R}^n$. Consider the homotopy $\zeta_t = (1-t)[\varphi - p] + t\psi$, then by Tietze's Extension Theorem there exists a continuous homotopy h_t on $\overline{\Omega} \times [0,1]$ with $h_t|_{\partial\Omega} = \zeta_t$. From the homotopy property of the degree it then follows that $W_{\partial\Omega}(\varphi,p) = \deg(h_0,\Omega,0) = \deg(h_1,\Omega,0) =: \deg(\psi,\partial\Omega,S^{n-1})$, and thus

$$\deg(\psi) := \deg\bigl(\psi,\partial\Omega,S^{n-1}\bigr) = W_{\partial\Omega}(\varphi - p,0),$$

which defines the degree for ψ. Homotopy defines an equivalence relation on mappings φ, called the homotopy type, and the degree only depends on the homotopy type of the map.

[3] The continuous image of a compact set is compact!

In order to derive an integral formula for the degree $\deg(\psi)$ assume now that f is a C^1-mapping on $\overline{\Omega}$ and $\partial\Omega$ is a piecewise C^1-boundary.

2.22 Theorem Let $\boldsymbol{\mu} \in \Gamma^{n-1}(S^{n-1})$, with $\int_{S^{n-1}} \boldsymbol{\mu} = 1$, and let ψ be as defined above. Then

$$\deg(\psi) = \int_{\partial\Omega} \psi^* \boldsymbol{\mu},$$

where $\psi^*\boldsymbol{\mu} \in \Gamma^{n-1}(\partial\Omega)$.

Proof. Let f be as before, and since $\deg(f,\Omega,p)$ does not depend on p, choose p to be a regular value, so that $f^{-1}(p) = \{x^j\}$ is a finite set. Let $B_\epsilon(p)$ be a sufficiently small ball in $\mathbb{R}^n \backslash f(\partial\Omega)$ such that $\overline{N} \subset \Omega$, where $N = f^{-1}(B_\epsilon(p))$. Since p is regular, $N = \cup_j N^j$ (finite), where the sets N^j are all mutually disjoint and diffeomorphic to $B_\epsilon(p)$. Consider the disjoint open sets $\Lambda = \Omega\backslash\overline{N} \subset \Omega$ and $N \subset \Omega$. Then $p \notin f\big(\overline{\Omega}\backslash(\Lambda \cup N)\big)$ and by Property 2.10 $\deg(f,\Omega,p) = \deg(f,\Lambda,p) + \deg(f,N,p)$. Since $p \notin f\big(\overline{\Lambda}\big)$ it follows from Property 2.5 that $\deg(f,\Lambda,p) = 0$ and therefore $\deg(f,\Omega,p) = \deg(f,N,p)$. According to the Definition in 1.5 and Property 2.10 the degree on N is given by $\deg(f,N,p) = \sum_j \deg(f,N^j,p)$and $\deg(f,N^j,p) = \operatorname{sign}\Big(J_f(x^j)\Big)$.

The mapping ψ has a C^1-extension to Λ denoted by Ψ and given by the formula in (2.2.2), i.e. $\Psi = (f-p)/|f-p|$. The restrictions to $\partial\Lambda$ is again denoted by ψ and the restriction to ∂N^j by ψ^j. Choose an $(n-1)$-form $\boldsymbol{\mu} \in \Gamma^{n-1}(S^{n-1})$, with $\int_{S^{n-1}} \boldsymbol{\mu} = 1$, which can be regarded as the restriction of $(n-1)$-form $\boldsymbol{\mu}$ on an open neighborhood of S^{n-1} in $\mathbb{R}^n$. The orientation on Λ induces the Stokes orientation[4] on $\partial\Lambda = \partial\Omega - \partial N$. By Stokes' Theorem

$$\int_{\partial\Lambda} \psi^*\boldsymbol{\mu} = \int_{\partial\Omega} \psi^*\boldsymbol{\mu} - \int_{\partial N} \psi^*\boldsymbol{\mu} = \int_\Lambda d(\Psi^*\boldsymbol{\mu}) = \int_\Lambda \Psi^* d\boldsymbol{\mu} = 0.$$

The latter follows from the fact that $\Psi^* d\boldsymbol{\mu} = 0$. Indeed,

$$(\Psi^* d\boldsymbol{\mu})_x(\xi^1, \cdots, \xi^n) = d\boldsymbol{\mu}_{f(x)}(\Psi_*\xi^1, \cdots, \Psi_*\xi^n) = 0,$$

since the set of tangent vectors $\big\{(\Psi_*\xi^1, \cdots, \Psi_*\xi^n\big\}$ are linearly dependent. Consequently,

$$\int_{\partial\Omega} \psi^*\boldsymbol{\mu} = \int_{\partial N} \psi^*\boldsymbol{\mu} = \sum_j \int_{\partial N^j} (\psi^j)^*\boldsymbol{\mu}.$$

Since $f|_{N^j}$ is a C^1-change of variables that is either orientation preserving or reversing, the same holds for the renormalized restrictions ψ^j via the Stokes

[4]The Stokes orientation is the induced orientation on the boundary using the outward pointing normal. The orientation of ∂N induced by Λ is opposite the orientation induced by N. This explains the notation $\partial\Omega - \partial N$.

orientation of ∂N^j. This yields the following identity

$$\int_{\partial N^j} (\psi^j)^* \boldsymbol{\mu} = \pm \int_{S^{n-1}} \boldsymbol{\mu} = \pm 1 = \operatorname{sign}\Big(J_f(x^j)\Big) = \deg(f, N^j, p).$$

Combining these identities finally gives

$$\begin{aligned} \int_{\partial\Omega} \psi^* \boldsymbol{\mu} &= \sum_j \int_{\partial N^j} (\psi^j)^* \boldsymbol{\mu} = \sum_j \deg(f, N^j, p) = \deg(f, N, p) \\ &= \deg(f, \Omega, p) = \deg(\psi), \end{aligned}$$

which proves the theorem. ■

■ **2.23 Remark** The integral representation can also be used to compute the degree of φ as defined in Definition 2.21. Let $\boldsymbol{\mu} \in \Gamma^{n-1}(\mathbb{R}^n \setminus \{p\})$ with $\int_{\partial\Omega} \boldsymbol{\mu} = 1$. Then $W_{\partial\Omega}(\varphi, p) = \int_{\partial\Omega} \varphi^* \boldsymbol{\mu}$. ■

■ **2.24 Remark** In the case that Ω and Ω' are bounded domains with smooth boundary and $f(\partial\Omega) \subset \partial\Omega'$ we have the following commuting diagram. By the latter condition f is a mappings of pairs, i.e. $f : (\overline{\Omega}, \partial\Omega) \to (\overline{\Omega}', \partial\Omega')$ and

$$\begin{array}{ccccccc} H^n_c(\Omega) & \xleftarrow{\cong} & H^n(\overline{\Omega}, \partial\Omega) & \xleftarrow{f^*} & H^n(\overline{\Omega}', \partial\Omega') & \xleftarrow{\cong} & H^n_c(\Omega') \\ & & \cong\big\downarrow & & \big\downarrow\cong & & \\ & & H^{n-1}(\partial\Omega) & \xleftarrow{\psi^*} & H^{n-1}(\partial\Omega') & & \end{array}$$

where $\psi = f|_{\partial\Omega}$. From this diagram it follows that $[f^*\boldsymbol{\omega}] = \deg(f)[\boldsymbol{\omega}]$ and $[\psi^*\boldsymbol{\theta}] = \deg(\psi)[\boldsymbol{\theta}]$ and thus $\deg(f) = \deg(\psi)$. In Section 2.2 we will come back to the boundary dependence of the degree. See also Sect. 2.1.c for a more detailed account on the algebraic topology. ■

2.2.b Winding numbers in the plane

Let $\Omega = B_1(0) \subset \mathbb{R}^2$, and let $f : B_1(0) \subset \mathbb{R}^2 \to \mathbb{R}^2$ be a continuous mapping. If $0 \neq f(\partial B_1)$, then the degree $\deg(f, B_1, 0)$ is well-defined. Denote the restriction of f to $\partial B_1 = S^1$ by φ, and the renormalization by $\psi = \frac{\varphi}{|\varphi|} : S^1 \to S^1$. Then, $\deg(\psi) = \deg(f, B_1, 0)$. The degree of ψ can be expressed as follows

$$\deg(\psi) = \int_{S^1} \psi^* \boldsymbol{\mu},$$

where $\boldsymbol{\mu}$ is a 1-form on S^1. The 1-forms on S^1 can be expressed as $\boldsymbol{\mu} = (c + h(\theta))d\theta$, where h is 2π-periodic. Via polar coordinates $x_1 = r\cos(\theta)$, $x_2 = r\sin(\theta)$, $\boldsymbol{\mu}$ extends to $\mathbb{R}^2 \backslash \{(0,0)\}$ and is given by

$$\boldsymbol{\mu} = c\frac{-x_2 dx_1 + x_1 dx_2}{x_1^2 + x_2^2} + dh(x_1, x_2).$$

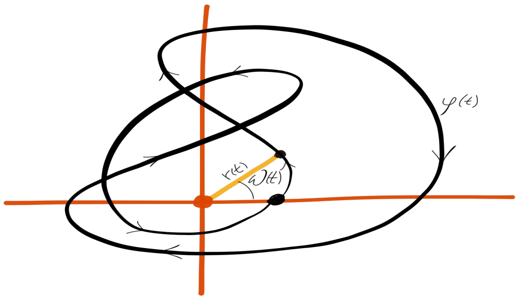


Figure 2.1: The curve represented by $\varphi(t)$ unwinds in $\mathbb{R}^2$. Polar coordinates are denoted by $r(t)$ and $\omega(t)$, which establishes the winding number.

By taking $c = 1/2\pi$ it follows that $\int_{S^1} \boldsymbol{\mu} = 1$. Let

$$\boldsymbol{\theta} = \frac{1}{2\pi} \frac{-x_2 dx_1 + x_1 dx_2}{x_1^2 + x_2^2}$$

be the standard 'volume' form on S^1. A direct calculation shows that $\int_{S^1} \psi^* \boldsymbol{\theta} = \int_{S^1} \varphi^* \boldsymbol{\theta}$ and therefore

$$W(\varphi, 0) := \frac{1}{2\pi} \int_{S^1} \varphi^* \boldsymbol{\theta} = \deg(\varphi) = \deg(\psi), \tag{2.2.3}$$

which is called the winding number φ about 0.

Conversely, starting with a mapping $\varphi : S^1 \to S^1$, Tietze's extension theorem yields that for any extension f to $B_1(0)$ the degree $\deg(f, B_1(0), 0)$ is given by the winding number defined in (2.2.3).

It remains to show that (2.2.3) represents the classical winding number for piecewise smooth curves in $\mathbb{R}^2$. The mapping $\varphi\colon S^1 \to \mathbb{R}^2 \setminus \{0\}$ represents a closed curve in the plane which avoids the origin. Assume without loss of generality that the curve is smooth. In terms of coordinates this yields

$$\varphi(t) = (\xi(t), \eta(t)), \quad t \in \mathbb{R},$$

where $\varphi(t + 2\pi) = \varphi(t)$ for all t and $r^2(t) := \xi^2(t) + \eta^2(t) \geq \delta >$ for all t. Via polar coordinates $\xi(t) = r(t)\cos(\omega(t))$ and $\eta(t) = \sin(\omega(t))$ define the winding number as:

$$W := \frac{\omega(2\pi) - \omega(0)}{2\pi}. \tag{2.2.4}$$

Since $\varphi(t + 2\pi) = \varphi(t)$ for all t if follows that $\omega(2\pi) = \omega(0) + 2k\pi$, $k \in \mathbb{Z}$, see Fig. 2.1. This implies that the winding number is an integer.

The normalized mapping $\psi\colon S^1 \to S^1$ is expressed in coordinates as follows:

$$\psi(t) = \big(\cos(\omega(t)), \sin(\omega(t))\big), \quad t \in \mathbb{R}.$$

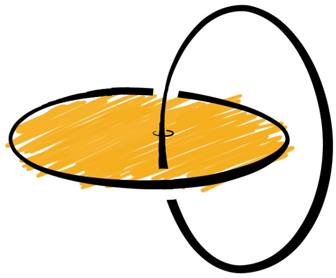


Figure 2.2: Two linking embeddings of S^1. One circle intersects any 'filing' of the other circle, which yields a non-zero linking number.

Now $\psi(t)$ represents a parametrized unit circle in $\mathbb{R}^2$ centered at the origin. Consider the integral

$$\begin{aligned} \deg(\varphi) = \deg(\psi) &= \frac{1}{2\pi}\int_{S^1}\psi^*\boldsymbol{\theta} = \frac{1}{2\pi}\int_0^{2\pi}(-\eta\xi' + \xi\eta')dt \\ &= \frac{1}{2\pi}\int_0^{2\pi}\omega'(t)dt = \frac{\omega(2\pi)-\omega(0)}{2\pi}, \end{aligned}$$

which show that the mapping degree is equal to the planar winding number defined in (2.2.4).

2.3 Linking numbers

The usual example of linking are two tangled closed loops in $\mathbb{R}^3$, but also the winding of a closed loop around a point in the plane is an example of linking in $\mathbb{R}^2$. Similarly, a compact orientable surface in $\mathbb{R}^3$ separating the inside from the outside is an example of lining in $\mathbb{R}^3$. The concept of linking can be formulated in terms of degree degree for objects of higher dimension as well.

Let $K, L \subset \mathbb{R}^n$ be smooth embedded manifolds of dimension k and ℓ respectively. Assume that both K and L are compact and orientable. Moreover $K \cap L = \varnothing$ and $k + \ell = n - 1$. Define the mapping

$$\Psi : K \times L \subset \mathbb{R}^{2n} \to S^{n-1} \subset \mathbb{R}^n, \quad (x,y) \mapsto \frac{y-x}{|y-x|},$$

which is a continuous mapping between orientable manifolds. The orientation on $K \times L$ is the product orientation and the orientation on S^{n-1} the orientation induced by $\mathbb{R}^n$.

2.25 Definition For two disjoint, smoothly embedded compact and orientable submanifolds K and L in $\mathbb{R}^n$, the linking number is defined by

$$\operatorname{link}(K,L) := \deg(\Psi),$$

provided that $k + \ell = n - 1$.

For the traditional linking of embedded circles in $\mathbb{R}^3$ we can compute some simple examples, see Fig. 2.2.

2.26 Example Consider embedded circles K and L in $\mathbb{R}^3$. In order to compute the linking number we need to compute the degree of the map $\Psi : K \times L \cong \mathbb{T}^2 \to S^2$. We start with a volume form on S^2. Define $\boldsymbol{\omega} = i_{\mathbf{n}}dx$, where $dx = dx_1 \wedge dx_2 \wedge dx_3$ and $\mathbf{n} = x_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2} + x_3\frac{\partial}{\partial x_3}$ the unit normal vector field to $S^2 \subset \mathbb{R}^3$, then

$$\boldsymbol{\omega} = x_1 dx_2 \wedge dx_3 - x_2 dx_1 \wedge dx_3 + x_3 dx_1 \wedge dx_2.$$

The integral $\int_{S^2} \boldsymbol{\omega} = 4\pi$ gives the area (volume) of S^2. The map Ψ is a composition of the $\Phi(x,y) = y - x : K \times L \to \mathbb{R}^3\backslash\{0\}$ and the retraction $\rho(x) = \frac{x}{|x|} : \mathbb{R}^3\backslash\{0\} \to S^2$.

Now

$$\begin{aligned}
\rho^*\boldsymbol{\omega}(\xi,\eta) &= \boldsymbol{\omega}(\rho_*(\xi),\rho_*(\eta)) \\
&= \frac{x_1}{|x|^3}dx_2 \wedge dx_3(\xi,\eta) - \frac{x_2}{|x|^3}dx_1 \wedge dx_3(\xi,\eta) + \frac{x_3}{|x|^3}dx_1 \wedge dx_2(\xi,\eta) \\
&= \frac{\det(x,\xi,\eta)}{|x|^3},
\end{aligned}$$

where we used the fact that for $\xi,\eta \in T_xS^2$ it holds that $\rho_*(\xi) = \frac{1}{|x|}\xi$ and $\rho_*(\eta) = \frac{1}{|x|}\eta$. Under the map Φ we obtain

$$\begin{aligned}
\Phi^*\boldsymbol{\omega}(\xi,\eta) &= \boldsymbol{\omega}(-\xi,\eta) = -\boldsymbol{\omega}(\xi,\eta) \\
&= -\det(y-x,\xi,\eta) = \det(x-y,\xi,\eta).
\end{aligned}$$

For the map Ψ this implies that

$$\Psi^*\boldsymbol{\omega}(\xi,\eta) = \frac{\det(x-y,\xi,\eta)}{|x-y|^3}.$$

Parametrize K and L and denote the parametrizations by κ and λ respectively. Then,

$$\int_{K\times L} \Psi^*\boldsymbol{\omega} = \int_0^{2\pi}\int_0^{2\pi} \frac{\det(\kappa(t)-\lambda(s),\kappa'(t),\lambda'(s))}{|\kappa(t)-\lambda(s)|^3}dtds.$$

The linking number is given by

$$\mathrm{link}(K,L) = \frac{1}{4\pi}\int_0^{2\pi}\int_0^{2\pi} \frac{\det(\kappa(t)-\lambda(s),\kappa'(t),\lambda'(s))}{|\kappa(t)-\lambda(s)|^3}dtds. \tag{2.3.5}$$

This integral may be hard to compute. Consider an example of two circles in the x_1,x_2-plane, then

$$t \mapsto (\cos(t),\sin(t),0), \quad s \mapsto (2\cos(s),2\sin(s),0),$$

and $\det(\kappa(t)-\lambda(s),\kappa'(t),\lambda'(s)) = 0$, which shows that $\mathrm{link}(K,L) = 0$.

Before doing some more elaborate examples let us derive some properties of the linking number.

2.27 Theorem The linking number satisfies the following properties:

(i) $\mathrm{link}(L,K) = (-1)^{(k+1)(\ell+1)}\mathrm{link}(K,L)$;

(ii) if K and L are separated by a hyperplane in $\mathbb{R}^n$, then $\mathrm{link}(K,L) = 0$;

(iii) let K_t and L_t be 1-parameter families of embedded circles such that $K_t \cap L_t = \varnothing$ for all $t \in [0,1]$, then $\mathrm{link}(K_0,L_0) = \mathrm{link}(K_1,L_1)$.

As a matter of fact the linking number is an isotopy invariant.

Proof. For the pair L,K we have the map $\widetilde{\Psi}(y,x) = \frac{x-y}{|x-y|}$. Define the maps $r(x,y) = (y,x)$ and $a(x,y) = (-x,-y)$. Then $\deg(r) = (-1)^{k\ell}$ and $\deg(a) = (-1)^{k+\ell+1}$. For the map $\widetilde{\Psi}$ it holds that $\widetilde{\Psi} = r^{-1} \circ \Psi \circ a$ and and by the composition property of the degree we derive the desired statement in (i).

As (ii) if a hyperplane exists then Ψ is not surjective onto S^{n-1} and therefore $\deg(\Psi) = 0$, which proves (ii).

Property (iii) is a direct consequence of he homotopy principle for the degree.

∎

2.28 Example Consider the circles $K = \{x \in \mathbb{R}^3 \mid x_1^2 + x_2^2 = 1,\ \ x_3 = 0\}$ and $L = \{x \in \mathbb{R}^3 \mid (x_2-1)^2 + x_3^2 = 1,\ x_1 = 0\}$. On K consider the orientation form $\theta_K = -x_2 dx_1 + x_1 dx_2$ and on L the orientation form $\theta_L = -x_3 dx_2 + (x_2-1)dx_3$. Choose the following parametrizations

$$t \mapsto (-\sin(t), \cos(t), 0), \quad s \mapsto (0, 1+\cos(s), \sin(s)),$$

again denoted by κ and λ respectively. Upon substitution in Equation (2.3.5) yields the following expression

$$\mathrm{link}(K,L) = \frac{1}{4\pi}\int_0^{2\pi}\int_0^{2\pi} \frac{\cos(s) - \cos(t)\cos(s) - \cos(t)}{\big(3 + 2\cos(s) - 2\cos(t)\cos(s) - 2\cos(t)\big)^{3/2}} dtds.$$

Under the mapping Ψ the inverse image of a value $p \in S^2$ is characterized by the following relation

$$\Psi^{-1}(p) = \{(x,y) \in K \times L \mid y - x = \mu p,\ \ \mu > 0\}.$$

Such a value is regular if $\Psi^*\boldsymbol{\omega}$ is nondegenerate at points in $\Psi^{-1}(p)$. By our previous calculations this means when $\det(x-y,\xi,\eta) \neq 0$, where $\xi \in T_xK$ and $\eta \in T_yL$. If we choose p to be a regular value, then the degree can be computed by adding the signs of the determinants at points in $\Psi^{-1}(p)$. Let us carry out this calculation for the above situation. Choose $p = (0,1,0)$, then $\Psi^{-1}(p)$

consists of the point pairs $(0,1,0) \in K$, $(0,2,0) \in L$, $(0,-1,0) \in K$, $(0,2,0) \in L$ and $(0,-1,0) \in K$, $(0,0,0) \in L$. The determinants are -1, $+2$ and -1 respectively, and therefore $\mathrm{link}(K,L) = -1$.

2.29 Remark In order to compute $\mathrm{link}(K,L)$ in Example 2.28 one can also try to evaluate the integral with brute force. We the help of Maple we obtain that

$$
\begin{aligned}
\mathrm{link}(K,L) &= \frac{1}{4\pi}\int_0^{2\pi}\int_0^{2\pi}\frac{\cos(s)-\cos(t)\cos(s)-\cos(t)}{\left(3+2\cos(s)-2\cos(t)\cos(s)-2\cos(t)\right)^{3/2}}dsdt \\
&= \frac{1}{4\pi}\int_0^{2\pi}\frac{2\cdot \mathrm{EllipticK}\left(2\sqrt{-1+\cos(t)}\right)}{5-4\cos(t)} \\
&\quad -\frac{6\cdot \mathrm{EllipticE}\left(2\sqrt{\cos(t)-1}\right)}{5-4\cos(t)}dt = -1,
\end{aligned}
$$

which follows by numerically integrating the elliptic integrals. ∎

2.4 The Brouwer fixed point theorem

A classical application of the Brouwer degree is the Brouwer fixed point theorem for continuous maps of the n-disc. A fixed point for a mapping $f : \mathbb{R}^n \to \mathbb{R}^n$ is a point $x \in \mathbb{R}^n$ which satisfies the equation

$$f(x) = x.$$

As a matter of fact the Brouwer fixed point theorem can be stated for sets homeomorphic to the n-disc, or closed unit ball $\overline{B_1(0)}$.

2.30 Theorem Let $\Omega \subset \mathbb{R}^n$ be an open subset such that $\overline{\Omega}$ is homeomorphic to $\overline{B_1(0)}$, and let $f : \overline{\Omega} \to \mathbb{R}^n$ be any continuous map. If $f(\overline{\Omega}) \subset \overline{\Omega}$, then f has a fixed point in $\overline{\Omega}$.

Proof. Let $\varphi : \overline{\Omega} \to \overline{B_1(0)}$ be a homeomorphism. Then the mapping $g := \varphi \circ f \circ \varphi^{-1} : \overline{B_1(0)} \to \overline{B_1(0)}$ is continuous. The maps f and g are conjugate and thus f has a fixed point if and only g has a fixed point.

2.31 Exercise Show the above claim for conjugate mappings. ∎

The Brouwer fixed point theorem can be proved by showing the theorem holds for g. Suppose that g has no fixed points in $\overline{B_1(0)}$, then $g(x) \neq x$, for all $x \in \overline{B_1(0)}$.

Consider the homotopy $h_t(x) = x - tf(x)$. In particular, $f(x) \neq x$ for $x \in \partial B_1(0)$. This implies that $0 \notin h_1(x)$. Observe that $tf(x) \in B_1(0)$ for all $0 \leq t < 1$ since $|tx|_2 < 1$. Consequently, if $x \in \partial B_1(0)$, then $h_t(x) \neq 0$ for all $0 \leq t \leq 1$. The Brouwer degree $\deg(h_t, B_1(0), 0)$ is well-defined and independent of $t \in [0,1]$. For $t = 0$, $\deg(h_0, B_1(0), 0) = \deg(\mathrm{Id}, B_1(0), 0) = 1$. On the other hand, since $f(x) \neq x$ for all $x \in \overline{B_1(0)}$ we have that $h_1^{-1}(0) = \emptyset$, which implies that $\deg(h_1, B_1(0), 0) = 0$, a contradiction. ■

Another proof of the Brouwer fixed point theorem is based on the observation that continuous mappings from $\overline{B_1(0)}$ to $\partial B_1(0)$, which are the identity on $\partial B_1(0)$ do not exit. This uses the boundary dependence property of the Brouwer degree discussed in Section 2.2, and holds in a much more general setting of bounded and open subset $\Omega \subset \mathbb{R}^n$.

2.32 Theorem There are no continuous maps $f : \Omega \to \partial\Omega$, with $f|_{\partial\Omega} = \mathrm{Id}$.

Proof. By Proposition 2.20 $\deg(f, \Omega, p) = \deg(\mathrm{Id}) = 1$, for any point $p \in \Omega$, which implies that the equation $f(x) = p$ has a solution, a contradiction. ■

Another theorem worth mentioning in this context is the Hairy Ball Theorem, which, in dimension two, asserts that a 2-sphere 'covered with hair' cannot be combed in a continuous manner. Here the theorem is formulated for the embedded sphere $S^{n-1} = \partial B_1(0)$. Consider a function $X : S^{n-1} \to \mathbb{R}^n$, with the property that $\langle X(x), x\rangle = 0$, for all $x \in S^{n-1}$. Such a function is called a tangent vector field on S^{n-1}.

2.33 Theorem — Theorem (Hairy Ball Theorem).. The $(n-1)$-sphere S^{n-1} allows a non-vanishing tangent vector field $X(x) \neq 0$ if and only if $n-1$ is odd.

Proof. If $n-1$ is odd a non-vanishing vector field is easily given:

$$X(x) = (-x_2, x_1, -x_4, x_3, \cdots, -x_n, x_{n-1}),$$

which is clearly tangent to S^{n-1} and non-vanishing.

As for the converse argue as follows. Suppose there exists a non-vanishing tangent vector field $X(x)$ on S^{n-1}, then normalization defines a unit tangent vector field $Y = X/\|X\|$. Consider

$$h_t = \cos(\pi t)x + \sin(\pi t)Y(x).$$

It is clear, since $\langle x, Y\rangle = 0$, that $\|h_t\| = 1$ and $h_t : S^{n-1} \to S^{n-1}$ for all $t \in [0,1]$. Moreover, $h_0 = \mathrm{Id}$ and $h_1 = -\mathrm{Id}$ and are thus homotopic mappings. From 2.13 and Proposition 2.20 it follows that $\deg(h_1) = \deg(-\mathrm{Id}) = (-1)^n$. By the homotopy invariance of the degree, $1 = \deg(\mathrm{Id}) = \deg(h_0) = \deg(h_1) = (-1)^n$ and thus $n-1$ is odd. ■

2.5 Homotopy types and Hopf's Theorem

For continuous mappings f from a compact domain $\overline{\Omega} \subset \mathbb{R}^n$ to $\mathbb{R}^n$, the question of solvability of the $f(x) = p$ is determined only by the mapping degree, when formulated in the following setting. It was proved in Section 2.2 that the degree $\deg(f, \Omega, p)$ is determined only by the degree of the boundary map $\varphi = f|_{\partial\Omega} : \partial\Omega \to \mathbb{R}^n \backslash \{p\}$. Non-triviality of $\deg(\varphi)$ implies that any continuous extension f of φ to $\overline{\Omega}$ has a solution to the equation $f(x) = p$. In this chapter it is proved that the converse also holds, i.e. if every continuous extension f of φ to $\overline{\Omega}$ has a solution to $f(x) = p$, then $\deg(\varphi) \neq 0$. This already indicates that the question of solvability is strongly related to the problem of extending a mapping φ to all of $\overline{\Omega}$. To be more precise, if $\varphi : \partial\Omega \subset \mathbb{R}^n \to \mathbb{R}^n \backslash \{p\}$ has a continuous extension $f : \overline{\Omega} \to \mathbb{R}^n \backslash \{p\}$, then the boundary map φ does not force solvability of $f(x) = p$ for all continuous extensions f, with $f|_{\partial\Omega} = \varphi$. In this case φ is said to be inessential with respect to Ω. When φ is not inessential with respect to Ω it is said to be essential with respect to Ω, which implies there are no continuous extension $f : \overline{\Omega} \to \mathbb{R}^n \backslash \{p\}$, and thus for continuous extension f takes values in $\mathbb{R}^n$ in general and the equation $f(x) = p$ has non-trivial solutions in Ω. A fundamental theorem by Hopf is used to prove that essential versus inessential is completely determined by the mapping degree of φ.

The goal of this chapter is to broaden the above question to cases where the degree cannot decide between essential versus inessential, or when the mapping degree is not defined. An important case is for mappings

$$f : \overline{\Omega} \subset \mathbb{R}^n \to \mathbb{R}^k,$$

where n is not necessarily equal to k. In this case the degree as introduced before is not defined. The question is whether $\varphi = f|_{\partial\Omega} : \partial\Omega \to \mathbb{R}^k \backslash \{p\}$ still determines the solvability if $f(x) = p$, for any continuous extension f of φ.

Consider mappings $\psi : \partial\Omega \subset \mathbb{R}^n \to S^{n-1}$, where $S^{n-1} \subset \mathbb{R}^n$ is the standard unit sphere. In this equal dimension situation an important version of the extension problem holds which can be regarded as a version of Hopf's Theorem.

2.34 Theorem Let $\Omega \subset \mathbb{R}^n$ be a connected, bounded domain. A continuous mapping $\psi : \partial\Omega \subset \mathbb{R}^n \to S^{n-1}$ extends to a continuous mapping $f : \overline{\Omega} \subset \mathbb{R}^n \to S^{n-1}$, with $f|_{\partial\Omega} = \psi$ if and only if $\deg(\psi) = 0$.

Theorem 2.34 is also referred to the extension problem for mappings $\psi : \partial\Omega \to S^{n-1} \subset \mathbb{R}^n$ and connected boundaries of bounded open sets $\Omega \subset \mathbb{R}^n$. In the forthcoming sections this problem will be put in a more general context. As explained above the extension problem is directly linked to the solvability problem, see Corollary 2.44.

2.35 Remark The connectivity condition Theorem 2.34 can be omitted by replacing the condition on the degree. Let $\psi^i = \psi|_{\partial\Omega^i}$, where Ω^i are the connected components of Ω, then the condition on the degree becomes $\deg(\psi^i) = 0$ for all connected components Ω^i of Ω. The proof is obvious by applying Theorem 2.34 to each component. ∎

2.36 Definition A family of mappings $\psi_t = \Psi(\cdot, t)$, with $\Psi : \partial\Omega \times [0,1] \to S^{n-1}$ is continuous, is called a homotopy between $\psi_0, \psi_1 : \partial\Omega \to S^{n-1}$. The mappings ψ_0 and ψ_1 are called homotopic.

Homotopy is an equivalence relation on $C^0(\partial\Omega; S^{n-1})$ and its equivalence classes are called homotopy types or homotopy classes. The homotopy type of a map ψ in $C^0(\partial\Omega; S^{n-1})$ is denoted by $[\psi]$ and the collection of all homotopy types or equivalence classes is denoted by $[\partial\Omega; S^{n-1}] = \{[\psi] \mid \psi \in C^0(\partial\Omega; S^{n-1})\}$.

2.37 Exercise Show that homotopy type introduced above defines an equivalence relation on $C^0(\partial\Omega; S^{n-1})$. ∎

Theorem 2.34 can be proved by using the following fundamental property of the generalized winding number (see Section 2.2), which is a special case of Hopf's Theorem.

2.38 Lemma A continuous mapping $\psi : S^{n-1} \to S^{n-1}$, where $S^{n-1} \subset \mathbb{R}^n$ is the standard unit sphere, has trivial homotopy type if and only if $\deg(\psi) = 0$.

Proof. The proof follows by combining Lemma **??** and Theorems **??** and **??**. ∎

2.39 Exercise Give an elemtary proof of Lemma 2.38 (Hint: use an induction argument in n). ∎

Proof of Theorem 2.34. If there exists a continuous extension $f : \overline{\Omega} \subset \mathbb{R}^n \to S^{n-1} \subset \mathbb{R}^n \backslash \{0\}$, then $\deg(f, \Omega, 0) = 0$, and thus by definition $\deg(\psi) = 0$.

Now suppose $\deg(\psi) = 0$. Let $g : \overline{\Omega} \subset \mathbb{R}^n \to \mathbb{R}^n$ be a continuous extension (use Tietze's Extension Theorem), with $g|_{\partial\Omega} = \psi$. By construction $g^{-1}(0) \subset U \subset \Omega$, where U is compact. Moreover, g can be chosen to be C^1 on U, and such that 0 is a regular value (see Chapter **??**). In this case $g^{-1}(0)$ is a finite set of points in $U \subset \Omega$. Now connect the points $x^j \in g^{-1}(0)$ via a path γ, such that γ has no self-intersections.

Since $\gamma \subset U$ is a compact set it can be covered by finitely many small open ball B^i, which yields a compact set $V \subset U$ which contains γ, and has a piecewise smooth boundary ∂V. Moreover, $\overline{V}$ is homeomorphic to the unit ball $\overline{B_1(0)}$, with

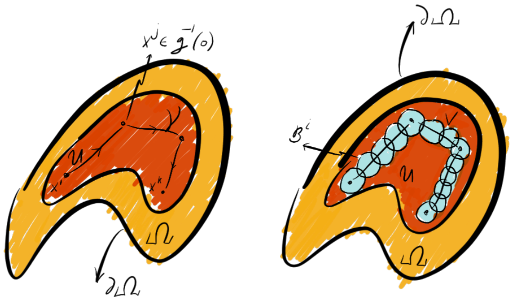


Figure 2.3: The zeroes of g are contained in $U \subset \Omega$ and are connected by a non-intersecting path γ [left]. The path $\gamma \subset U$ can be covered by a union of open balls $V \subset U$ [right].

homeomorphism $\alpha : \overline{V} \to \overline{B_1(0)}$. On the domain $\Omega' = \Omega - V$ it holds that

$$g : \overline{\Omega'} \subset \mathbb{R}^n \to \mathbb{R}^n \backslash \{0\},$$

and $\deg(g,\Omega',0) = 0$. Since $\partial\Omega' = \partial\Omega \cup \partial V$, Lemma (somewhere in Section 2.2) implies that

$$\deg(\psi) = \deg(g,\partial\Omega,S^{n-1}) = \deg(g,\partial V,\mathbb{R}^n\backslash\{0\}) = 0.$$

Now on ∂V, define $\psi' = g/|g|$, and $\psi' : S^{n-1} \to S^{n-1}$ is continuous. By Lemma 2.38, ψ' is homotopic to a constant map, and thus also $g|_{\partial V}$ is. Let $h : \partial V \times [0,1] \to \mathbb{R}^n\backslash\{0\}$ be a homotopy between $g|_{\partial V}$ and a constant map. Invoking the homeomorphism introduced above, then $k = h \circ \alpha^{-1} : S^{n-1} \times [0,1] \to \mathbb{R}^n\backslash\{0\}$ is also a homotopy. The map

$$p(tx) = k(x,t), \quad x \in S^{n-1},$$

defines an extension to $\overline{B_1(0)}$, and $p \circ \alpha$ is an extension of $g|_{\partial V}$ to V, and $p \circ \alpha : \overline{V} \to \mathbb{R}^n\backslash\{0\}$. Now adjust g with $p \circ \alpha$ on $\overline{V}$ to obtain an extension $\widetilde{g}$ to Ω, which takes values in $\mathbb{R}^n\backslash\{0\}$. The normalization $f = \widetilde{g}/|\widetilde{g}|$ yields the desired extension that maps from $\overline{\Omega}$ to S^{n-1}. ∎

The following theorem due to E. Hopf shows that the homotopy types in $C^0(\partial\Omega;S^{n-1})$ are characterized by the mapping degree, which is therefore the only homotopy invariant on $C^0(\partial\Omega;S^{n-1})$, which generalizes Lemma 2.38 and is a direct consequence of Theorem 2.34. Theorem 2.40 below is referred to as the classification problem and generalizations will be discussed in forthcoming sections.

2.40 Theorem Let $\partial\Omega \subset \mathbb{R}^n$ be compact, connected, smooth hypersurface.[a] Then, two continuous mappings $\psi_0, \psi_1 : \partial\Omega \to S^{n-1}$ are homotopic if and only if $\deg(\psi_0) = \deg(\psi_1)$.

[a] A smooth hypersurface is the level set $H^{-1}(0)$ of a smooth function $H : \mathbb{R}^n \to \mathbb{R}$, where 0 a regular value. Such a hypersurface is an embedded codimension-1 submanifold of $\mathbb{R}^n$. Moreover, $\partial\Omega$ is orientable.

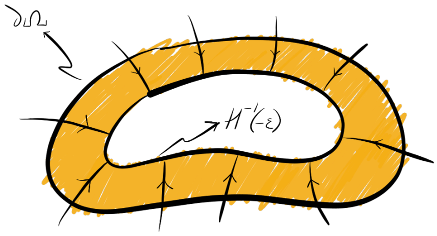


Figure 2.4: Deformation of $\partial\Omega$ via the normalized gradient flow on H.

Proof. Two mappings $\psi_0, \psi_1 : \partial\Omega \to S^{n-1}$ are homotopic if and only if there exists a homotopy $\psi_t = \Psi(\cdot, t)$ between ψ_0 and ψ_1, where $\Psi : \partial\Omega \times [0,1] \subset \mathbb{R}^{n+1} \to S^{n-1}$ is a continuous mapping. Let $F : \overline{\Omega} \times [0,1] \to \mathbb{R}^n$ be an extension of Ψ (use Tietze's Extension Theorem), then

$$\deg(f_t, \Omega, 0) = \deg(f_0, \Omega, 0) = \deg(f_1, \Omega, 0),$$

and therefore $\deg(\psi_0) = \deg(\psi_1)$.

Now suppose $\deg(\psi_0) = \deg(\psi_1)$. By assumption $\partial\Omega = H^{-1}(0)$ for some smooth function $H : \mathbb{R}^n \to \mathbb{R}$, with 0 a regular value. Therefore the interval $[-\epsilon, \epsilon]$, $\epsilon > 0$ sufficiently small, consists of regular values. The function is assumed to be negative on Ω, $H < 0$, and thus bounded from below. Define the domain

$$\Lambda := \{x \in \mathbb{R}^n \mid -\epsilon < H(x) < 0\},$$

is connected with $\partial\Lambda = \partial\Omega - H^{-1}(-\epsilon)$. The deformation lemma in Section 6.4 can be used now to show that there exists an isotopy[5] from $\partial\Omega = H^{-1}(0)$ to $H^{-1}(-\epsilon)$. Indeed, consider the normalized gradient flow

$$\frac{dx}{dt} = -\frac{\nabla H(x)}{|\nabla H(x)|^2},$$

The solution of the initial value problem for $x \in \partial\Omega$ is given by $\xi(x,t)$, with

$$\xi(x, 0) = x, \quad H(\xi(x,t)) = H(x) - t.$$

For details see Section 6.4. The mapping $\eta(x,t) = \xi\big(x, t(H(x) + \epsilon)\big)$ defines an isotopy from $\partial\Omega$ to $H^{-1}(-\epsilon)$;

$$\eta : \partial\Omega \times [0,1] \to \mathbb{R}^n,$$

where each $\eta_t(\cdot) = \eta(\cdot, t)$ is diffeomorphism from $\partial\Omega$ to $H^{-1}(-\epsilon t)$. Let ψ be a mapping from $\partial\Lambda$ to S^{n-1} defined as ψ_0 on $\partial\Omega$ and $\psi_1 \circ \eta_1^{-1}$ on $H^{-1}(-\epsilon)$. By Theorem 2.22

$$\deg(\psi) = \int_{\partial\Lambda} \psi^* \boldsymbol{\mu} = \int_{\partial\Omega} \psi_0^* \boldsymbol{\mu} - \int_{H^{-1}(-\epsilon)} \left(\psi_1 \circ \eta_1^{-1}\right)^* \boldsymbol{\mu}.$$

[5] An isotopy is a homotopy h_t for which h_t is a diffeomorphism for each $t \in [0,1]$.

Since η_1 is a diffeomorphism, it holds that $\int_{H^{-1}(-\epsilon)}(\psi_1\circ\eta_1^{-1})^*\boldsymbol{\mu} = \int_{\partial\Omega}\psi_1^*\boldsymbol{\mu}$, and therefore $\deg(\psi)=0$. By Theorem 2.34 there exists a continuous mapping $f:\overline{\Lambda}\subset\mathbb{R}^n\to S^{n-1}\subset\mathbb{R}^n$. Now define

$$\Psi(x,t)=f(\eta(x,t)):\partial\Omega\times[0,1]\to S^{n-1},$$

which is a homotopy between ψ_0 and ψ_1, and therefore proves the theorem. ∎

2.41 Corollary There exists a mapping $\psi:\partial\Omega\to S^{n-1}$ of any degree $m\in\mathbb{Z}$. In particular $[\partial\Omega,S^{n-1}]\cong\mathbb{Z}$.

Proof. Under construction. ∎

Theorem 2.40 is derived from the extension problem in Theorem 2.34. On the other hand Theorem 2.34 can be derived from the classification problem in Theorem 2.40 in the special case when $\partial\Omega=S^{n-1}$.

■ **2.42 Remark** Hopf's Theorem (Theorem 2.40) still holds in the case that $\partial\Omega$ is a triangulable set. Recall that a set is triangulable if it is homeomorphic to n-dimensional simplicial complex Δ_n. The result also holds for maps $\psi:X\to S^{n-1}$, where X is a triangulable topological space, with $\dim(X)=n-1$, see [14]. In particular (abstract) smooth manifolds M are triangulable, and therefore Hopf's Theorem extends to maps $\psi:M\to S^{n-1}$. In Chapter **??** the notion of degree for maps between smooth manifolds will be introduced. ■

2.43 Example Consider the annulus $\Omega=\{(x,y)\in\mathbb{R}^2\mid 1\le x^2+y^2\le 2\}$, and the mapping

$$f(x,y)=\begin{pmatrix} -y/\sqrt{x^2+y^2}\\ x/\sqrt{x^2+y^2}\end{pmatrix},$$

acting from $\overline{\Omega}$ to $\mathbb{R}^2\backslash\{0\}$. The boundary $\partial\Omega$ of the annulus is disconnected and $\deg(\psi)=0$, where $\psi=f|_{\partial\Omega}:\partial\Omega\to S^1$. Clearly, $[\psi]\neq 0$, which shows that the connectivity condition in Hopf's Theorem cannot be removed.

Connectivity of $\partial\Omega$ is not required for Theorem 2.34. The degree gives the proper invariant and and a straightforward calculation shows that $\deg(\psi)=0$, which is in compliance with the extension f.

The above theorem states that ψ is inessential with respect to Ω if and only if $\deg(\psi)=0$. The extension problem in Theorem 2.34 can be rephrased into a solvability property for the equation $f(x)=p$, with $f:\overline{\Omega}\subset\mathbb{R}^n\to\mathbb{R}^n$, and $\varphi=f|_{\partial\Omega}$. This problem will be referred to as the solvability problem. In the latter case the boundary mapping is denoted by $\varphi:\partial\Omega\subset\mathbb{R}^n\to\mathbb{R}^k\backslash\{p\}$, and the extension problem becomes; given φ, does there exist a continuous extension f :

$\overline{\Omega} \subset \mathbb{R}^n \to \mathbb{R}^n$, with $\varphi = f|_{\partial\Omega}$. This version of the extension problem is equivalent to the version in 2.34. Indeed, a normalized mapping $\psi = \frac{\varphi - p}{|\varphi - p|} : \partial\Omega \subset \mathbb{R}^n \to S^{k-1}$ is inessential with respect to Ω — there exists a continuous extension $g : \overline{\Omega} \subset \mathbb{R}^n \to S^{k-1}$ — if and only if $\varphi : \partial\Omega \subset \mathbb{R}^n \to \mathbb{R}^k \backslash \{p\}$ is inessential — there exists a continuous extension $f : \overline{\Omega} \subset \mathbb{R}^n \to \mathbb{R}^k \backslash \{p\}$. Indeed, if φ is inessential then $g = \frac{\varphi - p}{|\varphi - p|}$ gives the desired extension for ψ, and conversely, if ψ is inessential, then $f = r \cdot g + p$ is the desired extension for φ, where $r : \overline{\Omega} \subset \mathbb{R}^n \to \mathbb{R}^+$ is a continuous extension of $\rho = |\varphi - p| : \partial\Omega \subset \mathbb{R}^n \to \mathbb{R}^+$ via Tietze's Extension Theorem.

2.44 Corollary Let $\Omega \subset \mathbb{R}^n$ be a connected domain. A continuous mapping $\varphi : \partial\Omega \subset \mathbb{R}^n \to \mathbb{R}^n \backslash \{p\}$ is essential with respect to Ω if and only if $\deg(\varphi) \neq 0$.

Proof. By the discussion above and Theorem 2.34 φ is inessential with respect to Ω if and only if $\deg(\varphi) = 0$. Therefore, φ is essential with respect to Ω if and only if $\deg(\varphi) \neq 0$. ∎

▪ **2.45 Remark** If Ω is not necessarily connected, then the condition on the degree has to be replaced with $\deg(\varphi^i) \neq 0$, for some i, where $\varphi^i = f|_{\partial\Omega^i}$, and Ω^i are the connected components of Ω. ▪

2.6 Problems

2.46 Problem (Borsuk's Theorem) Let $\Omega \subset \mathbb{R}^n$ be a bounded domain satisfying the property that $x \in \Omega$ implies that $-x \in \Omega$. Let $\varphi : \partial\Omega \subset \mathbb{R}^n \to \mathbb{R}^n \backslash \{0\}$ such that $\varphi(-x) = -\varphi(x)$. Prove that for any continuous extension $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$ of φ it holds that $\deg(f, \Omega, 0)$ is an odd integer.

2.47 Problem Give a homological description of the local mapping degree for continuous mappings $f\colon N \to M$ between closed manifolds (compare [9], pp. 266).

2.48 Problem Formulate and prove the axioms of degree theory for continuous mapping $f\colon N \to M$.

Theorem 2.40 can be generalized to continuous mappings $f\colon N \to S^n$, where N is a closed, connected, orientable manifold of dimension n, i.e. two mappings $f, g\colon N \to M$ are homotopic if and only if $\deg(f) = \deg(g)$, and is called the *Hopf Degree Theorem*.

2.49 Problem Prove the Hopf Degree Theorem.

The following result implies an extension of the Hairy Ball Theorem and will be addressed in (5.8.10) and Problems 5.48 and 5.49.

2.50 Problem Let N be a smooth, closed, connected, orientable manifold with Euler characteristic $\chi(N) = 0$. Show that N allows smooth *non-vanishing* vector field $f\colon N \to TN$ (Hint: Use the Hopf Degree Theorem).

3 — The Leray-Schauder Degree

A natural question to ask is if there exists a degree theory for mappings on infinite dimensional spaces? The answer to this question is not so straightforward as the following example will show. Consider the space of sequences defined by $\ell^2 := \{x = (x_1, x_2, \cdots) \mid \sum_i x_i^2 < \infty\}$. The space $\ell^2 \cong \mathbb{R}^\infty$ has a natural norm $\|x\|_{\ell^2}^2 := \sum_i x_i^2$ and inner product $\langle x, y\rangle := \sum_i x_i y_i$ and is a complete normed linear space — a Hilbert space. Let $B^\infty = \{x \in \ell^2 \mid \|x\|_{\ell^2} \leq 1\}$ and define a mapping f as follows:

$$f(x) = \left(\sqrt{1 - \|x\|_{\ell^2}^2}, x_1, x_2, \cdots\right),$$

which is a continuous mapping from B^∞ to $\partial B^\infty =: S^\infty$. The mapping f has no fixed points in B^∞. Indeed, for $x \in S^\infty$ it holds that $f(x) = (0, x_1, x_2, \cdots) \neq x$. On the other hand $f(B^\infty) \subset B^\infty$ and the mapping f satisfies the requirements of the Brouwer fixed point theorem, which therefore does not holds for continuous mappings on ℓ^2.

3.1 Notation

The infinite dimensional spaces under consideration in this chapter are normed linear vector spaces are their subsets. Extensions will be discussed in the next chapter. Let X be a (real) linear vector space. On X we define a norm $\|\cdot\|_X$, or $\|\cdot\|$ for short which satisfies the following hypotheses:

(i) $\|x + y\| \leq \|x\| + \|y\|$, for all $x, y \in X$,
(ii) $\|\lambda x\| = |\lambda|\|x\|$, for all $x \in X$, and for all $\lambda \in \mathbb{R}$,
(iii) $\|x\| = 0$ if and only if $x = 0$.

If there is no ambiguity about the space involved we simply write $\|\cdot\|$.

The combination $(X, \|\cdot\|)$ is called a normed linear vector space. If in addition X is complete it is called a Banach space. A normed linear space is complete if every Cauchy sequence has a limit in X; $\{x^n\} \subset X$, with $\|x^n - x^m\| \to 0$, as $n, m \to \infty$, implies that there exists a $x \in X$ such that $\|x^n - x\| \to 0$, as $n \to \infty$. Normed vector spaces and Banach spaces are examples of metric and complete metric spaces respectively, where the metric is given by

$$d(x,y) := \|x - y\|.$$

For the remainder of this chapter X is assumed to be complete, i.e. a Banach space.

As in the previous chapter $\Omega \subset X$ denotes an open and bounded subset of X. The closure in X is denoted by $\overline{\Omega}$ and the boundary is given by $\partial\Omega = \overline{\Omega}\backslash\Omega$.

3.1.a Continuity

Throughout this section X, and Y are (real) Banach spaces with norms $\|\cdot\|_X$ and $\|\cdot\|_Y$ respectively. We omit the subscripts is there is no ambiguity about the notation.

3.1 Definition A mapping $f : X \to Y$ is continuous if $x^n \to x$ (in X) implies that $f(x^n) \to f(x)$ (in Y). A map is uniformly continuous on X, if for for any $\epsilon > 0$ there exists a $\delta_\epsilon > 0$ such that $\|x - y\| < \delta$ implies that $\|f(x) - f(y)\| < \epsilon$. The latter can also be defined with respect to a closed subset $A \subset X$.

A continuous function $f : X \to Y$ is bounded if $f(\Omega) \subset Y$ is bounded for any bounded subset $\Omega \subset X$. Continuous mappings on $\mathbb{R}^n$ are necessarily bounded, i.e. bounded sets in $\mathbb{R}^n$ are mapped to bounded set under f. This is however not the case in general Banach spaces.

3.2 Exercise Give an example of continuous map between Banach spaces that is not bounded. ■

3.3 Lemma A uniformly continuous map is bounded.

Proof. We need to show that for any bounded set $A \subset X$ the image $f(A) \subset Y$ is also bounded. Choose $R > 0$ such that $A \subset B_R(0)$, and let $n > \frac{2R}{\delta}$. Then for any two points $x, y \in A$ it holds that $\|x - y\| \le 2R$, and one can define the line-segment $x^t = x + t(y - x)$, $t \in [0,1]$, in $B_R(0)$. For $t_i = \frac{i}{n}$ we obtain point $x^{t_i} \subset B_R(0)$, with $\|x^{t_i} - x^{t_{i+1}}\| < \delta$, by the choice of n. Since f is uniformly continuous it follows that $\|f(x^{t_i}) - f(x^{t_{i+1}})\| < \epsilon$, for all i. From the triangle inequality we then get

$$\|f(x) - f(y)\| \le \sum_i \|f(x^{t_i}) - f(x^{t_{i+1}})\| < n\epsilon,$$

which proves the boundedness of f. ■

The space of continuous mappings from X to Y is denoted by $C^0(X,Y)$. For mappings on bounded domains $f : \overline{\Omega} \subset X \to Y$ we write $f \in C^0(\overline{\Omega};Y)$ or $C^0(\overline{\Omega})$ in the case that $X = Y$. Furthermore, define

$$C^0_b(X,Y) := \{f \colon X \to Y \ : \ \sup_{x\in X} \|f(x)\|_Y < \infty\}.$$

In case of $f\colon \overline{\Omega} \subset X \to Y$ we have $C^0_b(\overline{\Omega};Y)$, or $C^0_b(\overline{\Omega})$. On C^0_b the following norm is defined

$$\|f\|_{C^0_b} := \sup_{x\in\overline{\Omega}} \|f(x)\|_Y,$$

which makes $C^0_b(\overline{\Omega})$ a normed linear space.

3.1.b Differentiability

3.4 Definition A mapping $f \in C(X,Y)$ is called Fréchet differentiable at a point $x_0 \in X$, if there exists a bounded linear map $A : X \to Y$ such that

$$\|f(x) - f(x_0) - A(x - x_0)\| = o(\|x - x_0\|),$$

in a neighborhood N of x_0.

We use the notation $A = f'(x_0)$ for the Fréchet derivative. If the map $x \mapsto f'(x)$ is continuous as a map from X to $B(X,Y)$, then f is of class C^1; notation $f \in C^1(X,Y)$.

3.5 Definition A mapping $f \in C(X,Y)$ is called Gateaux differentiable in the direction $h \in X$, at a point x_0, if there exists a $y \in Y$ such that

$$\lim_{t\to 0} \|f(x_0 + th) - f(x_0) - ty\| = 0,$$

with $x_0 + th$ defined in a neighborhood N of x_0.

The Gateaux derivative at at point is usually denoted by $df(x_0,h)$ and is commonly referred to as the directional derivative in the direction h. In $\mathbb{R}^n$ is notion is known as partial derivative and is a weaker notion of differentiability.

3.6 Exercise Give an example of a function that is Gateaux differentiable in a point, but is not Fréchet differentiable. ■

For functions on $\mathbb{R}^n$ there is an important relation between the two notions of differentiability, i.e. the partial derivatives exist and are continuous, then the function is differentiable. In the Banach space setting the same result holds.

3.7 Theorem If $f \in C(X,Y)$ is Fréchet differentiable at a point x_0, then f is Gateaux differentiable at x_0. Conversely, if a function $f \in C(X,Y)$ is Gateaux differentiable at x_0 for all directions $h \in X$, and the mapping $x \mapsto df(x,\cdot) \in B(X,Y)$ is continuous at x_0, then f is Fréchet differentiable at x_0.

In the latter case we write, by the linearity of df in h, that $df(x,h) = df(x_0)h = f'(x)h$.

Proof. The first claim of the theorem simply follows from the definition of the Fréchet derivative. For the converse we argue as follows. By assumption the map $f(x_0+th)$ is differentiable in t (sufficiently small), and $df(x_0+th,h) = df(x_0+th)h$ is continuous in t. Therefore,

$$f(x_0+h) - f(x) = \int_0^1 df(x_0+th)hdt.$$

Using this identity we find the following estimate:

$$\begin{aligned}
\|f(x_0+h) - f(x_0) - df(x_0)h\| &= \|\int_0^1 (df(x_0+th) - df(x_0),h)dt\| \\
&\le \int_0^1 \|(df(x_0+th) - df(x_0),h)\|dt \\
&\le \int_0^1 \|df(x_0+th) - df(x_0)\|_{B(X,Y)}\|h\|_X dt \\
&= o(\|h\|),
\end{aligned}$$

by the continuity of $df(x_0+th)$. ■

The notions differentiability can be further extended to higher derivatives and we leave this to the reader. Furthermore, one can easily prove various basic properties of derivatives:

3.8 Exercise Prove the chain rule: Let $f : X \to Y$, $g : Y \to Z$, and f and g are differentiable at x_0 and $y_0 = f(x_0)$ respectively. Then $\big(g(f(x_0))\big)' = g'(f(x_0)) \cdot f'(x_0)$. ■

3.9 Exercise Prove the product rule: Let $f : X \to \mathbb{R}$, and $g : X \to Y$ be differentiable at x_0, then $f \cdot g$ is differentiable at x_0, and $(f \cdot g)'(x_0)h = f'(x_0)h \cdot g(x_0) + f(x_0) \cdot g'(x_0)h$. ■

A value $p \in Y$ is called a regular value if $f'(x) \in B(X,Y)$ is surjective for all $x \in f^{-1}(p)$ and p is singular, or critical if it is not a regular value. A point $x \in X$ is called regular if $f'(x)$ is surjective and otherwise a point is called singular, or critical point.

3.2 Compact and finite rank maps

An important subspace of continuous mappings are the compact mappings. A mapping $k\colon X \to X$ is compact if $\overline{k(\Omega')}$ is compact for any bounded subset $\Omega' \subset X$. Compact mappings are bounded since $\overline{k(\Omega')}$ is bounded for any bounded set $\Omega' \subset X$. The space of compact mappings on X is denoted by $K(X)$. This definition of compact mappings also holds for continuous mappings on subsets of X.

3.10 Definition A continuous map $k\colon \overline{\Omega} \subset X \to X$ is called compact if $k(\overline{\Omega})$ is relatively compact.

In particular for any $\Omega' \subset \overline{\Omega}$ it holds that $\overline{k(\Omega')}$ is compact since $k(\Omega') \subset k(\overline{\Omega})$. The space of compact continuous mappings on $\overline{\Omega}$ is denoted by $K(\overline{\Omega}) \subset C^0_b(\overline{\Omega})$. Compact maps are examples of mappings which are close to mappings in finite dimensional Euclidean space.

The following lemma explains how compact maps can be approximated by maps of finite rank. To be more precise, a finite rank map is a mapping whose range is contained in a finite dimensional subspace of X. The subspace of finite ranks mappings is denoted by $F(\overline{\Omega}; X) \subset C^0(\overline{\Omega}; X)$.

3.11 Lemma Let $k \in K(\overline{\Omega})$, then for any $\epsilon > 0$, there exists a finite rank map $k^\epsilon \in F(\overline{\Omega})$ such that $\|k - k^\epsilon\|_{C^0_b} < \epsilon$.

Proof. We follow the proof given by Berger[7]. Since $\overline{k(\overline{\Omega})}$ is compact it can be covered by finitely many balls $B_\epsilon(x^i)$, with $x^i \in \overline{k(\overline{\Omega})}$. Define

$$\mu^i(x) = \frac{\lambda^i(x)}{\sum_j \lambda^j(x)},$$

where $\lambda^i(x) = \max(0, \epsilon - \|k(x) - x^i\|)$. This maximum is zero whenever $k(x) \notin \overline{B_\epsilon(x^i)}$ and therefore $\mu^i(x) = 0$, unless $\|k(x) - x^i\| < \epsilon$. Set

$$k^\epsilon(x) = \sum_i \mu^i(x) x^i.$$

Now $k^\epsilon(\overline{\Omega}) \subset \operatorname{span}(x^i)$. As for the approximation we obtain

$$\|k - k^\epsilon\|_{C^0_b} = \|k - \sum_i \mu^i x^i\|_{C^0_b} = \Big\|\sum_i \mu^i (k - x^i)\Big\|_{C^0_b},$$

using the fact that $\sum_i \mu^i = 1$. By construction $\|\mu^i(x)(k(x) - x^i)\| < \mu^i(x)\epsilon$ and thus

$$\begin{aligned} \|k - k^\epsilon\|_{C^0_b} &= \sup_{x\in\overline{\Omega}} \Big\|\sum_i \mu^i(x)(k(x) - x^i)\Big\| \\ &\leq \sup_{x\in\overline{\Omega}} \sum_i \Big\|\mu^i(x)(k(x) - x^i)\Big\| < \sum_i \mu^i(x)\epsilon = \epsilon, \end{aligned}$$

which completes the proof. ■

The converse of this lemma can be formulated as follows:

3.12 Lemma For any sequence $\{k^\epsilon\} \subset F(\overline{\Omega}) \cap C^0_b(\overline{\Omega})$, with $k^\epsilon \to k$ in $C^0_b(\overline{\Omega})$, as $\epsilon \to 0$, it holds that $k \in K(\overline{\Omega})$.

Proof. See Berger[7] for a detailed proof. ■

Finally, using the above characterization of compact mappings, it is worth mentioning a version of Tietze's extension theorem for compact mappings.

3.13 Proposition Every compact mapping $k \in K(\overline{\Omega})$ extends to a compact mapping $\tilde{k} \in K(X)$.

For differentiable mappings k the derivative inherits the compactness properties of k which play in important role in the generic treatment of the Leray-Schauder degree.

3.14 Proposition Let $k \in K(\overline{\Omega}) \cap C^1(\Omega)$, then for any $x \in \Omega$ the linear operator $k'(x) : X \to X$ is compact.

3.3 Definition of the Leray-Schauder degree

We define a degree theory which extends the Brouwer degree on subclass of continuous mappings.

3.3.a Infinite dimensional spheres are contractible

In the introduction to this chapter we gave an example of a fixed point free mapping satisfying the conditions of the Brouwer fixed point theorem. Following the proof of the Brouwer fixed point theorem, define the continuous mapping $r(x) = x + \lambda_-(x)(f(x) - x)$, where $\lambda_-(x) \le 0$.[1] Since f has no fixed points it holds that $r(B^\infty) = S^\infty$.

A consequence of the above construction is that two mappings g_1, g_2 from S^∞ to S^∞ are homotopic. Indeed, $h(x,t) = r\big((1-t)g_1(x) + tg_2(x)\big)$[2] gives a homotopy bewteen g_1 and g_2. This implies in particular that S^∞ is contractible. This is far from the situation in finite dimensions.

The problem with degree theory in infinite dimensional spaces is that homotopy invariance, a basic property of the degree, prevents the existence of a

[1] Continuity can be proved in a similar way as in Theorem 2.30.

[2] One easily verifies that

$$\|(1-t)g_1 + tg_2\|^2_{\ell^2} \le 1,$$

provided $\|g_i\|^2_{\ell^2} = 1$, $i = 1,2$.

non-trivial degree theory (compare the axioms for degree theory, Section 2.1). We can alter the notion of homotopy invariance in order to build a degree theory, or limit the types of maps for which a degree is well-defined. The Leray-Schauder degree does both by considering specific types of mappings, namely mappings of the form

$$f = \mathrm{id} - k,$$

where id is the identity map on X and $k \in K(\overline{\Omega})$. Homotopies are considered in the same class.

3.3.b The Leray-Schauder degree

Denote the function class by $\mathcal{C}^0_{\mathrm{Id}}(\overline{\Omega}) = \{f = \mathrm{id} - k \mid k \in K(\overline{\Omega})\}$ and by $\mathcal{C}^0_{\mathrm{Id}}(X)$ for mapping defined on X. These classes are affine subspaces of $C^0_b(\overline{\Omega})$ and $C^0_b(X)$ respectively.

3.15 Lemma Let $\Omega \subset X$ be a bounded set. Then, $\partial\Omega$ is a closed and bounded set in X. Due to the specific form of f the set $f(\partial\Omega)$ is also closed and bounded.

Proof. Indeed, let $x^n \in \partial\Omega$ such that $f(x^n) \to x^*$. Since k is compact we have that $k(x^n)$ has a convergent subsequence and $k(x^{n_k}) \to x^{**}$.[3] Therefore, $x^{n_k} = f(x^{n_k}) + k(x^{n_k}) \to x^* + x^{**} = x \in \partial\Omega$,[4] which, by continuity, implies that $x^* = x - k(x) = f(x)$, proving the closedness of $f(\partial\Omega)$. For the boundedness we argue as follows: $k(\partial\Omega)$ is pre-compact and thus $f(\partial\Omega)$ is bounded. ∎

3.16 Lemma For $p \notin f(\partial\Omega)$ we have

$$\inf_{x\in\partial\Omega} \|p - f(x)\| = \inf_{y\in f(\partial\Omega)} \|p - y\| = \delta > 0.$$

Proof. Indeed, if not, there exists a minimizing sequence $x^n \in \partial\Omega$ such that $f(x^n) \to p$. By the closedness of $f(\partial\Omega)$ then $p \in f(\partial\Omega)$, a contradiction. ∎

3.17 Definition Let f be a continuous map of the form $f = \mathrm{id} - k$, with $k \in K(\overline{\Omega})$, and let $p \notin f(\partial\Omega)$. Let k^ϵ be a finite rank perturbation with $\|k - k^\epsilon\|_{C^0_b} < \epsilon$ and $\epsilon < \delta/2$ (δ as given above) and with $k^\epsilon(\overline{\Omega}) \subset Y^\epsilon \subset X$ (subspace). Then for any finite dimensional subspace X^ϵ containing both Y^ϵ and p, define the Leray-

[3] This uses the boundedness of $\partial\Omega$.
[4] Due to the closedness of $\partial\Omega$.

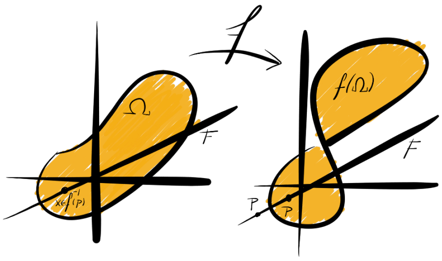


Figure 3.1: Zeroes for p-values restricted to the subspace F.

Schauder degree as

$$\deg_{LS}(f,\Omega,p) := \deg(f^\epsilon,\Omega\cap X^\epsilon,p),$$

where $f^\epsilon = \mathrm{id} - k^\epsilon$.

The remainder of this section is devoted to showing that the Leray-Schauder degree is well-defined. By the choice of domain $\Omega\cap X^\epsilon$ it follows that $f^\epsilon : \overline{\Omega}\cap X^\epsilon \to X^\epsilon$ and $p\in X^\epsilon$. Moreover, if $p\notin f(\partial\Omega)$, Lemma 3.16 implies

$$\begin{aligned}\inf_{x\in\partial\Omega}\|p-f^\epsilon(x)\| &= \inf_{x\in\partial\Omega}\|p-x+k^\epsilon(x)\|\\ &\geq \inf_{x\in\partial\Omega}\|p-x+k(x)\| - \delta/2 > \delta/2 > 0,\end{aligned}$$

which proves that $p\notin f^\epsilon(\partial\Omega)$ and in particular $p\notin f^\epsilon(\partial\Omega\cap X^\epsilon)$.[5] Consequently, the degree $\deg(f^\epsilon,\Omega\cap X^\epsilon,p)$ is well-defined. We show now that the definition is independent of the chosen subspace X^ϵ and approximation k^ϵ.

Let $\psi = \mathrm{id} - \phi$, where $\phi : \overline{\Omega}\subset\mathbb{R}^n \to F\subset\mathbb{R}^n$, and F a linear subspace of dimension $m\leq n$. The restriction to of ψ to F is denoted by $\psi_F : \overline{\Omega}\cap F\to F$. We define $\deg(\psi_F,\Omega\cap F,p)$ as the degree of $q\circ\psi\circ q^{-1}$, where $q(F) = \mathbb{R}^m\oplus 0$ is a linear change of variables, see **??** and 2.1.b.

3.18 Lemma Let $p\in F\setminus\psi(\partial\Omega)$, then $\deg(\psi,\Omega,p) = \deg(\psi_F,\Omega\cap F,p)$.

Proof. Let $q(x) = y$ be a linear change of variables on $\mathbb{R}^n$ such that $q(F) = \mathbb{R}^m\oplus 0$. The transformed function $q\circ\psi\circ q^{-1} = \mathrm{id} - q\circ\phi\circ q^{-1}$ is a again denoted by $\psi = \mathrm{id} - \phi$, and we prove the lemma for $F = \mathbb{R}^m\oplus 0$. Identify $\mathbb{R}^m\oplus 0$ with $\mathbb{R}^m$. On $\mathbb{R}^n$ we use coordinates (ξ,η), with $\xi\in\mathbb{R}^m$ and $\eta\in\mathbb{R}^{n-m}$. For $p\in\mathbb{R}^m$ we have that $x\in\psi^{-1}(p)$ satisfies: $x = p+\phi(x)\in\mathbb{R}^m$ and thus $x\in\Omega\cap\mathbb{R}^m$. This shows that $\psi^{-1}(p) = \psi_F^{-1}(p)$.

In order to compute and compare the degree we assume without loss of generality that ψ is C^1 and p is regular. The mapping ψ is given by $\psi(\xi,\eta) =$

[5] For a linear subspace $X^\epsilon\subset X$ it holds that $\partial(\Omega\cap X^\epsilon) = \partial\Omega\cap X^\epsilon$.

$(\xi + \phi(\xi,\eta),\eta)$ and the derivative at zeroes $x = (\xi,0)$ is

$$\psi'(\xi,\eta) = \begin{pmatrix} I_{\mathbb{R}^m} + d_\xi\phi(\xi,0) & d_\eta\phi(\xi,0) \\ 0 & I_{\mathbb{R}^{n-m}} \end{pmatrix} = \begin{pmatrix} \psi_F'(\xi,0) & * \\ 0 & I_{\mathbb{R}^{n-m}} \end{pmatrix}$$

This expression shows that the signs of the determinants are the zeroes are the same for both $\psi'(\xi,0)$ and $\psi_F'(\xi,0)$, and since $\psi^{-1}(p) = \psi_F^{-1}(p)$ we obtain the desired identity. ■

Let $\Omega \subset X$ be a bounded set and let $f = \mathrm{id} - \phi$, where $\phi : \overline{\Omega} \subset X \to E \subset X$, with $E \subset X$ a finite dimensional subspace of X. Suppose $p \notin f(\partial\Omega)$, then define $F \subset X$ to be a finite dimensional subspace that contains both E and p. The restriction $f_F = f|_{\overline{\Omega}\cap F} = \mathrm{id} - \phi|_{\overline{\Omega}\cap F}$ is a mapping $f_F : \overline{\Omega} \cap F \subset F \to F$. We may think of F as $\mathbb{R}^n$ via a linear isomorphism. The degree is then defined for the transformed mapping and the degree does not depend on the particular choice of the linear isomorphism. We have

3.19 Lemma The degree $\deg(f_F, \Omega \cap F, p)$ is independent of the choice of subspace $F \subset X$.

Proof. Let $\widetilde{F}$ be another subspace that contains both E and p, then also $F \cap \widetilde{F}$ suffices. Note that $\phi(\overline{\Omega}) \subset F \cap \widetilde{F}$ and $p \in (F \cap \widetilde{F}) \setminus f(\partial\Omega)$. Then by Lemma 3.18 we have that

$$\deg(f_F, \Omega \cap F, p) = \deg(f|_{\overline{\Omega}\cap F \cap \widetilde{F}}, \Omega \cap F \cap \widetilde{F}, p).$$

The same identity holds for the degree $\deg(f_{\widetilde{F}}, \Omega \cap \widetilde{F}, p)$, which proves the lemma. ■

We return to the mapping $f^\epsilon : \overline{\Omega} \subset X \to X$, with $p \notin f^\epsilon(\partial\Omega)$, which follows from the assumptions on k^ϵ with $\epsilon \leq \delta/2$.

3.20 Lemma Let $X^\epsilon, \widetilde{X}^\epsilon \subset X$ be finite dimensional subspaces such that $Y^\epsilon \subset X^\epsilon, \widetilde{X}^\epsilon$ and $p \in X^\epsilon, \widetilde{X}^\epsilon$. Then $\deg(f^\epsilon, \Omega \cap \widetilde{X}^\epsilon, p) = \deg(f^\epsilon, \Omega \cap X^\epsilon, p)$.

Proof. Apply Lemma 3.19 to $\phi = k^\epsilon$, $f_F = f^\epsilon$ and $F = X^\epsilon$. ■

The final step is to show that Definition 3.17 is independent of the chosen approximation k^ϵ.

3.21 Lemma Let k^ϵ and $\widetilde{k}^\epsilon$ both be finite rank approximations for k with $\|k - k^\epsilon\|_{C_b^0} < \epsilon$, $\|k - \widetilde{k}^\epsilon\|_{C_b^0} < \epsilon$ and $\epsilon < \delta/2$. Then

$$\deg(f^\epsilon, \Omega \cap X^\epsilon, p) = \deg(\widetilde{f}^\epsilon, \Omega \cap \widetilde{X}^\epsilon, p),$$

for any subspaces X^ϵ and $\widetilde{X}^\epsilon$ containing both p and the ranges of k^ϵ and $\widetilde{k}^\epsilon$ respectively.

Proof. Let $Z^\epsilon \subset X$ be a finite dimensional linear subspace containing both X^ϵ and $\widetilde{X}^\epsilon$. From Lemma 3.20 it follows that

$$\deg(f^\epsilon,\Omega\cap X^\epsilon,p) = \deg(f^\epsilon,\Omega\cap Z^\epsilon,p),$$
$$\deg(\widetilde{f}^\epsilon,\Omega\cap \widetilde{X}^\epsilon,p) = \deg(\widetilde{f}^\epsilon,\Omega\cap Z^\epsilon,p).$$

Consider the compact homotopy $k_t^\epsilon = (1-t)k^\epsilon + t\widetilde{k}^\epsilon$, which yields a homotopy $f_t^\epsilon = \mathrm{id} - k_t^\epsilon$. By the choices of k_ϵ and $\widetilde{k}_\epsilon$, the homotopy k_t^ϵ is a proper homotopy.[6] By Property (ii) of Section 2.1 then $\deg(f^\epsilon,\Omega\cap Z^\epsilon,p) = \deg(\widetilde{f}^\epsilon,\Omega\cap Z^\epsilon,p)$ which then proves that

$$\deg(f^\epsilon,\Omega\cap X^\epsilon,p) = \deg(\widetilde{f}^\epsilon,\Omega\cap \widetilde{X}^\epsilon,p).$$

The Leray-Schauder degree is well-defined. ■

3.4 Properties of the Leray-Schauder degree

The properties of the (Brouwer) degree listed in Section 2.1 hold equally well for the Leray-Schauder degree. For this list we refer to Section 2.1. For proving these properties for the Leray-Schauder degree one has to make sure that approximations are constructed such that the conditions for the Brouwer degree are met.

3.4.a Validity of the Leray-Schauder degree

An important property is the validity property.

3.22 Proposition If $p \notin f(\overline{\Omega})$, then $\deg_{LS}(f,\Omega,p) = 0$.

Proof. By the same token as Lemma 3.15 the set $f(\overline{\Omega})$ is closed and since $p \notin f(\overline{\Omega})$ we have that $\inf_{y\in f(\overline{\Omega})} \|p-y\| \ge \delta > 0$. Let f^ϵ be an approximation for f as described in the definition of the Leray-Schauder degree (Definition 3.17) and with X^ϵ such that $f^\epsilon(\overline{\Omega}) \subset X^\epsilon$ and $p \in X^\epsilon$. If we choose $\epsilon > 0$ small enough, i.e. $\epsilon \le \delta/2$, then for all $\|f-f^\epsilon\|_{C_0^b} < \epsilon$ it holds that $\inf_{y\in f^\epsilon(\overline{\Omega}\cap X^\epsilon)} \|p-y\| \ge \inf_{y\in f(\overline{\Omega})} \|p-y\| - \delta/2 \ge \delta/2 > 0$ and $p \notin f^\epsilon(\overline{\Omega}\cap X^\epsilon)$. From the properties of the Brouwer degree we now have that $\deg(f^\epsilon,\Omega\cap X^\epsilon,p) = 0$. ■

As an immediate consequence it now holds that

$$\deg_{LS}(f,\Omega,p) \neq 0,$$

implies that $f^{-1}(p) \neq \varnothing$. Indeed, if $f^{-1}(p) = \varnothing$, then $d_{LS}(f,\Omega,p) = 0$, a contradiction.

[6] Estimate $\|f_t^\epsilon(x) - p\|$, $x \in \partial\Omega$ by same token as in 3.3.b.

3.4.b Degree theories

Another way to treat the Leray-Schauder degree is to show that the axioms of a degree theory are satisfied and derive the properties from that. We start with the Leray-Schauder degree theory and explain the axioms in a more general context later on.

Consider triples (f,Ω,p) with $\Omega \subset X$ a bounded and open set in a Banach space $(X,\|\cdot\|)$, $f \in C^0_{\mathrm{Id}}(\overline{\Omega})$ and $p \in X\backslash f(\partial\Omega)$. For such triples we assign the Leray-Schauder degree

$$(f,\Omega,p) \mapsto \deg_{LS}(f,\Omega,p).$$

3.23 Theorem The Leray-Schauder degree satisfies the following properties:

(A1) if $p \in \Omega$, then $\deg_{LS}(\mathrm{id},\Omega,p) = 1$;

(A2) for $\Omega^1,\Omega^2 \subset \Omega$, disjoint open subsets of Ω, and $p \notin f\big(\overline{\Omega}\backslash(\Omega_1 \cup \Omega_2)\big)$, it holds that $\deg_{LS}(f,\Omega,p) = \deg_{LS}(f,\Omega^1,p) + \deg_{LS}(f,\Omega^2,p)$;

(A3) for any continuous paths $t \mapsto f_t = \mathrm{id} - k_t$, $k_t \in K(\overline{\Omega})$ and $p \notin f_t(\partial\Omega)$, it holds that $\deg_{LS}(f_t,\Omega,p_t)$ is independent of $t \in [0,1]$;

(A4) $\deg_{LS}(f,\Omega,p) = \deg_{LS}(f-p,\Omega,0)$.

and $\deg_{LS}$ is called a degree theory.

Proof. Under construction. ∎

As in the case of the Brouwer degree the essential properties of the Leray-Schauder degree follow from (A1)-(A3). In Section 2.1 we choose to prove these properties of the degree using only the axioms. Therefore most properties hold also for the Leray-Schauder degree with the same proofs. There are some differences though.

3.4.c Properties

Let us go through the list in Section 2.1 and point out the differences.

3.24 Property (Validity of the degree)If $p \notin f(\overline{\Omega})$, then $\deg_{LS}(f,\Omega,p) = 0$. Conversely, if $\deg_{LS}(f,\Omega,p) \neq 0$, then there exists a $x \in \Omega$, such that $f(x) = p$. ▪

Proof. Under construction. ∎

3.25 Property (Continuity of the degree)The degree $\deg_{LS}(f,\Omega,p)$ is continuous in $f = \mathrm{id} - k$, i.e. there exists a $\delta = \delta(p,f) > 0$, such that for all $g = \mathrm{id} - \tilde{k}$ satisfying $\|k - \tilde{k}\|_{C^0_b} < \delta$, it holds that $p \notin g(\partial\Omega)$ and $\deg(g,\Omega,p) = \deg(f,\Omega,p)$.

Proof. Under construction. ■

3.26 Property (Dependence on path components)The degree only depends on the path components $D \subset X \backslash f(\partial\Omega)$, i.e. for any two points $p, q \in D \subset X \backslash f(\partial\Omega)$ it holds that $\deg_{LS}(f, \Omega, p) = \deg_{LS}(f, \Omega, q)$. For any path component $D \subset X \backslash f(\partial\Omega)$ this justifies the notation $\deg_{LS}(f, \Omega, D)$.

Proof. Under construction. ■

3.27 Property (Translation invariance)The degree is invariant under translation, i.e. for any $q \in X$ it holds that $\deg_{LS}(f - q, \Omega, p - q) = \deg_{LS}(f, \Omega, p)$.

Proof. Under construction. ■

3.28 Property (Excision)Let $\Lambda \subset \Omega$ be a closed subset in Ω and $p \notin f(\Lambda)$. Then, $\deg_{LS}(f, \Omega, p) = \deg_{LS}(f, \Omega \backslash \Lambda, p)$.

Proof. Under construction. ■

3.29 Property (Additivity)Suppose that $\Omega^i \subset \Omega$, $i = 1, \cdots, k$, are disjoint open subsets of Ω, and $p \notin f\big(\overline{\Omega} \backslash (\cup_i \Omega_i)\big)$, then $\deg_{LS}(f, \Omega, p) = \sum_i \deg_{LS}(f, \Omega^i, p)$.

Proof. Under construction. ■

As for the Brouwer degree the Leray-Schauder degree can also be defined in the C^1-case. Let $p \in X \backslash f(\partial\Omega)$ be a regular value then by the Inverse Function Theorem the set $f^{-1}(p)$ consists of isolated points. Let $x_n \in f^{-1}(p)$, then $x_n = p + k(x_n)$ which has a convergent subsequence by the compactness of k and therefore $f^{-1}(p)$ is compact. Combined with isolation this yields that $f^{-1}(p)$ is a finite set. Using the excision and additivity Properties 3.28 and 3.29 we derive that $\deg_{LS}(f, \Omega, p) = \sum_j \deg_{LS}(f, N_\epsilon(x^j), p)$, $x^j \in f^{-1}(p)$. It holds that

$$\deg_{LS}(f, N_\epsilon(x^j), p) = \deg_{LS}(f, B_{\epsilon'}(x^j), p) =: i(f, x^j),$$

for all for any $0 < \epsilon'$ sufficiently small. The integer $i(f, x^j)$ is called the index of an isolated zero. For the Brouwer degree the index is given by the sign of the Jacobian at x^j. Since for each $x \in f^{-1}(p)$ the operator $f'(x) = I - k'(x) \in B(X)$[7] is invertible

[7]The identity operator $x \mapsto x$ is denoted by id and it linearization by I.

and the index given by the following spectral formula. Let $\lambda > 1$ be an eigenvalue of $A = I - k'(x)$, i.e. $A\xi = \lambda\xi$, $X \ni \xi \neq 0$, then

$$n_\lambda = \dim\Big(\bigcup_{k=1}^{\infty} \ker(\lambda I - A)^k\Big) < \infty,$$

by the compactness of $k'(x)$.

3.30 Lemma Let $x \in \Omega$ be a regular point of $f = C^0_{\mathrm{Id}}(\overline{\Omega})$, then

$$i(f,x) = (-1)^\beta,$$

where $\beta = \sum_{\lambda>1} n_\lambda$.

■ **3.31 Remark** The formula for the index can also be given for the Brouwer degree because $(-1)^\beta = \operatorname{sign}(J_f(x))$ is the finite dimensional case. The above consideration also shows how the Leray-Schauder degree is defined axiomatically and leads to a similar expression as a sum of indices in the C^1-case. The latter can also be used as a first definition and we can mimic the steps in Sect. 1.2 to construct a C^1-Leray-Schauder degree. One can then show that this yields the same degree as constructed here. ■

3.5 The Schauder fixed point theorem

In this section we give an extension of the Brouwer fixed point theorem to Banach spaces due to Schauder, cf.[21] A open set $\Omega \subset X$ is convex if $(1-t)x + ty \in \overline{\Omega}$ for all $x, y \in \overline{\Omega}$ and all $t \in [0,1]$. For convex sets we have the following analogue of the Brouwer fixed point theorem.

3.32 Theorem Let $\Omega \subset X$ be a bounded, open and convex set with $0 \in \Omega$ and let $g\colon \overline{\Omega} \subset X \to X$ be compact mapping. If $g(\overline{\Omega}) \subset \overline{\Omega}$, then g has at least one fixed point.

Proof. We argue by contradiction, i.e. suppose $g(x) \neq x$ for all $x \in \overline{\Omega}$. Consider the homotopy $h_t(x) = x - tg(x)$. In particular, $g(x) \neq x$ for $x \in \partial\Omega$. This implies that $0 \notin h_1(x)$.

Observe, since Ω is open, contains 0 and is convex, that $tg(x) \in \Omega$ for all $0 \le t < 1$ and for all $x \in \overline{\Omega}$.

3.33 Exercise Prove the above statement. ■

Consequently, if $x \in \partial\Omega$, then $h_t(x) \neq 0$ for all $0 \le t \le 1$. The Leray-Schauder degree $\deg_{LS}(h_t, \Omega, 0)$ is well-defined and independent of $t \in [0,1]$. For $t = 0$, $\deg_{LS}(h_0, \Omega, 0) = \deg_{LS}(\mathrm{id}, \Omega, 0) = 1$. On the other hand, since $g(x) \neq x$ for all $x \in$

$\overline{\Omega}$ we have that $h_1^{-1}(0) = \emptyset$, which implies that $\deg_{LS}(h_1, \Omega, 0) = 0$, a contradiction. ■

■ **3.34 Remark** The Schauder point theorem formulated above is also true if we replace the condition $0 \in \Omega$, by $\Omega \neq \varnothing$. In these let $x_0 \in \Omega$ be an interior point. Define $y = x - x_0$ and $\tilde{g}(y) = g(y + x_0) - x_0$. Now apply Theorem 3.32 to $\tilde{g}$ on the domain $\tilde{\Omega} = \Omega - x_0$. ■

As an application of the Schauder fixed point theorem we now consider the Peano's Theorem which guarantees the existence of solutions to initial value problems in $\mathbb{R}^n$. Consider the equation

$$\dot{y}(t) = f(t, y(t)), \quad y(t_0) = y_0 \in \mathbb{R}^n, \quad t \in \mathbb{R}, \tag{3.5.1}$$

where $y\colon \mathbb{R} \to \mathbb{R}^n$ and $f\colon \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is a continuous function. Note that we only assume continuity of f.

3.35 Theorem — Peano's theorem. Consider Equation (3.5.1) with f continuous. Let $y_0 \in \mathbb{R}^n$. Then there exists a time $\tau > 0$ and C^1-function $y\colon [t_0 - \tau, t_0 + \tau] \to \mathbb{R}^n$ which satisfies Equation (3.5.1).

Proof. We will restrict the proof to the case $t_0 = 0$ and $y_0 = 0$ without loss of generality. We start with reformulating Equation (3.5.1) in terms of an integral equation. Integration of (3.5.1) over $[0, t]$ yields the variation of constants equation:

$$y(t) = \int_0^t f(s, y(s))ds =: g(y(t)), \tag{3.5.2}$$

which provides a fixed point problem for the mapping g. Let $Y = C_b^0([-\tau, \tau]; \mathbb{R}^n)$ with norm $\|y\| := \sup_{t\in[-\tau,\tau]} |y(t)]$ and $\overline{\Omega} = \{y \in Y : \|y\| \leq M\}$ is closed, bounded subset containing 0. The mapping g maps from $\overline{\Omega}$ in to Y. We first need to choose the constants τ and M such that $g(\overline{\Omega}) \subset \overline{\Omega}$. We have for $y \in \overline{\Omega}$

$$|g(y)| \leq \int_0^t |f(s, y(s))| ds \leq |t||f(s, y(s))| \leq \tau N,$$

where $N = \max_{(s,y)\in[-\tau,\tau]\times[-M,M]} |f(s, y)|$. This implies $\|g(y)\| \leq \tau N$. If we choose $\tau \leq M/N$, then $g(\overline{\Omega}) \subset \overline{\Omega}$.

Finally the mapping g is compact. Indeed, $g(y)(t)$ is continuously differentiable and $(d/dt)g(y)(y) = f(t, y(t))$. We now use the fact that the embedding $C_b^1([-\tau, \tau]) \hookrightarrow C_b^0(-\tau, \tau])$ is compact. Therefore, g maps bounded sets to pre-compact sets in Y. The existence of a solution to $y = g(y)$ is now guaranteed by the Schauder fixed point theorem (Theorem 3.32), which completes the proof. ■

■ **3.36 Remark** In order to prove Peano's Theorem for every t_0 and y_0 we can use the extension of the Schauder fixed point theorem discussed in Remark 3.34. ■

3.6 Semi-linear elliptic equations and a priori estimates

In this section we will give a application of the Leray-Schauder degree in the context of nonlinear elliptic equations. We follow the notes by L. Nirenberg. The methods that we discuss apply in general for elliptic differential operator of any order. In order to simplify matter here we will restrict ourselves to the Laplace operator with Dirichlet boundary data. Let $D \subset \mathbb{R}^n$ be a bounded domain with smooth boundary ∂D. Consider the problem

$$-\Delta u = g(x,u,\nabla u), \quad u = 0, \quad x \in \partial D.$$

For the nonlinearity g we assume that C^∞-function of arguments, i.e. $g \in C^\infty(\overline{D} \times \mathbb{R} \times \mathbb{R}^n)$, and

$$|g(x,u,\nabla u)| \le C + C|\nabla u|^\gamma, \quad \gamma < 1,$$

uniformly in $x \in \overline{D}$, and $u \in \mathbb{R}$. Under these conditions we can prove the following result.

3.37 Theorem Under the assumptions on g the above elliptic equation has a solution $u \in C^\infty(\overline{D})$. Moreover, if $g(x,0,0) \not\equiv 0$, then the solution u is not identically zero.

Proof. The idea behind the proof is the formulate the above elliptic equation as a problem of finding zeroes of an appropriate function f on a (infinite dimensional) Banach space. Let us start with choosing an appropriate space in which to work. Define $X = H^2 \cap H_0^1(D)$ to be the intersection of two Sobolev spaces. For details on Sobolev space we refer to the next chapter. We will use the implications of this choice with respect to the well-defined of the elliptic equation, and postpone to proofs to the next chapter. The space $H^2 \cap H_0^1$ is a Hilbert space with norm $\|u\|_X = \int_D |\Delta u|^2 dx$. Due to the Dirichlet boundary conditions the Laplace operator[8] $-\Delta : H^2 \cap H_0^1(D) \subset L^2(D) \to L^2(D)$ has a compact inverse $(-\Delta)^{-1} : L^2(D) \to L^2(D)$. We rewrite the elliptic equation as

$$u - (-\Delta)^{-1} g(x,u,\nabla u) = 0. \tag{3.6.3}$$

The above equation can be regarded as a seeking zeroes of the (Nemytskii) mapping $f(u) = u - (-\Delta)^{-1} g(x,u,\nabla u)$ on $H^2 \cap H_0^1(D)$. By the estimate on g we have that

$$\begin{aligned} \int_D |g(x,u(x),\nabla u(x)|^2 dx &\le C \int_D \Big[1 + |\nabla u(x)|^{2\gamma}\Big] dx \\ &\le C' \Big(\int_D \Big[1 + |\nabla u(x)|^2\Big] dx\Big)^\gamma \\ &\le C\Big(1 + \|u\|_{H_0^1}^2\Big)^\gamma, \end{aligned}$$

[8] To get a sense of these fact on may consider the one-dimensional problem $-u_{xx} = f$ on $D = (0,1)$ and use the sine-Fourier series to derive the properties for Laplacian, see Sect. 3.7.

which proves that for $u \in X$, $g(x,u,\nabla u)(x)$ is an L^2-function. Consequently, the composition $(-\Delta)^{-1}\big[g(x,u,\nabla u)\big] \in X$, proving that $f: X \to X$ is well-defined. The latter follows from the fact that $R\big((-\Delta)^{-1}\big) = H^2 \cap H_0^1(D)$. As a map from L^2 to $H^2 \cap H_0^1$, the inverse Laplacian is an isometry. Concerning the continuity of this substitution map we refer to the next section. If we define $Y = H_0^1(\Omega)$ then f is a map from Y to Y, and $f = \mathrm{id} - k$, where $k: Y \to Y$ is a compact map. Indeed, k is a composition of the Nemytskii map $u \mapsto g(x,u,\nabla u)$ (from Y to L^2), the inverse Laplacian $(-\Delta)^{-1}$ (from L^2 to X), and the compact embedding $X \hookrightarrow Y$, which proves the compactness of k. This brings us into the realm of the Leray-Schauder degree.

Suppose $u \in X$ is a solution of the equation (3.6.3), then the estimate on $g(x,u,\nabla u)$ can be used now to obtain an a priori estimate on the solutions:

$$\begin{aligned} \|u\|_Y^2 &\leq C\|u\|_X^2 = C\|g(x,u,\nabla u)\|_{L^2}^2 \\ &\leq C\Big(1 + \|u\|_Y^2\Big)^{\gamma}, \end{aligned}$$

which, since $\gamma < 1$, implies that $\|x\|_Y \leq R$.

3.38 Exercise Prove the inequalities $\|u\|_{L^2} \leq C\|u\|_X$, and $\|u\|_{H_0^1} \leq C\|u\|_X$, for all $u \in X$. ■

Define the domain $\Omega = B_{2R}(0) \subset Y$. Clearly, f is a continuous map from $\overline{\Omega}$ into Y, which is of the form identity minus compact. Due to the above a priori estimate $f^{-1}(0) \subset \overline{B_R(0)} \subset Y$, and therefore $0 \notin f(\partial B_{2R}(0))$. Consequently, the Leray-Schauder degree

$$\deg_{LS}(f,\Omega,0),$$

is well-defined.

In order to compute this degree we consider the following homotopy:

$$f_t(u) = u - t(-\Delta)^{-1}[g(x,u,\nabla u)], \quad t \in [0,1].$$

Notice, for $t \in [0,1]$ we have via the same a priori estimates, that $f_t^{-1}(0) \subset \overline{B_R(0)}$, and therefore $0 \notin f_t(\partial\Omega)$ for all $t \in [0,1]$. Homotopy invariance of the Leray-Schauder degree then yields

$$\deg_{LS}(f,\Omega,0) = \deg_{LS}(\mathrm{id},\Omega,0) = 1,$$

which implies, by validity property of the Leray-Schauder degree, that $f^{-1}(0) \neq \varnothing$. Equation (3.6.3) thus has a solution $u \in Y$. The equation yields $u = (-\Delta)^{-1}[g(x,u,\nabla u)] \in X$, which implies that the solution also lies in X.

To prove regularity we use a bootstrapping argument. The integral estimates on g can be adjusted to L^p-estimates. This gives, by the Sobolev embeddings that:

$$u \in H^{1,p} \implies g(x,u,\nabla u) \in L^p \implies u \in H^{2,p} \implies u \in H^{1,p'},$$

where $\frac{1}{p'} = \frac{1}{p} - \frac{1}{n}$, provided $n > p$. This yields the recurrence relation

$$\frac{1}{p_{k+1}} = \frac{1}{p_k} - \frac{1}{n}.$$

The starting point is $p = 2$. We can repeat these recurrent steps until k times until $2(k+1) > n > 2k$, and then $u \in H^{2,p_k}$, where $p_k = \frac{2n}{n-2k}$. Again by the Sobolev embeddings, we have that

$$H^{2,p_k}(D) \hookrightarrow C^{1,\alpha}(\overline{D}),$$

where $\alpha = 1 - \frac{n}{p_k}$, since $\frac{n}{p_k} = \frac{n}{2} - k$, and $k+1 > \frac{n}{2} > k$, it holds that $0 < \alpha < 1$. We now repeat the bootstrapping in the Hölder space:

$$u \in C^{1,\alpha} \implies g(x,u,\nabla u) \in C^{0,\gamma\alpha} \implies u \in C^{2,\alpha'},$$

where $\alpha' = \gamma\alpha$. The idea now is the use the elliptic regularity theory for the Laplacian by differentiation the equation. Let $v_i = \frac{\partial u}{\partial x_i}$, then

$$-\Delta v_i = \partial_{x_i} g + (\partial_u g) v_i + \sum_j \partial_{v_j} g \frac{\partial v_j}{\partial x_i}.$$

Since g is a C^∞-function of its arguments, and $u \in C^{2,\alpha'}$, the right hand side is in $C^{0,\alpha'}$, implying that $v_i \in C^{2,\alpha'}$, and thus $u \in C^{3,\alpha'}$. We can repeat this process indefinitely, which proves that $u \in C^\infty(\overline{D})$.

If $g(x,0,0) \not\equiv 0$, then $u = 0$ cannot be a solution, and thus $u \not\equiv 0$. ∎

3.7 Problems

3.39 Problem Prove Proposition 3.14.

3.40 Problem Remark 3.31 describes a construction of a C^1-mapping degree for mappings f of the form $f = \mathrm{id} - k$, k compact. Carry the construction and show that this defines a degree theory, cf. Sect. 1.2.

3.41 Problem Consider the equation $-u_{xx} = f$ on $D = (0,1)$ with the Dirichlet boundary conditions $u(0) = u(1) = 0$.

(i) Use the spectral theorem to expand f and u;
(ii) Compute $\left(\frac{d^2}{dx^2}\right)^{-1}$ it terms of the basis obtained in (a);
(iii) Describe the Sobolev spaces $L^2(D)$, $H_0^1(D)$ and $H^2 \cap H_0^1(D)$ is terms of the basis;
(iv) Show that $\left(\frac{d^2}{dx^2}\right)^{-1}$ is an isometry from $L^2(D)$ to $H^2 \cap H_0^1(D)$ where $\|u\|_{H^2 \cap H_0^1} = \|u''\|_{L^2}$.
(v) Show that the embeddings $H^2 \cap H_0^1(D) \hookrightarrow H_0^1(D)$ and $H_0^1(D) \hookrightarrow L^2(D)$ are compact.

4 — Dynamical Systems

Prototypical examples of dynamical systems are systems of ordinary differential equations. But also iterations of mappings or homeomorphisms define dynamical systems (discrete time). In this part we will consider a topological theory of dynamical systems designed to understand global features of the dynamics. There are also links with degree theory. For example the problem of finding solutions for equation $f(x) = 0$, where $f : \mathbb{R}^n \to \mathbb{R}^n$, can be reformulated in terms of dynamical systems. The mapping f is regarded as a vector field on $\mathbb{R}^n$ and integrating the vector field yields the differential equation $x' = f(x)$, $x(0) = x_0$. The solution operator $x(t; x_0)$ is denoted by $\varphi(t,x)$ and satisfies the properties that: $\varphi(0,x) = x$ and $\varphi(t+s,x) = \varphi(t,\varphi(s,x))$, for all $x \in \mathbb{R}^n$ and all $s,t \in \mathbb{R}$. We do assume here that f is for example Lipschitz and bounded (otherwise consider $f/(1+|f|)$ as vector field). The mapping φ is called a flow on $\mathbb{R}^n$ and fixed points of φ correspond to zeroes of f.

4.1 Preliminaries and notation

Let (X,d) be a metric space[1] X with metric d and the space of time variables is $\mathbb{R}$.

4.1 Definition A (continuous time) *dynamical system*, or *flow* on X is a continuous mapping $\varphi : \mathbb{R} \times X \to X$ that satisfies the following two properties:

(i) $\varphi(0,x) = x$ for all $x \in X$, and

(ii) $\varphi(t,\varphi(s,x)) = \varphi(t+s,x)$ for all $s,t \in \mathbb{R}$ and all $x \in X$.

The latter is referred to as the group property for φ.

[1] Most of the concepts in this section and chapter work for more general topological spaces X.

From the group property it follows that a dynamical system is a one-parameter family of homeomorphisms of X and $\varphi^{-1}(t,\cdot) = \varphi(-t,\cdot)$. A continuous time dynamical is also referred to as a (global) flow on X. A vector field generates a local flow (local in the time variable). By renormalizing the vector field one obtains a global flow (reparametrizing time).

4.2 Exercise Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a Lipschitz continuous and bounded on $\mathbb{R}^n$. Show that $x' = f(x)$ generates a dynamical system on $X = \mathbb{R}^n$, satisfying Definition 4.1. ▪

4.3 Definition An orbit through $x \in X$ is the image of the function $t \mapsto \varphi(t,x)$ and is denoted by γ_x.

▪ **4.4 Remark** An orbit is also referred to as trajectory and for simplicity of notation, γ_x will be used both to denote the function $\gamma_x : \mathbb{R} \to X$ which defines a trajectory through x and its image $\{\gamma_x(t) \in X \mid t \in \mathbb{R}\}$ which is the trajectory. ▪

4.1.a Invariant sets

Complete orbits are examples of subsets of X that have the property that the set is invariant under the action of φ. In other words, complete orbits are invariant under the dynamics of φ. This leads to the following definition.

4.5 Definition A set $S \subset X$ is *invariant* if $\varphi(t,S) = S$ for all $t \in \mathbb{R}$.

The set of all invariant sets of a dynamical system is denote by

$$\mathsf{Invset}(X,\varphi) := \{S \subset X \mid \varphi(t,S) = S,\ \forall t \in \mathbb{R}\},$$

If there is no ambiguity about the dynamical system φ we write $\mathsf{Invset}(X)$. Observe that for any dynamical system, the empty set is an invariant set, i.e $\varnothing \in \mathsf{Invset}(X)$.

4.6 Exercise Show that S is an invariant set for φ if and only if $\varphi(t,S) = S$ for all $t \in \mathbb{R}^+$. ▪

If $S \subset X$ is an invariant set, then the restriction of φ to S is defined as $\varphi\|_S(t,x) = \varphi(t,x)$ for all $x \in S$, $t \in \mathbb{R}$, and the mapping $\varphi\|_S : \mathbb{R} \times S \to S$ is a dynamical system on S.

4.7 Exercise Show that S is an invariant set if and only $S \subset \varphi(t,S)$, for all $t \in \mathbb{R}$. Equivalently, show that S is invariant if and only if $\varphi(t,S) \subset S$ for all $t \in \mathbb{R}$. ▪

A set S is forward invariant if $\varphi(t,S) \subset S$ for all $t \in \mathbb{R}^+$ and the set of forward invariant sets is denoted by $\mathsf{Invset}^+(X,\varphi)$. A set S is backward invariant if $\varphi(t,S) \subset S$ for all $t \in \mathbb{R}^-$ and set of backward invariant sets is denoted by

$\mathsf{Invset}^-(X,\varphi)$. For flows it holds that a sets $S \subset X$ is invariant if and only if it is both forward and backward invariant, $\mathsf{Invset}(X,\varphi) = \mathsf{Invset}^+(X,\varphi) \cap \mathsf{Invset}^-(X,\varphi)$. The following lemma is additional characterizations of invariant sets.

4.8 Proposition The following statements are equivalent:

(i) S is invariant;
(ii) $S = \bigcap_{t\in\mathbb{R}} \varphi(t,S)$ and $\varphi\|_S$ is surjective;
(iii) for all $x \in S$ there exists an orbit $\gamma_x \subset S$;
(iv) $\varphi(t,S) = S$ for all $t \in (0,\tau]$, for some $\tau > 0$.

Similar characterizations can be given for forward/backward invariant sets.

4.9 Exercise Prove Proposition 4.8. ■

Set inclusion $\subseteq$ induces a partial order on the set of (forward/backward) invariant sets (see also Appendix D.1). From the definition of invariant set one derives that unions and intersections of invariant sets are again invariant. As a consequence $\mathsf{Invset}(X,\varphi)$ is a lattice.

4.10 Theorem The set $\big(\mathsf{Invset}(X,\varphi),\vee,\wedge\big)$ with

$$S \vee S' = S \cup S', \quad S \wedge S' = S \cap S, \tag{4.1.1}$$

is a bounded distributive lattice. The sets $\varnothing$ and X are the neutral elements.

4.11 Exercise Show that the sets $\mathsf{Invset}^+(X,\varphi)$ and $\mathsf{Invset}^-(X,\varphi)$ are bounded, distributive lattices with respect to $\cap$ and $\cup$, and $\varnothing$ and X are the neutral elements. ■

The mapping $S \mapsto S^c$ is a lattice anti-isomorphism between $\mathsf{Invset}^+(X,\varphi)$ and $\mathsf{Invset}^-(X,\varphi)$.

4.12 Corollary The set invariant sets $\mathsf{Invset}(X,\varphi)$ is a Boolean algebra.

The above described algebraic structures become interesting if consider special invariant sets such as attractors and repeller, which will be discussed in the next chapter.

4.1.b Asymptotic limit sets

A point $x \in X$ need not have a limit under φ as $t \to \infty$. For example points $y = \varphi(t,x)$ approaching a periodic orbit. In order to describe the limiting behavior of a point x one can consider limiting behaviors for arbitrary time sequences $t_n \to \infty$.

4.13 Definition A point y is called a omega limit point of a set $U \subset X$ under φ if there exist times $t_n \to \infty$ and points $x_n \in U$ such that $\lim_{n\to\infty} \varphi(t_n, x_n) = y$. The set of all omega limit points y is called the omega limit set of U and is denoted by $\omega(U, \varphi)$.

4.14 Exercise Construct an example to show that in general $\omega(U) \neq \bigcup_{x\in U} \omega(x)$. ▪

If there is not ambiguity about the dynamical system φ, the short hand notation $\omega(U)$ is used. The following lemma gives a convenient characterization of omega limit sets and which is sometimes used as a definition.

4.15 Proposition Let $U \subset X$ be a non-empty set. Then

$$\omega(U) = \bigcap_{t\geq 0} \mathrm{cl}\big(\varphi([t,\infty), U)\big). \tag{4.1.2}$$

The omega limit set $\omega(U)$ is invariant, and contained in $\mathrm{cl}(\Gamma^+(U))$.[a] If $U \subset X$ is forward invariant, then

$$\omega(U) = \bigcap_{t\geq 0} \mathrm{cl}\big(\varphi(t, U)\big), \tag{4.1.3}$$

and $\omega(U) = \mathrm{cl}(U)$. If U is connected, then also $\omega(U)$ is connected.

[a] The τ-forward image if a set U is defined as $\Gamma^+_\tau(U) := \varphi([\tau,\infty), U)$. The τ-backward image is defined similarly.

4.16 Exercise Show that $\Gamma^+_\tau(U) \in \mathsf{Invset}^+(X, \varphi)$. ▪

4.17 Proposition Suppose $\Gamma^+_\tau(U)$ is precompact for some $\tau \geq 0$. Then,
(i) $\omega(U, \varphi)$ is compact.
(ii) $U \neq \varnothing$, implies $\omega(U, \varphi) \neq \varnothing$;
(iii) U connected implies that $\omega(U, \varphi)$ is connected;
(iv) if $\varphi(t, U) \subset U$ for all $t \geq \tau \geq 0$, then $\omega(U, \varphi) = \mathrm{Inv}\big(\mathrm{cl}(U)\big)$, in particular, for general U, $\omega(U, \varphi) = \mathrm{Inv}\big(\mathrm{cl}(\Gamma^+_\tau(U))\big)$;
(v) for all $x \in U$, $d(\varphi(t, x), \omega(U, \varphi)) \to 0$, as $t \to \infty$.

Let $U \subset X$, then the maximal invariant set in U is defined as

$$\mathrm{Inv}(U, \varphi) := \{x \in U \mid \exists \gamma_x \subset U\}.$$

The omega limit set of U has the property that $\omega(U) = \mathrm{Inv}\big(\mathrm{cl}(\Gamma^+_\tau(U))\big)$.

4.18 Exercise Prove Proposition 4.15. ▪

The following lemma provides an additional list of useful properties of omega limit sets.

4.19 Proposition Let $U, V \subset X$, then the omega limit sets satisfy the following list of properties:

(i) if $V \subset U$, then $\omega(V) \subset \omega(U)$;
(ii) $\omega(U \cup V) = \omega(U) \cup \omega(V)$ and $\omega(U \cap V) \subset \omega(U) \cap \omega(V)$;
(iii) if $V \subset \omega(U)$, then $\omega(V) \subset \omega(U)$;
(iv) $\omega(U) = \omega(\mathrm{cl}(U))$, i.e. $\mathrm{cl}(\omega(U)) = \omega(\mathrm{cl}(U))$;
(v) $\omega(U) = \omega\big(\varphi(t, U)\big)$ for all $t \in \mathbb{T}$.
(vi) if there exists a backward orbit $\gamma_x^- \subset U$, then $x \in \omega(U)$.

4.20 Exercise Prove Proposition 4.19. ▪

Describing the limiting behavior of φ as $t \to -\infty$ is identical to $t \to \infty$ by simply reversing time $t \to -t$.

4.21 Definition A point y is called a alpha limit point of a set $U \subset X$ under φ if there exist times $t_n \to -\infty$ and points $x_n \in U$, $y_n \in \varphi(t_n, x_n)$ such that $\lim_{n\to\infty} y_n = y$. The set of all alpha limit points y is called the alpha limit set of U and is denoted by $\alpha(U, \varphi)$.

As for omega limit sets we have a similar characterization for alpha limit sets.

4.22 Lemma Let $U \subset X$ be a non-empty set. Then,

$$\alpha(U) = \bigcap_{t\leq 0} \mathrm{cl}\big(\varphi((-\infty, t], U)\big). \tag{4.1.4}$$

The alpha limit set $\alpha(U)$ is non-empty, compact, invariant, and contained in $\mathrm{cl}(\varphi(,\mathbb{R}^-, U))$. If $U \subset X$ is backward invariant, then

$$\alpha(U) = \bigcap_{t\leq 0} \mathrm{cl}\big(\varphi(t, U)\big), \tag{4.1.5}$$

and $\alpha(U) = \mathrm{cl}(U)$. If U is connected, then also $\alpha(U)$ is connected.

4.23 Exercise Prove Lemma 4.22. ▪

It follows from the invertibility of φ that $\alpha(U, \varphi) = \omega(U, \varphi^{-1})$. This way the analogue of Proposition 4.19 holds for alpha limit sets.

4.1.c Stable and unstable sets and connecting orbits

Related to the idea of a limit set is the notion of a stable or unstable set of an equilibrium point or more generally of an invariant set.

4.24 Definition Let $S \subset X$ be an invariant set, then

$$\begin{aligned} W^s(S,\varphi) &= \{x \in X \mid \lim_{t\to\infty} d\big(\varphi(t,x),S\big) = 0\} \\ W^u(S,\varphi) &= \{x \in X \mid \lim_{t\to-\infty} d\big(\varphi(t,x),S\big) = 0\}, \end{aligned}$$

are called the stable and unstable sets for S respectively.

By definition, S is always included in $W^s(S)$ and $W^u(S)$, and it can happen that both $W^s(S) = W^u(S) = S$. In the case that $S = \{x\}$ is a hyperbolic fixed point for a smooth dynamical system on $\mathbb{R}^n$, then the stable and unstable sets are immersed submanifolds. If x is an attracting fixed point, then $W^u(S) = S$. If x is a repelling fixed point, then $W^s(S) = S$.

4.25 Proposition Let $S \subset X$ be an invariant set for φ, then both $W^s(S)$ and $W^u(S)$ are invariant.

4.26 Exercise Prove Proposition 4.25 ■

The space of connecting orbits between two invariant sets S and S' are points that lie in the stable set of S and the unstable set of S'. More precisely:

4.27 Definition For two invariant sets S and S' the set of connecting orbits from S' to S is defined as $C(S',S) := W^s(S) \cap W^u(S',\varphi)$.

For each $x \in C(S',S)$ there exists an orbit $\gamma_x \subset C(S',S)$ such that both $\lim_{t\to\infty} d(\gamma_x(t),S) = 0$ and $\lim_{t\to-\infty} d(\gamma_x(t),S') = 0$. Such an orbit is called a connecting orbit from S' to S.

4.28 Proposition Let $S, S' \subset X$ be invariant sets, then
(i) $C(S',S) = W^s(S) \cap W^u(S')$ is invariant;
(ii) $S \cap S' = \varnothing$, implies that $C(S',S) \cap S' = \varnothing$ and $C(S',S) \cap S = \varnothing$;
(iii) if the sets S and S' are compact, then the connecting orbits are characterized by $C(S',S) = \{x \mid \omega(x) \subset S,\ \text{ and } \ \alpha(x) \subset S'\}$.

This yields the decomposition $X = S \sqcup S' \sqcup C(S',S) \sqcup C(S,S')$.

4.29 Exercise Prove Proposition 4.28. ■

4.2 Attractors and repellers

In order to obtain robust decompositions of dynamical systems we need to study the behavior of φ as $t \to \pm\infty$. We define sets that absorb trajectories as $t \to \infty$.

4.2.a Attracting neighborhoods

4.30 Definition A subset $U \subset X$ is called an *attracting neighborhood* if there there exists a $\tau > 0$ such that $\varphi(t, \mathrm{cl}(U)) \subset \mathrm{int}(U)$, for all $t \geq \tau$. The set of all such subset in X is denoted by $\mathsf{ANbhd}(X, \varphi)$.

By reversing time $t \mapsto -t$ the above definitions are use to define repelling neighborhoods and repelling blocks. These sets are denoted by $\mathsf{RNbhd}(X, \varphi)$ and $\mathsf{RBlock}(X, \varphi)$ respectively. Since the definitions of these lattice depend only on time-reversal the same properties hold.

Attracting neighborhoods have have natural binary operations which play a crucial role in the theory of decompositions.

4.2.b Binary operations

From the definitions above it follows that $\mathsf{ABlock}(X, \varphi) \subset \mathsf{ANbhd}(X, \varphi)$. Another property that follows is that unions and intersections of attracting neighborhoods and blocks are again attracting neighborhoods and blocks. Indeed, let $U, U' \in \mathsf{ANbhd}(X, \varphi)$, then[2]

$$\begin{aligned}\varphi(t, \mathrm{cl}(U \cup U')) &\subset \varphi(t, \mathrm{cl}(U) \cup \mathrm{cl}(U')) = \varphi(t, \mathrm{cl}(U)) \cup \varphi(t, \mathrm{cl}(U'))\\ &\subset \mathrm{int}(U) \cup \mathrm{int}(U') \subset \mathrm{int}(U \cup U'),\end{aligned}$$

for all $t \geq \max\{\tau, \tau'\}$, which shows that $U \cup U' \in \mathsf{ANbhd}(X, \varphi)$. Similarly,[3]

$$\begin{aligned}\varphi(t, \mathrm{cl}(U \cap U')) &\subset \varphi(t, \mathrm{cl}(U) \cap \mathrm{cl}(U')) \subset \varphi(t, \mathrm{cl}(U)) \cap \varphi(t, \mathrm{cl}(U'))\\ &\subset \mathrm{int}(U) \cap \mathrm{int}(U') = \mathrm{int}(U \cap U'),\end{aligned}$$

for all $t \geq \max\{\tau, \tau'\}$, which shows that $U \cap U' \in \mathsf{ANbhd}(X, \varphi)$.

The same follows for attracting blocks and therefore $U, U' \in \mathsf{ABlock}(X, \varphi)$ implies that $U \cup U', U \cap U' \in \mathsf{ABlock}(X, \varphi)$.

Summarizing, $\mathsf{ANbhd}(X, \varphi)$ and $\mathsf{ABlock}(X, \varphi)$ are *bounded, distributive lattices*, cf. Sect. D.1.

Worth mentioning are the natural mapping:

$$^c : \mathsf{ANbhd}(X, \varphi) \longrightarrow \mathsf{RNbhd}(X, \varphi),$$

4.31 Proposition The above defined mapping is an involutive lattice anti-isomorphisms.

[2]We use that $\mathrm{cl}(U \cup U') = \mathrm{cl}(U) \cup \mathrm{cl}(U')$ and $\mathrm{int}(U) \cup \mathrm{int}(U') \subset \mathrm{int}(U \cup U')$.
[3]We use that $\mathrm{cl}(U \cap U') \subset \mathrm{cl}(U) \cap \mathrm{cl}(U')$ and $\mathrm{int}(U) \cap \mathrm{int}(U') = \mathrm{int}(U \cap U')$.

4.32 Exercise Prove Proposition 4.31. ■

4.2.c Attractors

The link with invariant sets is not immediately clear in general metric spaces X. For example, consider parallel flow on $X = \mathbb{R}^n$ generated by the differential equations $\dot{x}_1 = 1$ and $\dot{x}_i = 0$, $i \geq 2$. Then, $N = \{x \,:\, x_1 \geq 0\}$ is an attracting block and $\mathrm{Inv}(N, \varphi) = \varnothing$. In general, for an isolating neighborhood $U \subset X$ we may consider consider $\mathrm{Inv}(U, \varphi) = \omega(U)$, which is invariant but may be the empty set.

4.33 Definition A *trapping region* is a forward invariant set $U \in \mathsf{Invset}^+(X)$, such that $\varphi(\tau, \mathrm{cl}(U)) \subset \mathrm{int}(U)$ for some $\tau > 0$.

4.34 Lemma A neighborhood $U \subset X$ is a trapping region if and only if U is a forward invariant attracting neighborhood.

Proof. If $U \subset X$ is a forward invariant attracting neighborhood, then $\varphi(t, U) \subset U$ for all $t \geq 0$, and $\varphi(t, \mathrm{cl}(U)) \subset \mathrm{int}(U)$ for all $t \geq \tau > 0$ by definition. This implies that U is a trapping region.

If U is a trapping region, then U is forward invariant, and there exists a $\tau > 0$ such that $\varphi(\tau, \mathrm{cl}(U)) \subset \mathrm{int}(U)$. By the group property

$$\begin{aligned} \varphi(t + \tau, \mathrm{cl}(U)) &= \varphi(\tau, \varphi(t, \mathrm{cl}(U))) \subset \varphi(\tau, \mathrm{cl}(\varphi(t, U))) \\ &\subset \varphi(\tau, \mathrm{cl}(U)) \subset \mathrm{int}(U) \quad \forall\, t \geq 0, \end{aligned}$$

which proves that U is an attracting neighborhood. ■

We write

$$\mathsf{TrapR}(X, \varphi) := \mathsf{ANbhd}(X, \varphi) \cap \mathsf{Invset}^+(X, \varphi).$$

A *repelling region* is defined by reversing time and the repelling sets and given by

$$\mathsf{RepR}(X, \varphi) := \mathsf{RNbhd}(X, \varphi) \cap \mathsf{Invset}^-(X, \varphi).$$

As before $\mathsf{TrapR}(X, \varphi)$ and $\mathsf{RepR}(X, \varphi)$ are bounded distributive lattice with binary operations $\cap$ and $\cup$.

By definition trapping regions are attracting neighborhoods and the same holds for repelling regions. The next lemma shows that trapping/repelling regions exist whenever attracting/repelling neighborhoods exist. Attracting and repelling blocks are special trapping and repelling region respectively.

4.35 Lemma Let $U \in \mathsf{ANbhd}(X,\varphi)$, then there exists a closed trapping region $U' \subset U$. Similarly, if $U \in \mathsf{RNbhd}(X,\varphi)$, then there exists a closed repelling region $U' \subset U$.

Proof. See proof of Theorem **??**. ∎

Since trapping region and attracting blocks 'contract' onto their maximally contained invariant set we introduce the notion of attractor via trapping regions. The same is carried for repelling regions and repellers.

4.36 Definition A set $A \subset X$ is called an *attractor* if there exists a trapping region $U \in \mathsf{TrapR}(X,\varphi)$ such that $A = \mathrm{Inv}(U,\varphi)$. A set $R \subset X$ is called a *repeller* if there exists a repelling region $U \in \mathsf{RepR}(X,\varphi)$ such that $R = \mathrm{Inv}(U,\varphi)$.

We denote the set of attractors by $\mathsf{Att}(X,\varphi)$ and the set of repellers by $\mathsf{Rep}(X,\varphi)$. Since an attractor is given by $A = \mathrm{Inv}(U,\varphi) = \omega(U)$ it is a closed invariant set (possibly the empty set) and

$$A \cup A' = \omega(U) \cup \omega(U') = \omega(U \cup U').$$

The union $U \cup U'$ is again an attracting neighborhood and therefore $A \cup A'$ is an attractor. Because also $A \cap A'$ is closed an invariant we have that

$$A \cap A' = \omega(A \cap A') = \omega(U \cap U') \subset \omega(U) \cap \omega(U') = A \cap A',$$

which proves that $A \cap A'$ is an attractor and $\omega(\cdot)$ is a lattice homomorphism. The same holds for repellers. The sets $\mathsf{Att}(X,\varphi)$ and $\mathsf{Rep}(X,\varphi)$ are therefore bounded distributive lattices with binary operations $\cap$ and $\cup$.

The mapping $U \mapsto U^c = X \setminus U$ is a lattice anti-isomorphism. Therefore, trapping regions are mapped to repelling regions and vice versa. Let U be a trapping region, then

$$\varphi(\tau, (\mathrm{cl}(U))^c) = \varphi(\tau, \mathrm{cl}(U))^c \supset (\mathrm{int}(U))^c,$$

and thus $(\mathrm{cl}(U))^c \supset \varphi(-\tau, (\mathrm{int}(U))^c)$. Using the fact that $(\mathrm{cl}(U))^c = \mathrm{int}(U^c)$ and $(\mathrm{int}(U))^c = \mathrm{cl}(U^c)$ we obtain

$$\varphi(-\tau, \mathrm{cl}(U^c)) \subset \mathrm{int}(U^c),$$

which proves that U^c is a repelling region. The same holds for repelling regions U.

The following commuting diagram relates the trapping/repelling regions and attracting/repelling neighborhoods.

$$\begin{tikzcd}
\mathsf{TrapR}(X,\varphi) \arrow[r, leftrightarrow, "c"] \arrow[d, "i"'] & \mathsf{RepR}(X,\varphi) \arrow[d, "i"] \\
\mathsf{ANbhd}(X,\varphi) \arrow[r, leftrightarrow, "c"] & \mathsf{RNbhd}(X,\varphi)
\end{tikzcd}$$

4.3 Lyapunov functions and blocks

Attractors and repellers characterize directionality of φ. Lyapunov functions provide an alternative method to characterize order in dynamics by mapping orbits into $\mathbb{R}$ monotonically. The following definition is a general one that allows us to establish orders in the global dynamics of φ.

4.3.a Lyapunov functions

4.37 Definition Let $\mathsf{U} = \{U_i\}_{i\in I}$ be a set of pairwise disjoint subsets of X. A continuous function $J : X \to \mathbb{R}$ is a *Lyapunov function* for φ *relative* to U, or a Lyapunov function for U for short, if

(i) for each forward orbit $\gamma_x^+ \subset X$ the function $J \circ \gamma_x^+ \colon \mathbb{R}^+ \to \mathbb{R}$ is non-increasing;

(ii) J is constant on each U_i;

(iii) $J(\gamma_x^+(t)) < J(x)$ for all $t > 0$ and $x \in X \setminus (\bigcup_i U_i)$.

Moreover, if a Lyapunov function exists with respect to U, the dynamics is said to be gradient-like on $X \setminus (\bigcup_i U_i)$.

■ **4.38 Remark** In the above definition the space X may be replaced by an arbitrary set $Y \subset X$. Under the same conditions (i)-(iii), the notion of Lyapunov function for (Y, U) is well-defined, with $U_i \subset Y \subset X$. If $Y = X$ we simply say a Lyapunov function for U instead of (X, U). ■

4.39 Proposition Suppose J is a Lyapunov function for U and $\gamma_x \colon \mathbb{R} \to X$ is a complete orbit. Then $J \circ \gamma_x \colon \mathbb{R} \to \mathbb{R}$ is decreasing. Moreover, $J \circ \gamma_x$ is strictly decreasing on any interval $I \in \mathbb{R}$ with $\gamma_x^+(I) \subset X \setminus (\bigcup_i U_i)$.

Proof. By part (i) of the definition, $J \circ \gamma_x$ is decreasing on $[\tau, \infty)$ for all $\tau \in \mathbb{R}$, and hence decreasing on all of $\mathbb{R}$. Suppose $I = [a, b]$ and choose $t_0, t_1 \in [a, b]$. Consider the orbit $\gamma_{\gamma_x(t_0)}$. Then $J \circ \gamma_x$ on $[t_0, t_1]$ is the same as $J \circ \gamma_{\gamma_x(t_0)}$ on $[0, t_1 - t_0]$. By part (iii) of the definition,

$$J \circ \gamma_x(t_0) = J \circ \gamma_{\gamma_x(t_0)}(0) > J \circ \gamma_{\gamma_x(t_0)}(t_1 - t_0) = J \circ \gamma_x(t_1),$$

which completes the proof. ■

If we do not specify the sets U then J is a function which satisfies the requirements of Definition 4.37 for some U. This could be $\mathsf{U} = \{X\}$ in which case J is a constant function and called the trivial Lyapunov function. When $\mathsf{U} = \{\varnothing\}$, then J is everywhere strictly decreasing along the flow.

A value $a \in \mathbb{R}$ is called a *regular value* for a Lyapunov function J if $J(\gamma_x^+(t)) < J(x)$ for all $x \in X$ such that $J(x) = a$ and for all $t > 0$. We say that a Lyapunov is *nontrivial* if there exists a regular value for J.

Whether non-trivial Lyapunov functions exist depends on φ. For example for $\varphi(t,x) = x$, the trivial Lyapunov is the only one. For example if U is a finite set of subsets $U_i \neq X$, then every number $a \in \mathbb{R} \setminus \bigcup_i J(U_i)$ is a regular value.

4.40 Exercise Prove the above statement. ▪

4.3.b Attracting and repelling blocks

4.41 Definition A closed subset $N \subset X$ is called an *attracting block* if $\varphi(t,N) \subset \mathrm{int}(N)$, for all $t > 0$. The set of all such subset in X is denoted by $\mathsf{ABlock}(X,\varphi)$.

Note that both $\varnothing$ and X are both elements of $\mathsf{ANbhd}(X,\varphi)$ and $\mathsf{ABlock}(X,\varphi)$. This implies that $\mathsf{ANbhd}(X,\varphi)$ and $\mathsf{ABlock}(X,\varphi)$ are always non-trivial. One can find an example where both $\mathsf{ANbhd}(X,\varphi)$ and $\mathsf{ABlock}(X,\varphi)$ have two elements, eg. consider a rotation on a disc X. Repelling blocks are defined in a similar way and are denoted by $\mathsf{RBlock}(X,\varphi)$.

▪ **4.42 Remark** By definition $\varphi(t,N) \neq \varnothing$ if $N \neq \varnothing$ and therefore $\mathrm{int}(N) \neq \varnothing$, which shows that attracting blocks are neighborhoods. ▪

Non-trivial Lyapunov functions can be used to construct attracting/repelling blocks.

4.43 Proposition Let J be a non-trivial Lyapunov function for φ and let $a \in \mathbb{R}$ be a regular value. Then, $N = J^a := \{x \in X \mid J(x) \leq a\}$ is an attracting block for φ. The set $N^\# = J_a = \{x \in X \mid J(x) \geq a\}$ is a repelling block for φ.

Proof. By definition $N = \mathrm{int}(N) \cup \partial N$, with $\mathrm{int}(N) = \{x \in N \ : \ J(x) < a\}$ and $\partial N = \{x \in X : J(x) = a\}$. By Definition 4.37(i) $J(\varphi(t,x))$ is non-increasing and thus if $x \in \mathrm{int}(N)$, then $J(\varphi(t,x)) \leq J(x) < a$ and therefore $\varphi(t, \mathrm{int}(N)) \subset \mathrm{int}(N)$ for all $t \geq 0$. If $x \in \partial N$, then, since a is a regular value it follows that $J(\varphi(t,x)) < J(x) = a$ for all $t > 0$. These facts combined yield that $\varphi(t,N) \subset \mathrm{int}(N)$ for all $t > 0$ and thus N is an attracting block.

As for $N^\# = \mathrm{cl}(N^c)$ we have $N^c = \{x \in X \ : \ J(x) > a\}$ and $\mathrm{cl}(N^c) = \{x \in X : J(x) \geq a\}$. This is a repelling block by definition since $^\#$ is an anti-isomorphism. One may also prove this via time reversal. ∎

Proposition 4.44 shows that non-trivial Lyapunov functions define attracting and repelling blocks. The converse can also be obtained.

4.44 Proposition Let $N \in \mathsf{ABlock}(X,\varphi)$ be an attracting block, then there exists a non-trivial Lyapunov function J such that N occurs as a sub-level set, i.e. $N = J^a$ for some regular value $a \in \mathbb{R}$.

Proof. Use the construction via exit time. Under construction. Us the fact that exit times are well-defined as continuous functions. ∎

4.3.c Existence of Lyapunov functions and blocks

It is clear that attracting blocks are trapping regions and therefore also attracting neighborhoods. We now reverse the question. Given an attracting neighborhood U, is there an attracting block? From Lemma 4.35 we have a trapping region $U' \subset U$. The next step is to find an attracting block $N \subset U' \subset U$.

4.45 Proposition Let $U \subset X$ be a trapping region. Then there exists an attracting block $N \subset U$. Similarly, for a repelling region U there exists a repelling block $N \subset U$.

Proof. The idea of the proof is to construct a Lyapunov function that allows the definition of an attracting block. Since U is a trapping region we have that $\varphi(\tau, \mathrm{cl}(U)) \subset \mathrm{int}(U)$ and

$$\begin{aligned} \varphi(t+\tau, \mathrm{cl}(U)) &= \varphi(\tau, \varphi(t, \mathrm{cl}(U))) \subset \varphi(\tau, \mathrm{cl}(\varphi(t, U))) \\ &\subset \varphi(\tau, \mathrm{cl}(U)), \quad \forall\, t \geq 0, \end{aligned}$$

which implies that $A = \varphi(\tau, \mathrm{cl}(U)) \subset \mathrm{int}(U)$ is a closed forward invariant. By the same token $B = \varphi(-\tau, \mathrm{cl}(U^c)) \subset \mathrm{int}(U^c)$ is closed backward invariant. Moreover, $A \cap B = \varnothing$.

We now claim the following property for points $x \in X \backslash (A \cup B)$:

$$x \in X \backslash (A \cup B) \quad \Longrightarrow \quad \varphi(2\tau, x) \in A, \text{ and } \varphi(-2\tau, x) \in B. \tag{4.3.6}$$

Indeed, if $x \in U$, then $\varphi(\tau, x) \in A$ by definition, and since A is forward invariant also $\varphi(2\tau, x) \in A$. If $x \in \mathrm{cl}(U^c) \setminus B$, then $\varphi(\tau, x) \in U$. Suppose $\varphi(\tau, x) \in \mathrm{cl}(U^c)$, then $x \in \varphi(-\tau, \mathrm{cl}(U^c)) = B$, which is a contradiction. Therefore, $\varphi(2\tau, x) \in \varphi(\tau, U) \subset A$. Summarizing, $x \in X \setminus B$, then $\varphi(2\tau, x) \subset A$. Via time-reversal we also prove that $x \in X \setminus A$ implies that $\varphi(-2\tau, x) \in B$.

Now consider the distance potential

$$\boldsymbol{\delta}(x) := \frac{d(x, A)}{d(x, A) + d(x, B)},$$

which is a continuous function $\boldsymbol{\delta}\colon X \to [0,1]$ which satisfies $\boldsymbol{\delta}^{-1}(0) = A$, $\boldsymbol{\delta}^{-1}(1) = B$ and $\boldsymbol{\delta}(x) \in (0,1)$ for $x \in X \setminus (A \cup B)$. Define

$$\triangle(x) := \sup_{t \geq 0} \boldsymbol{\delta}(\varphi(t, x)), \quad \triangledown(x) := \inf_{t \leq 0} \boldsymbol{\delta}(\varphi(t, x)).$$

Due to Property (4.3.6) we have that $\triangle(x) = \max_{t \in [0, 2\tau]} \boldsymbol{\delta}(\varphi(t, x))$ and $\triangledown(x) = \min_{t \in [-2\tau, 0]} \boldsymbol{\delta}(\varphi(t, x))$. Since $\boldsymbol{\delta} \circ \varphi\colon \mathbb{R} \times X \to \mathbb{R}$ is continuous the maximum of

$t \in [0,2\tau]$ and minimum over $t \in [-2\tau,0]$ are continuous function of $x \in X$.

4.46 Exercise Prove the above statements. ■

Consequently, $\triangle(x)$ and $\triangledown(x)$ are continuous functions on X with values in $[0,1]$. We have that $\triangle^{-1}(0) = A$ and $\triangle^{-1}(1) = B$. Since A is forward invariant it follows that $\triangle(A) = 0$ and thus $A \subset \triangle^{-1}(0)$. On the other hand if $x \in X \setminus A$, then $\triangle(x) > 0$, which shows that $\triangle^{-1}(0) \subset A$ and therefore $\triangle^{-1}(0) = A$. Since $\triangle(x)$ is defined as a maximum and $\boldsymbol{\delta}(x)$ is at most 1 we have that if $x \in B$, then $\triangle(x) = 1$ and thus $B \subset \triangle^{-1}(1)$. Since B is backward invariant we have that $B^c = X \setminus B$ is forward invariant. This yields $x \in B^c$ implies $\varphi(t,x) \in B^c$ for all $t \geq 0$ and therefore $\boldsymbol{\delta}(\varphi(t,x) < 1$ and $\triangle(x) < 1$ for all $t \geq 0$. Consequently, $\triangle^{-1}(1) \subset B$ and thus $\triangle^{-1}(1) = B$. Similarly, $\triangledown^{-1}(0) = A$ and $\triangledown^{-1}(1) = B$.

Another property that follows immediately from the definition is:

$$\begin{aligned} \triangle(\varphi(t,x)) \leq \triangle(x), \qquad & \forall t \geq 0, \\ \triangledown(x) \leq \triangledown(\varphi(t,x)), \qquad & \forall t \leq 0. \end{aligned}$$

We start with the definition of the following function:

$$J(x) := \int_0^\infty e^{-s} \triangle(\varphi(s,x)) ds,$$

which defines a continuous function $J \colon X \to [0,1]$. The next property follows from the fact that $\triangle(\varphi(t,x))$ is non-increasing in $t \geq 0$:

$$J(x) = \int_0^\infty e^{-s} \triangle(\varphi(s,x)) ds \leq \int_0^\infty e^{-s} \triangle(x) ds = \triangle(x).$$

We also define the function:

$$I(x) := \int_{-\infty}^0 e^{s} \triangle(\varphi(s,x)) ds,$$

which also defines a continuous function $I \colon X \to [0,1]$. Since $\triangledown(\varphi(t,x))$ is non-decreasing in $t \leq 0$ we obtain:

$$\triangledown(x) = \int_{-\infty}^0 e^{s} \triangledown(x) ds \leq \int_{-\infty}^0 e^{s} \triangledown(\varphi(s,x)) ds = I(x).$$

We now show that both J and I are non-trivial Lyapunov functions for φ. We show that J is a Lyapunov function and the proof for I is identical. By definition

$$J(\varphi(t,x)) - J(x) = \int_0^\infty e^{-s} \Big(\triangle(\varphi(t+s,x)) - \triangle(\varphi(s,x) \Big) ds \leq 0.$$

For $x \in A$ it holds that $J(\varphi(t,x)) = 0$ for all $t \geq 0$ by the forward invariance of A. If $x \in \mathrm{Inv}(B,\varphi) \subset B$, then $J(\varphi(t,x)) = 1$ by the invariance of $\mathrm{Inv}(B,\varphi)$. Now suppose $x \in X \setminus (A \cup \mathrm{Inv}(B,\varphi))$, then $\varphi(s,x) \notin \mathrm{Inv}(B,\varphi)$ and therefore exists $\tau' > 0$ such that $\varphi(\tau',x) \in B^c$. by the previous $\varphi(2\tau + \tau', x) \in A$. Consequently, there exists a $s > 0$ such that $\triangle(\varphi(t+s,x)) = 0$ and $\triangle(\varphi(s,x)) > 0$. This proves that $\triangle(\varphi(t+s,x)) - \triangle(\varphi(s,x)$ is not identically equal to 0 on $\mathbb{R}^+$ and therefore

$$J(\varphi(t,x)) < J(x), \quad \forall x \in X \setminus (A \cup \mathrm{Inv}(B,\varphi)), \text{ and } \forall t > 0,$$

which proves that J is a non-trivial Lyapunov function. In the same way we prove that

$$I(\varphi(t,x)) < I(x), \quad \forall x \in X \setminus (\mathrm{Inv}(A,\varphi) \cup B), \text{ and } \forall t > 0.$$

We can use both functions now to construct specific attracting and repelling blocks. By Proposition 4.44 we have that I^ϵ, $0 < \epsilon < 1$, is an attracting block and since $\triangledown(x) \leq I(x) \leq \epsilon$, it holds that $I^\epsilon \subset \triangledown^\epsilon$. Since $\triangledown^\epsilon \cap B = \varnothing$ we have by (4.3.6) that $\varphi(2\tau, \triangledown^\epsilon) \subset U$ and therefore also $N := \varphi(2\tau, I^\epsilon) \subset U$. By construction N is an attracting block.

In exactly the same way we use J to construct a repelling block inside U^c. ∎

4.4 Sublattices and filtrations

In the previous sections we established two important lattice embeddings:

$$\mathsf{TrapR}(X,\varphi) \subset \mathsf{ANbhd}(X,\varphi) \subset \mathsf{Set}(\mathsf{X}),$$

where $\mathsf{Set}(\mathsf{X})$ is the Boolean algebra of subsets of X and the binary operations are $\cap$ and $\cup$.

For arbitrary metric spaces X the lattices and Boolean algebras described above may be very big. In order to extract robust information from a dynamical system one considers finite sublattices. Recall that a sub-lattices are always assumed to contain X and $\varnothing$. Finite sub-lattices allow a representation theory which we summarize now.

4.4.a Birkhoff's Representation Theorem

Consider a poset $(\mathsf{P},\leq)$. A subset $A \subset \mathsf{P}$ is *attracting* if $x \in A$ and $y \leq x$ implies that $y \in A$. Under the operations of intersection and union the collection of attracting subsets of $(\mathsf{P},\leq)$ defines the *lattice of attracting sets* denoted by $\mathsf{O}(\mathsf{P})$. Observe that $\varnothing, \mathsf{P} \in \mathsf{O}(\mathsf{P})$ and act as the 0 and 1 elements, respectively.

4.47 Example Consider the poset $\mathsf{P} = \{1,2,3,4\}$ with the order indicated in Figure 4.1. Then the lattice of attracting sets is given by

$$\mathsf{O}(\mathsf{P}) = \Big\{\varnothing, \{1\}, \{1,2\}, \{1,3\}, \{1,2,3\}, \{1,2,3,4\}\Big\}$$

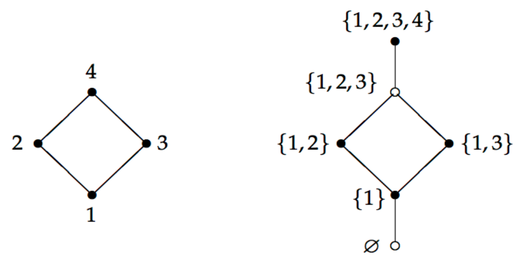


Figure 4.1: The linking sets S and ∂Q.

and can be visualized as in Figure 4.1.

4.48 Exercise Prove that $\mathsf{O}(\mathsf{P})$ is a distributive lattice with operations given by intersection and union. ■

Notice that given any $x \in \mathsf{P}$, the set $\downarrow x := \{y \in \mathsf{P} \mid y \leq x\} \in \mathsf{O}(\mathsf{P})$. To see that not every element of $\mathsf{O}(\mathsf{P})$ has this form, consider the element $\{1,2,3\} \in \mathsf{O}(\mathsf{P})$ from Example 4.47. As the following proposition indicates in the setting of finite posets these sets describe the join-irreducible elements of $\mathsf{O}(\mathsf{P})$.

4.49 Theorem Let P be a finite poset. The function

$$\begin{array}{rcl} \downarrow : (\mathsf{P}, \leq) & \to & (\mathsf{J}(\mathsf{O}(\mathsf{P})), \subseteq) \\ p & \mapsto & \downarrow p = \{q \in \mathsf{P} \mid q \leq p\} \end{array}$$

is a poset isomorphism.

Before beginning the proof we state the following lemma which is of interest in its own right.

4.50 Lemma If $I \in \mathsf{O}(\mathsf{P})$, then $I = \bigvee_{p \in \max(I)} \downarrow(p)$.

4.51 Exercise *Prove Lemma 4.50.* ■

Proof of Theorem 4.49. We first show that $\downarrow(\mathsf{P}) \subset \mathsf{J}(\mathsf{O}(\mathsf{P}))$. Let $p \in \mathsf{P}$ and assume $\downarrow(p) = \mathsf{B} \vee \mathsf{C} = \mathsf{B} \cup \mathsf{C}$ where $\mathsf{B}, \mathsf{C} \in \mathsf{O}(\mathsf{P})$. Observe that if $\mathsf{B} \neq \downarrow(p)$, then $\mathsf{B} \subset \downarrow(p)$. Thus there exists $x \in \downarrow(p) \setminus B$. The fact that B is attracting implies that $p \notin \mathsf{B}$. Since $p \in \downarrow(p)$, either $B = \downarrow(p)$ or $C = \downarrow(p)$ and hence $\downarrow(p) \in \mathsf{J}(\mathsf{O}(\mathsf{P}))$. The opposite inclusion, $\mathsf{J}(\mathsf{O}(\mathsf{P})) \subset \downarrow(\mathsf{P})$, follows from Lemma 4.50. ■

Notice that given any $p \in \mathsf{P}$, the set $\downarrow p := \{q \in \mathsf{P} \mid q \leq p\} \in \mathsf{O}(\mathsf{P})$. To see that not every element of $\mathsf{O}(\mathsf{P})$ has this form, consider the element $\{1,2,3\} \in \mathsf{O}(\mathsf{P})$ from Example 4.47. In $\mathsf{O}(\mathsf{P})$ sets of the form $\downarrow p$ are examples of join-irreducible elements defined as follows. An element $c \in \mathsf{L}$ is *join-irreducible* if

(i) $c \neq 0$ and
(ii) $c = a \vee b$ implies $c = a$ or $c = b$ for all $a, b \in \mathsf{L}$.

The set of join-irreducible elements in L is denoted by $\mathsf{J}(\mathsf{L})$. Similarly, an element $x \in \mathsf{L}$ is *meet-irreducible* if (i)-(ii) are satisfied with $\vee$ replaced by $\wedge$. The set of meet-irreducible elements in L is denoted by $\mathsf{J}^*(\mathsf{L})$. The sets $\mathsf{J}(\mathsf{L})$ and $\mathsf{J}^*(\mathsf{L})$ are posets as subsets of L. An element $b \in \mathsf{L}$ is join-irreducible if and only if it has a unique predecessor a in the covering relation, i.e. $a \prec b$, and hence we can define the unique predecessor map $\overleftarrow{\cdot} : \mathsf{J}(\mathsf{L}) \to \mathsf{L}$. An element $a \in \mathsf{L}$ is meet-irreducible if and only if it has a unique successor b in the covering relation which leads the unique successor map $\overrightarrow{\cdot} : \mathsf{J}^*(\mathsf{L}) \to \mathsf{L}$. A representation $a = \bigvee_i c_i$ is redundant if $a = \bigvee_{i \neq j} c_i$ for some j, otherwise the the representation is irredundant.

4.52 Theorem Let L be a finite distributive lattice. For every $a \in \mathsf{L}$ there exists a unique set of irredundant join-irreducible elements $\iota^\vee(a) \subset \mathsf{J}(\mathsf{L})$ and a unique set of irredundant meet-irreducible elements $\iota^\wedge(a) \subset \mathsf{J}^*(\mathsf{L})$ such that

$$a = \bigvee_{b \in \iota^\vee(a)} b = \bigwedge_{c \in \iota^\wedge(a)} c.$$

Proof. See Roman [**?**], p. 124, Theorem 4.30. ∎

When P is a finite poset, then the irredundant meet- and join-irreducible representations of elements in the lattice $\mathsf{O}(\mathsf{P})$ are characterized as follows. Let $I \in \mathsf{O}(\mathsf{P})$, then $I = \bigvee_p \downarrow p$, where the join is taken over all maximal $p \in I$. Similarly, $I = \bigwedge_p (\uparrow p)^c$, where the meet is taken over all minimal $p \in I^c$. Fundamental in the theory of finite distributive lattices is Birkhoff's Representation Theorem, which states that all finite distributive lattices are of the form $\mathsf{O}(\mathsf{P})$ for some finite poset P.

Now consider a finite distributive lattice $(\mathsf{L}, \vee, \wedge)$. Using the order relation on L the set of join-irreducible elements $\mathsf{J}(\mathsf{L})$ is a finite poset. This in turn can be used to define the lattice $\mathsf{O}(\mathsf{J}(\mathsf{L})$. In analogy with the case of posets we can ask what is the relationship between L and $\mathsf{O}(\mathsf{J}(\mathsf{L})$?

4.53 Theorem (Birkhoff's Representation Theorem) Let L be a finite distributive lattice and let P be finite partially ordered set. Then

$$\downarrow : \mathsf{P} \to \mathsf{J}(\mathsf{O}(\mathsf{P}))$$

is a poset isomorphism and

$$\begin{aligned} \downarrow^{\vee}\colon \mathsf{L} &\to \mathsf{O}(\mathsf{J}(\mathsf{L})) \\ x &\mapsto \downarrow^{\vee}(x) := \{y \in \mathsf{J}(\mathsf{L}) \mid y \leq x\} \end{aligned}$$

is a lattice isomorphism.

Proof. To be typed. ∎

We will often have need move between posets and lattices, and so to maintain clarity we adopt the language of category theory. Let $\mathfrak{Lat}_D^F$ denote the category of finite distributive lattices, whose morphisms are $(0,1)$-homomorphisms, and $\mathfrak{Poset}_F$ denote the category of finite posets, whose morphisms are order-preserving mappings. The following two results follow from [**?**, Theorem 8.24]. If L and K are objects in $\mathfrak{Lat}_D^F$ and $f\colon \mathsf{L} \to \mathsf{K}$ is a lattice homomorphism. Then

$$\begin{aligned} \mathsf{J}(f)\colon \mathsf{J}(\mathsf{K}) &\to \mathsf{J}(\mathsf{L}) \\ a &\mapsto \min\{b \in \mathsf{J}(\mathsf{L}) \mid a \leq f(b)\} \end{aligned}$$

is an order-preserving. If P and Q are objects in $\mathfrak{Poset}_F$ and $\psi\colon \mathsf{P} \to \mathsf{Q}$ is an order-preserving mapping. Then

$$\begin{aligned} \mathsf{O}(\psi)\colon \mathsf{O}(\mathsf{Q}) &\to \mathsf{O}(\mathsf{P}) \\ I &\mapsto \psi^{-1}(I) = \bigcup_{p\in I} \psi^{-1}(p) \end{aligned}$$

is a lattice homomorphism. Consequently we have the following theorem.[**?**, **?**]

4.54 Theorem The mappings $\mathsf{J} : \mathfrak{Lat}_D^F \to \mathfrak{Poset}_F$ and $\mathsf{O} : \mathfrak{Poset}_F \to \mathfrak{Lat}_D^F$ define contravariant functors.

In purely combinatorial setting, Birkhoff's Representation Theorem provides a precise description of the relation between posets and lattices. Our interest in this theorem is that it provides a tool by which we can relate objects of dynamical interest such as Morse decompositions and attractors.

4.4.b Booleanization and duality

The *spectral functor*[4] $\mathsf{J}\colon$ FDLAT $\to$ FPOSET is a contravariant functor that assigns the poset of join-irreducible elements $(\mathsf{J}(\mathsf{L}), \subset)$ to a finite distributive lattice L. The *down-set functor* $\mathsf{O}\colon$ FPOSET $\to$ FDLAT is a contravariant functor that assigns a finite distributive lattice $(\mathsf{O}(\mathsf{P}), \cap, \cup)$ to a finite poset P. By Birkhoff's representation theorem for finite distributive lattices we have that $\mathsf{O}(\mathsf{J}(\mathsf{L})) \cong \mathsf{L}$ and $\mathsf{J}(\mathsf{O}(\mathsf{P})) \cong \mathsf{P}$

[4] We use the fact that the spectrum of a finite distributive lattice can be represented in terms of the join-irreducible elements in the lattice.

are *natural isomorphisms*. Consider the forgetful functor $\mathsf{F}\colon \mathrm{FPOSET} \to \mathrm{FSET}$ and define the covariant functor $\mathsf{B} := \mathsf{O} \circ \mathsf{F} \circ \mathsf{J}$ and B is a functor from the category FDLAT to the category of finite Boolean algebras FBOOL.

4.55 Proposition For every lattice homomorphism $h\colon \mathsf{L} \to \mathsf{C}$, with C a Boolean algebra, there exists a lattice monomorphism $j\colon \mathsf{L} \to \mathsf{B}(\mathsf{L}) = \mathbf{2}^{\mathsf{J}(\mathsf{L})}$ and a unique lattice homomorphism $\mathsf{B}(h)\colon \mathsf{B}(\mathsf{L}) \to \mathsf{C}$, such that the diagram

$$\begin{tikzcd} \mathsf{L} \arrow[r, "h"] \arrow[d, tail, "j"'] & \mathsf{C} \\ \mathsf{B}(\mathsf{L}) \arrow[ur, "\mathsf{B}(h)"'] & \end{tikzcd} \tag{4.4.7}$$

commutes. The mapping $\mathsf{B}(h) : \mathsf{B}(\mathsf{L}) \to \mathsf{C}$ is Boolean.

The convex sets in a poset $(\mathsf{P}, \leq)$ play a central role in (Boolean) decompositions; e.g. the subsets $\{p\}$ are convex. The meet semi-lattice of convex subsets in P is denoted by $\mathsf{Convex}(\mathsf{P}) = \{\alpha \setminus \beta \mid \alpha, \beta \in \mathsf{O}(\mathsf{P})\}$. From Booleanization we have that the mapping

$$\alpha \setminus \beta \mapsto \mathsf{B}(h)(\alpha \setminus \beta) = h(\alpha) \setminus h(\beta) = e_{\alpha\setminus\beta}, \quad \alpha, \beta \in \mathsf{O}(\mathsf{P}),$$

from $\mathsf{Convex}(\mathsf{P})$ to C is a well-defined meet semi-lattice homomorhphism, i.e. $\alpha \setminus \beta = \alpha' \setminus \beta'$ implies $e_{\alpha\setminus\beta} = e_{\alpha'\setminus\beta'}$ and the mapping is meet semi-lattice homomorphism since $\mathsf{B}(h)$ is Boolean.

The fundamental commuting diagram for dynamics and Morse operations can be formalized by the following *duality* diagrams:

$$\begin{tikzcd} \mathsf{K} \arrow[r, leftrightarrow, "*"] \arrow[d, "h"'] & \mathsf{K}^* \arrow[d, "h^*"] \\ \mathsf{L} \arrow[r, leftrightarrow, "*"] & \mathsf{L}^* \end{tikzcd} \qquad \begin{tikzcd} \mathsf{K} \times \mathsf{K}^* \arrow[r, "\sqcap"] \arrow[d, "h \times h^*"'] & \mathsf{Convex} \arrow[d, "g"] \\ \mathsf{L} \times \mathsf{L}^* \arrow[r, "\sqcap"] & \mathsf{J} \end{tikzcd} \tag{4.4.8}$$

where Convex and J are semi-lattices and $g\colon \mathsf{Convex} \to \mathsf{J}$ is a semi-lattice homomorphism. Combining the duality with Booleanization yields the cob-web diagram:

$$\begin{tikzcd}
\mathsf{C} \arrow[rrr, leftrightarrow, "c"] \arrow[ddd, "\bar{h}"'] & & & \mathsf{C} \arrow[ddd, "\bar{h}^*=\bar{h}"] \\
& \mathsf{K} \arrow[ul, hook] \arrow[r, leftrightarrow, "*"] \arrow[d, "h"'] & \mathsf{K}^* \arrow[ur, hook'] \arrow[d, "h^*"] & \\
& \mathsf{L} \arrow[dl, hook'] \arrow[r, leftrightarrow, "*"] & \mathsf{L}^* \arrow[dr, hook] & \\
\mathsf{D} \arrow[rrr, leftrightarrow, "c"] & & & \mathsf{D}
\end{tikzcd} \tag{4.4.9}$$

where $\bar{h} = \mathsf{B}(h)$. We assume that a Morse operation yields the following commuting diagram with respect to Booleanization:

$$\begin{tikzcd}
\mathsf{K} \times \mathsf{K}^* \arrow[r, "\sqcap"] \arrow[d, "k \times k^*"'] & \mathsf{Convex} \arrow[d] \\
\mathsf{B} \times \mathsf{B} \arrow[r, "\cap"] & \mathsf{B}
\end{tikzcd} \tag{4.4.10}$$

An important consequence of the above diagrams is the following characterization of $\bar{h}$ on the *convex* sets in C.

4.56 Proposition Let $a \sqcap b^* = a' \sqcap b'^*$, then $h(a) \sqcap h^*(b^*) = h(a') \sqcap h^*(b'^*)$.

In the forthcoming sections we will use this theory in order to provide a lattice-order theoretic setting for the dynamical Morse decompositions.

4.4.c Filtrations and tilings

Let $\mathsf{U} \subset \mathsf{TrapR}(X, \varphi)$ be a finite sub-lattice. Then Booleanization yields a tiling of X denoted by $\mathsf{T}(\mathsf{U})$. The elements in T are isolating neighborhoods. If U is a totally ordered sub-lattice then it is called a *filtration* of trapping regions. The associated tiling also have a canonical linear order.

If we carry out the same procedure with $\mathsf{U} \subset \mathsf{ABlock}(X, \varphi)$, or $\mathsf{U} \subset \mathsf{TrapR}_{\mathscr{R}}(X, \varphi)$, then the tiles, i.e. the isolating neighborhoods, are regular closed sets.

4.5 Regular closed sets and attracting blocks

It turns out that attracting/repelling blocks have special properties which are advantages for decomposing X via blocks. We start with an intermezzo on regular open and closed sets.

4.5.a Regular open and closed sets

A closed sets $A \subset X$ is a *regular closed set* if $A = \mathrm{cl}(\mathrm{int}(A))$. An open set $A \subset X$ is an *regular open set* if $A = \mathrm{int}(\mathrm{cl}(A))$. Denote the class of regular closed subsets in X by $\mathscr{R}(X)$ and the regular open sets by $\mathscr{R}^c(X)$.

For regular closed sets we define an alternate complement:

$$A^{\#} := \mathrm{cl}(A^c). \tag{4.5.11}$$

For regular open sets we define

$$A^{\perp} = (\mathrm{cl}(A))^c. \tag{4.5.12}$$

Recall the relation between int and cl: $\mathrm{cl}(A^c) = (\mathrm{int}(A))^c$ and $\mathrm{int}(A^c) = (\mathrm{cl}(A))^c$. For alternate complements we have:

$$\begin{aligned}\mathrm{cl}(\mathrm{int}(A^{\#})) &= \mathrm{cl}(\mathrm{int}(\mathrm{cl}(A^c))) = \mathrm{cl}(\mathrm{int}((\mathrm{int}(A))^c)) = \mathrm{cl}((\mathrm{cl}(\mathrm{int}(A)))^c)\\ &= \mathrm{cl}(A^c) = A^{\#},\end{aligned}$$

which shows that $A^{\#} \in \mathscr{R}(X)$. In same way it follows that $\mathrm{int}(\mathrm{cl}(A^{\perp})) = A^{\perp}$ and thus $A \in \mathscr{R}^c(X)$.

4.57 Lemma A closed set $A \subset X$ is a regular closed set if and only if $A^{\#\#} = A$.

Proof. By definition $A^{\#\#} = (A^{\#})^{\#}$ and therefore

$$A^{\#\#} = \mathrm{cl}((\mathrm{int}(A))^{cc}) = \mathrm{cl}(\mathrm{int}(A)),$$

which proves the lemma. ∎

For regular open set a similar statement follows along the same lines.

4.58 Lemma An open set $A \subset X$ is a regular open set if and only if $A^{\perp\perp} = A$.

Proof. By definition $A^{\perp\perp} = (A^{\perp})^{\perp}$ and therefore

$$A^{\perp\perp} = \mathrm{int}((\mathrm{cl}(A))^{cc}) = \mathrm{int}(\mathrm{cl}(A)),$$

which proves the lemma. ∎

The application

$$A \mapsto A^c,$$

defines in involution from $\mathscr{R}(X)$ to $\mathscr{R}^c(X)$. Let $A \in \mathscr{R}(X)$, then $A^c = (\mathrm{cl}(\mathrm{int}(A)))^c = \mathrm{int}((\mathrm{int}(A))^c) = \mathrm{int}(\mathrm{cl}(A^c))$, which proves that A^c is a regular open set. The same holds for the mapping $^c : \mathscr{R}^c(X) \to \mathscr{R}(X)$.

4.59 Proposition — Walker [35]. Let X be a topological space. The family $\mathscr{R}(X)$ of regular closed subsets of X is a Boolean algebra with the following operations:

(i) $A \le A'$ if and only if $A \subset A'$;

(ii) $A \vee A' := A \cup A'$;

(iii) $A \wedge A' := (A \cap A')^{\#\#} = \mathrm{cl}(\mathrm{int}(A \cap A'))$;

(iv) $A^{\#} := \mathrm{cl}(A^c)$;

where $0 = \varnothing$ and $1 = X$.

■ **4.60 Remark** In [35], Proposition 2.3, it is proved that $\mathscr{R}(X)$ is a complete Boolean algebra, i.e. $\bigvee_\alpha A_\alpha := \mathrm{cl}\left(\bigcup_\alpha \mathrm{int}(A_\alpha)\right)$ and $\bigwedge_\alpha A_\alpha := \mathrm{cl}\left(\mathrm{int}(\bigcap_\alpha A_\alpha)\right)$ are well-defined. the same holds for $\mathscr{R}^c(X)$ by duality. ■

If we utilize the duality between $\mathscr{R}(X)$ and $\mathscr{R}^c(X)$ we can prove a similar structure for $\mathscr{R}^c(X)$ and show that $A \mapsto A^c$ is a Boolean (anti-)isomorphism. Let $A, A' \in \mathscr{R}^c(X)$, then

$$(A \cap A')^c = A^c \cup A'^c = A^c \vee A'^c,$$

and

$$\begin{aligned}(\mathrm{int}(\mathrm{cl}(A \cup A')))^c &= \mathrm{cl}((\mathrm{cl}(A \cup A'))^c) = \mathrm{cl}(\mathrm{int}((A \cup A')^c)) = \mathrm{cl}(\mathrm{int}(A^c \cap A'^c)) \\ &= A^c \wedge A'^c\end{aligned}$$

For $A, A' \in \mathscr{R}^c(X)$ this motivates the definition $A \vee A' := \mathrm{int}(\mathrm{cl}(A \cup A'))$ and $A \wedge A' := A \cap A'$. To complete the statement we have that

$$(A^{\#})^c = (\mathrm{cl}(A^c))^c = \mathrm{int}(A) = (A^c)^{\perp},$$

which proves that ^{c} commutes with the operations $^{\#}$ and $^{\perp}$ in $\mathscr{R}(X)$ and $\mathscr{R}^c(X)$ respectively. We have the following result for regular open sets.

4.61 Proposition Let X be a topological space. The family $\mathscr{R}^c(X)$ of regular open subsets of X is a Boolean algebra with the following operations:

(i) $A \le A'$ if and only if $A \subset A'$;

(ii) $A \vee A' := (A \cup A')^{\perp\perp} = \mathrm{int}(\mathrm{cl}(A \cup A'))$;

(iii) $A \wedge A' := A \cap A'$;

(iv) $A^{\perp} := (\mathrm{cl}(A))^c$;

where $0 = \varnothing$ and $1 = X$.

In the Boolean algebras $\mathscr{R}(X)$ and $\mathscr{R}^c(X)$ we can also define the alternate difference of sets.

4.62 Lemma Let $A, A' \in \mathscr{R}(X)$. Then $A - A' := A \wedge (A')^{\#} = \mathrm{cl}(A \setminus A')$.

Proof. By definition $A - A' := A \wedge A'^{\#} = \mathrm{cl}(\mathrm{int}(A \cap A'^{\#})) = \mathrm{cl}(\mathrm{int}(A \cap \mathrm{cl}(A'^{c})))$. Since A' is a regular closed set the complement A'^{c} is a regular open set and therefore $\mathrm{int}(\mathrm{cl}(A'^{c})) = A'^{c}$. This yields

$$A - A' = \mathrm{cl}(\mathrm{int}(A \cap \mathrm{cl}(A'^{c}))) = \mathrm{cl}(\mathrm{int}(A) \cap \mathrm{int}(\mathrm{cl}(A'^{c}))) = \mathrm{cl}(\mathrm{int}(A) \cap A'^{c}).$$

Finally, since A is a regular closed set we have that $\mathrm{cl}(\mathrm{int}(A)) = A$ and thus $\mathrm{cl}(\mathrm{int}(A) \cap A'^{c}) = \mathrm{cl}(A \cap A'^{c})$, see [35]. Combining this with the previous we obtain

$$A - A' = \mathrm{cl}(\mathrm{int}(A) \cap A'^{c}) = \mathrm{cl}(A \cap A'^{c}) = \mathrm{cl}(A \setminus A'),$$

which proves the lemma. ∎

The notion of difference of sets is easier yo characterize in $\mathscr{R}^{c}(X)$.

4.63 Lemma Let $A, A' \in \mathscr{R}^{c}(X)$. Then $A - A' := A \cap (A')^{\perp} = A \setminus \mathrm{cl}(A')$.

Proof. By definition

$$A - A' := A \cap A'^{\perp} = A \cap (\mathrm{cl}(B)^{c} = A \cap (X \cap (\mathrm{cl}(B))^{c}) = A \cap (\mathrm{cl}(B))^{c} = A \setminus \mathrm{cl}(B),$$

which proves the lemma. ∎

4.64 Lemma Let $A, A' \in \mathscr{R}(X)$, then $A \wedge A' = \varnothing$ if and only if $A \cap \mathrm{int}(A') = \varnothing$.

Proof. By definition

$$A \wedge A' = \mathrm{cl}(\mathrm{int}(A \cap A')).$$

Using the property $\mathrm{int}(A \cap A') = \mathrm{int}(A) \cap \mathrm{int}(A')$,

$$A \wedge A' = \mathrm{cl}(\mathrm{int}(A) \cap \mathrm{int}(A')).$$

Also, if $U \subset X$ is open and $B, B' \subset X$ with $\mathrm{cl}(B) = \mathrm{cl}(B')$, then $\mathrm{cl}(B \cap U) = \mathrm{cl}(B' \cap U)$. Taking $U = \mathrm{int}(A')$, $B = \mathrm{int}(A)$, and $B' = A$ implies

$$A \wedge A' = \mathrm{cl}(A \cap \mathrm{int}(A')).$$

Therefore

$$A \wedge A' = \varnothing \quad \text{iff} \quad \mathrm{cl}(A \cap \mathrm{int}(A')) = \varnothing \quad \text{iff} \quad A \cap \mathrm{int}(A') = \varnothing,$$

which proves the equivalence. ∎

Sets $A, A' \subset X$ for which $A \wedge A' = \varnothing$ will be referred to as *regularly disjoint sets*.

4.5.b Attracting and repelling blocks

Attracting and repelling block as introduced before are regular closed set in canonical way.

4.65 Lemma $\mathsf{ABlock}(X,\varphi) \subset \mathscr{R}(X)$ and the inclusion is a lattice embedding.

Proof. Let $N \in \mathsf{ABlock}(X,\varphi)$, then $\mathrm{cl}(\mathrm{int}(N)) \subset N$. Let $x \in N \setminus \mathrm{cl}(\mathrm{int}(N))$. By assumption $y_n := \varphi(t_n, x) \in \mathrm{int}(N)$ and $y_n \to x$ as $n \to \infty$. This implies that $x \in \mathrm{cl}(\mathrm{int}(N))$, a contradiction. ∎

We will regard $\mathsf{ABlock}(X,\varphi)$ with the binary operation $\vee$ and $\wedge$ defined above unless specified otherwise. With respect to these binary operations $\mathsf{ABlock}(X,\varphi)$ does not embed into $\mathsf{ANbhd}(X,\varphi)$.

If we consider a different algebra of sets, i.e. the regular closed sets $\mathscr{R}(X)$ then the same question can be formulated for attracting and repelling blocks.

4.66 Proposition Let $N \in \mathsf{ABlock}(X,\varphi)$ and $M \in \mathsf{RBlock}(X,\varphi)$, then $N \wedge M \in \mathsf{INbhd}(X,\varphi)$.

Proof. We can follow the proof of Proposition 5.7 which yields

$$\Lambda_{\tau''}(N \wedge M) \subset \mathrm{int}(N \cap M).$$

It remains to show that $\mathrm{int}(N \cap M) \subset \mathrm{int}(N \wedge M)$. Observe that

$$\mathrm{int}(N \cap M) \subset \mathrm{cl}(\mathrm{int}(N \cap M)) = N \wedge M,$$

and since $\mathrm{int} \circ \mathrm{int} = \mathrm{int}$ we have $\mathrm{int}(N \cap M) \subset \mathrm{int}(N \wedge M)$, which concludes the proof. ∎

This property plays a role latter on when we discussed isolating blocks.

Summarizing, the operations

$$\cap\colon \mathsf{ANbhd}(X,\varphi) \times \mathsf{RNbhd}(X,\varphi) \longrightarrow \mathsf{INbhd}(X,\varphi),$$
$$\wedge\colon \mathsf{ABlock}(X,\varphi) \times \mathsf{RBlock}(X,\varphi) \longrightarrow \mathsf{INbhd}(X,\varphi),$$

define well-defined mappings and are called *Morse product*. The Morse product is a useful operation to find isolating neighborhoods.

Isolating neighborhoods that arise as the Morse product of an attracting and repelling neighborhood are called *Morse neighborhoods* and are denoted by $\mathsf{MNbhd}(X,\varphi)$. Regular closed Morse neighborhoods are denoted by $\mathsf{MNbhd}_{\mathscr{R}}(X,\varphi)$. In particular, the Morse product of an attracting block and a repelling block yields a regular closed Morse neighborhood.

We can also discuss the above theory within the Boolean algebra of regular closed subsets of X in which case we obtain the following commuting diagram:

$$\begin{array}{ccc}
\mathsf{TrapR}_{\mathscr{R}}(X,\varphi) & \overset{\#}{\longleftrightarrow} & \mathsf{RepR}_{\mathscr{R}}(X,\varphi) \\
\Big\downarrow{\scriptstyle i} & & \Big\downarrow{\scriptstyle i} \\
\mathsf{ANbhd}_{\mathscr{R}}(X,\varphi) & \overset{\#}{\longleftrightarrow} & \mathsf{RNbhd}_{\mathscr{R}}(X,\varphi)
\end{array}$$

Here we use the definitions:

$$\mathsf{TrapR}_{\mathscr{R}}(X,\varphi) := \mathsf{ANbhd}(X,\varphi) \cap \mathsf{Invset}^{+}(X,\varphi) \cap \mathscr{R}(X),$$

$$\mathsf{RepR}_{\mathscr{R}}(X,\varphi) := \mathsf{RNbhd}(X,\varphi) \cap \mathsf{Invset}^{-}(X,\varphi) \cap \mathscr{R}(X).$$

5 — Conley Index

In this chapter we will discuss the Conley index for isolated neighborhoods and for isolated invariant sets if the necessary compact is assumed. The treatment of the Conley in this chapter is based on Benci's approach to the Conley index which allows the theory to be developed in arbitrary metric spaces. Conley index theory unifies the topological tools in this book and we will explain how for degree theory, variational methods and Morse theory are connected.

5.1 Isolating neighborhoods

In order to better understand the behavior of φ on attracting neighborhoods and blocks we define the forward-backward image mapping. Following Benci[5] we introduce the following notion of isolating neighborhood.

5.1 Definition A subset $U \subset X$ and $\tau > 0$, then

$$\Lambda_\tau(U) := \bigcap_{t\in[-\tau,\tau]} \varphi(t,\mathrm{cl}(U)), \tag{5.1.1}$$

which is a closed subset of X by definition.[a]

[a] Arbitrary intersections of closed sets are closed.

If $\tau = \infty$, then define

$$\Lambda_\infty(U) := \bigcap_{\tau>0} \Lambda_\tau(U) = \mathrm{Inv}(\mathrm{cl}(U),\varphi).$$

Note that if $U \in \mathsf{Invset}^+(X,\varphi)$, then $\Lambda_\infty(U) = \omega(U,\varphi)$.[1] For $U \in \mathsf{Invset}^-(X,\varphi)$ we obtain that $\Lambda_\infty(U) = \alpha(U,\varphi)$.

5.2 Lemma Let $U, U' \subset X$. Then,

(i) $U \subset U'$ implies $\Lambda_\tau(U) \subset \Lambda_\tau(U')$ for all $\tau > 0$;

(ii) $\tau > \tau' > 0$ implies $\Lambda_\tau(U) \subset \Lambda_{\tau'}(U)$;

(iii) $\Lambda_\tau(\Lambda_{\tau'}(U)) = \Lambda_{\tau+\tau'}(U)$;

(iv) $\Lambda_\tau(U) \subset \mathrm{int}(U)$ implies $\Lambda_{2\tau}(U) \subset \mathrm{int}(\Lambda_\tau(U)) \subset \mathrm{int}(U)$.

Proof. Under construction. ■

A consequence of (ii)-(iii) above is that if $\Lambda_\tau(U) \subset \mathrm{int}(U)$, then $\Lambda_{t+\tau}(U) \subset \mathrm{int}(U)$ for all $t \geq 0$. In particular $\mathrm{Inv}(\mathrm{cl}(U),\varphi) = \Lambda_\infty(U) \subset \mathrm{int}(U)$. Since we do not suppose any compactness properties on U and X, the invariant set $\Lambda_\infty(U)$ may be the empty set. The same holds for $\Lambda_\tau(U)$. The set $\Lambda_\infty(U)$ is a closed invariant set.

Define the set

$$\mathsf{INbhd}(X,\varphi) := \{U \subset X \;:\; \Lambda_\tau(U) \subset \mathrm{int}(U), \text{ for some } \tau > 0\}.$$

Moreover, $\mathsf{INbhd}_\infty(X,\varphi) := \{U \subset X \;:\; \mathrm{Inv}(\mathrm{cl}(U),\varphi) \subset \mathrm{int}(U)\}$ and

$$\mathsf{INbhd}_\infty(X,\varphi) \subset \mathsf{INbhd}(X,\varphi).$$

In particular $\mathsf{ANbhd}(X,\varphi), \mathsf{RNbhd}(X,\varphi) \subset \mathsf{INbhd}_\infty(X,\varphi)$.

5.3 Exercise Prove the above inclusions. ■

5.4 Exercise Let $U \in \mathsf{INbhd}_\infty(X,\varphi)$. Show that $\mathrm{Inv}(\mathrm{cl}(U),\varphi) = \mathrm{Inv}(U,\varphi)$. ■

Let $U, U' \in \mathsf{INbhd}(X,\varphi)$, then there exist $\tau, \tau' > 0$ be such that $\Lambda_\tau(U) \subset \mathrm{int}(U)$ and $\Lambda_{\tau'}(U') \subset \mathrm{int}(U')$. Define $\tau'' = \max\{\tau, \tau'\}$, then by Lemma 5.2(i)-(ii) we have that $\Lambda_{\tau''}(U \cap U') \subset \Lambda_{\tau''}(U)$ and $\Lambda_{\tau''}(U \cap U') \subset \Lambda_{\tau''}(U')$ and $\Lambda_{\tau''}(U) \subset \Lambda_\tau(U)$ and $\Lambda_{\tau''}(U') \subset \Lambda_{\tau'}(U')$. Consequently,

$$\begin{aligned}\Lambda_{\tau''}(U \cap U') &\subset \Lambda_{\tau''}(U) \cap \Lambda_{\tau''}(U') \subset \Lambda_\tau(U) \cap \Lambda_{\tau'}(U') \\ &\subset \mathrm{int}(U) \cap \mathrm{int}(U') = \mathrm{int}(U \cap U'),\end{aligned}$$

which shows that $U \cap U' \in \mathsf{INbhd}(X,\varphi)$ and $\cap$ is a well-defined binary operation on $\mathsf{INbhd}(X,\varphi)$.

5.5 Proposition The set $\mathsf{INbhd}(X,\varphi)$ is a bounded $\cap$-semi-lattice.

[1]We use the fact that $\mathrm{cl}(\varphi(t,U)) = \varphi(t,\mathrm{cl}(U))$, since $\varphi(t,\cdot)$ is a homeomorphism for all $t \in \mathbb{R}$.

5.6 Exercise Show that $\mathsf{INbhd}(X,\varphi)$ is not closed under $\cup$ in general. ▪

The complement operation are not well-defined on $\mathsf{INbhd}(X,\varphi)$ in general. However, another important operation can be defined. Let $U \in \mathsf{ANbhd}(X,\varphi)$ and $V \in \mathsf{RNbhd}(X,\varphi)$, then $\varphi(t,\mathrm{cl}(U)) \subset \mathrm{int}(U)$ for all $t \geq \tau > 0$ and $\varphi(t,\mathrm{cl}(V)) \subset \mathrm{int}(V)$ for all $t \leq \tau' < 0$. Define $\tau'' = \max\{\tau, -\tau'\}$[2] As before we have

$$\begin{aligned}\varphi(t,\mathrm{cl}(U\cap V)) &\subset \varphi(t,\mathrm{cl}(U)) \subset \mathrm{int}(U), \quad \text{for } t \geq \tau > 0,\\ \varphi(t,\mathrm{cl}(U\cap V)) &\subset \varphi(t,\mathrm{cl}(V)) \subset \mathrm{int}(U), \quad \text{for } t \leq \tau' < 0,\end{aligned}$$

which implies

$$\begin{aligned}\Lambda_{\tau''}(U\cap V) &= \bigcap_{t\in[-\tau'',\tau'']} \varphi(t,\mathrm{cl}(U\cap V)) \subset \varphi(\tau,\mathrm{cl}(U)) \subset \mathrm{int}(U),\\ \Lambda_{\tau''}(U\cap V) &= \bigcap_{t\in[-\tau'',\tau'']} \varphi(t,\mathrm{cl}(U\cap V)) \subset \varphi(\tau',\mathrm{cl}(V)) \subset \mathrm{int}(V),\end{aligned}$$

from which we derive

$$\Lambda_{\tau''}(U\cap V) \subset \mathrm{int}(U)\cap\mathrm{int}(V) = \mathrm{int}(U\cap V).$$

We conclude

5.7 Proposition Let $U \in \mathsf{ANbhd}(X,\varphi)$ and $V \in \mathsf{RNbhd}(X,\varphi)$, then $U\cap V \in \mathsf{INbhd}(X,\varphi)$.

Summarizing, the operations

$$\cap\colon \mathsf{ANbhd}(X,\varphi)\times\mathsf{RNbhd}(X,\varphi) \longrightarrow \mathsf{INbhd}(X,\varphi),$$

define well-defined mappings and are called *Morse product*. The Morse product is a useful operation to find isolating neighborhoods.s

Isolating neighborhoods that arise as the Morse product of an attracting and repelling neighborhood are called *Morse neighborhoods* and are denoted by $\mathsf{MNbhd}(X,\varphi)$.

5.2 Index pairs

Let $U\cap V$, with $U \in \mathsf{TrapR}(X,\varphi)$ and $V \in \mathsf{RepR}(X,\varphi)$, be a Morse neighborhood, then

$$U\cap V = U\cap (V^c)^c = U\setminus V^c,$$

which is the difference of two trapping regions U and V^c. These sets are not necessarily nested, but the Morse neighborhood can also be realized by considering

[2]Note that we use τ and $-\tau'$!

a nest pair of trapping regions. Indeed, $U \cap V^c$ is an trapping region and $U \cap V^c \subset U$, and

$$U \setminus (U \cap V^c) = U \cap (U \cap V^c)^c = U \cap (U^c \cup V) = U \cap V \tag{5.2.2}$$

For a single Morse neighborhood $U \cap V$ we have the following filtration:

$$\varnothing \subset U \cap V^c \subset U \subset X, \tag{5.2.3}$$

which is a linearly ordered sub-lattice of $\mathsf{TrapR}(X,\varphi)$. This filtration for $U \cap V$ is also called the canonical *index pair* for $U \cap V$. The filtration in (5.2.3) is usually denote by $(U \cap V^c, U)$.

A pair of trapping regions defines a isolating neighborhood. We generalize the notion of index pair without assuming trapping regions.

5.8 Definition A pair (N_1, N_0) of closed subsets, with $N_0 \subset N_1 \subset X$, is called an *index pair* if

(i) $N_1 \setminus N_0$ is an isolating neighborhood;
(ii) $\varphi(t, N_0) \cap N_1 \subset N_0$ for all $t \geq 0$ — N_0 is *forward invariant relative to* N_1;
(iii) $x \in N_1$ and $\varphi(t,x) \notin N_1$ for some $t > 0$, implies there exists $t_0 \leq t_1$ such that $\varphi(t_0, x) \in N_0$ — N_0 is a *exit set* for N_1.

■ **5.9 Remark** If N_0 and N_1 are both forward invariant then (ii) and (iii) are automatically satisfied, and if (ii) and (iii) are satisfied then N_0 is forward invariant if and only N_1 is forward invariant. If $N_0 = \varnothing$, then (i) - (iii) implies that N_1 is trapping region. ■

5.10 Definition Let $U \subset X$ be an isolating neighborhood. An index pair (N_1, N_0) is an *index pair for* U if

$$\Lambda_\tau(U) \subset \mathrm{cl}(N_1 \setminus N_0) \subset \mathrm{cl}(U),$$

for some $\tau > 0$.

For an index pair (N_1, N_0) the set $N_1 \setminus N_0$ is an associated isolating neighborhood. The question now is. Given an isolating neighborhood $U \in \mathsf{INbhd}(X,\varphi)$, is there an index pair (N_1, N_0) for U, i.e. (5.2.2) is satisfied? Define

$$\Lambda_\tau^- := \{x \in \Lambda_\tau(U) \;:\; \gamma_x([0,\tau]) \cap \partial U \neq \varnothing\}. \tag{5.2.4}$$

5.11 Theorem Let $U \subset X$ be an isolating neighborhood. Then, there exists a time $\tau > 0$ such that $\big(\Lambda_\tau(U), \Lambda_\tau^-(U)\big)$ is an index pair for U.

Proof. We start by showing that $\Lambda_\tau^-(U)$ is a closed subset of $\partial\Lambda_\tau(U)$. Let $x^n \to x \in \Lambda_\tau^-(U)$ and let $t^n \in [0,\tau]$ be such that $\varphi(t^n, x^n) \in \partial U$. Then, along a subsequence $t^{n_k} \to t^* \in [0,\tau]$ and $\varphi(t^{n_k}, x^{n_k}) \to \varphi(t^*, x) \in \partial U$ by the continuity of φ and the closedness of ∂U. This proves that $\Lambda_\tau^-(U)$ is closed.

For every $x \in \Lambda_\tau^-(U)$ there exists a $t \in [0,\tau]$ such that $\varphi(t,x) \in \partial U$. Choose $x^n \in X \setminus U$ such that $x^n \to \varphi(t,x)$ and thus $\varphi(-t,x^n) \to x$. By definition $\varphi(-t,x^n) \in X \setminus U$ and $x \in \Lambda_\tau(U)$ and since $\partial\Lambda_\tau(U) = \Lambda_\tau(U) \cap \mathrm{cl}(X \setminus \Lambda_\tau(U))$ we conclude that $x \in \partial\Lambda_\tau(U)$. Consequently, $\Lambda_\tau^-(U) \subset \partial\Lambda_\tau(U)$.

The next step is the verify (i)-(iii) in Definition 5.8. To verify (i) observe that, since $\Lambda_\tau^-(U) \subset \partial\Lambda_\tau(U)$, we have $\mathrm{cl}\big(\Lambda_\tau(U) \setminus \Lambda_\tau^-(U)\big) \subset \Lambda_\tau(U)$. Therefore, by Lemma 5.2(iii) and (iv)

$$\Lambda_\tau\big(\Lambda_\tau(U) \setminus \Lambda_\tau^-(U)\big) \subset \Lambda_{2\tau}(U) \subset \mathrm{int}(\Lambda_\tau(U)) \subset \mathrm{int}\big(\Lambda_\tau(U) \setminus \Lambda_\tau^-(U)\big).$$

The latter uses the fact that $\mathrm{int}(\Lambda_\tau(U)) \subset \Lambda_\tau(U) \setminus \Lambda_\tau^-(U)$. This concludes the verification of (i). We also derive that

$$\Lambda_{2\tau}(U) \subset \mathrm{int}\big(\Lambda_\tau(U) \setminus \Lambda_\tau^-(U)\big) \subset \mathrm{cl}\big(\Lambda_\tau(U) \setminus \Lambda_\tau^-(U)\big),$$

which verifies Definition 5.10. ■

The above theorem states that index pair and isolating neighborhoods are equivalent, i.e. one implies the other and vice versa. In order to define invariant we introduce equivalence relations of index pairs and isolating neighborhoods.

5.12 Definition Two index pair (N_1, N_0) and $N_1', N_0')$ are equivalent if

$$\Lambda_\tau(N_1 \setminus N_0) \subset \mathrm{cl}(N_1' \setminus N_0'), \quad \text{and} \quad \Lambda_\tau(N_1' \setminus N_0') \subset \mathrm{cl}(N_1 \setminus N_0),$$

for some $\tau > 0$. The equivalence classes are denoted by $[N_1, N_0] = [N_1', N_0']$ and will be referred to as *index classes*. The set of isolating neighborhood classes is denoted by $\mathsf{IC}(X, \varphi)$.

A similar notion can be introduced for isolating neighborhoods.

5.13 Definition Two isolating neighborhoods U and U' equivalent if

$$\Lambda_\tau(U) \subset \mathrm{cl}(U'), \quad \text{and} \quad \Lambda_\tau(U') \subset \mathrm{cl}(U),$$

for some $\tau > 0$. The equivalence classes are denoted by $[U] = [U']$ and will be referred to as *isolating neighborhood classes*. The set of isolating neighborhood classes is denoted by $\mathsf{INC}(X, \varphi)$.

5.14 Lemma Let $(N_1,N_0) \sim (N_1',N_0')$ be equivalent index pairs. Then, $N_1 \setminus N_0 \sim N_1' \setminus N_0'$.

Proof. By assumption

$$\Lambda_\tau(N_1 \setminus N_0) \subset \mathrm{cl}(N_1' \setminus N_0'), \quad \text{and} \quad \Lambda_\tau(N_1' \setminus N_0') \subset \mathrm{cl}(N_1 \setminus N_0),$$

for some $\tau > 0$, which proves the lemma. ■

5.15 Lemma Let $U \sim U'$ be equivalent isolating neighborhoods with index pairs (N_1,N_0) and (N_1',N_0') respectively. Then, $(N_1,N_0) \sim (N_1',N_0')$.

Proof. By assumption there exist $\tau,\tau' > 0$ such that

$$\begin{aligned}\Lambda_\tau(U) &\subset \mathrm{cl}(N_1 \setminus N_0) \subset \mathrm{cl}(U),\\ \Lambda_{\tau'}(U') &\subset \mathrm{cl}(N_1' \setminus N_0') \subset \mathrm{cl}(U').\end{aligned}$$

This implies $\Lambda_\tau(N_1 \setminus N_0) \subset \mathrm{cl}(N_1 \setminus N_0) \subset \mathrm{cl}(U)$ and Lemma 5.2(i) and (iii) we have that $\Lambda_{2\tau}(N_1 \setminus N_0) \subset \Lambda_\tau(U) \subset \mathrm{cl}(U')$, since $U \sim U'$. In the same we prove that $\Lambda_{2\tau'}(N_1' \setminus N_0') \subset \Lambda_{\tau'}(U') \subset \mathrm{cl}(U)$. Now, Lemma 5.2(ii)

$$\Lambda_{2(\tau+\tau')}(N_1 \setminus N_0) \subset \Lambda_{2\tau+\tau'}(N_1 \setminus N_0) \subset \Lambda_{\tau'}(U') \subset \mathrm{cl}(N_1' \setminus N_0'),$$

and similarly

$$\Lambda_{2(\tau+\tau')}(N_1' \setminus N_0') \subset \Lambda_{\tau+2\tau'}(N_1' \setminus N_0') \subset \Lambda_\tau(U) \subset \mathrm{cl}(N_1 \setminus N_0),$$

which proves the equivalence of (N_1,N_0) and (N_1',N_0'). ■

We can now define the following mapping from $\imath\colon \mathsf{IC}(X,\varphi) \to \mathsf{INC}(X,\varphi)$:

$$[N_1,N_0] \mapsto [N_1 \setminus N_0],$$

which is well-defined by Lemma 5.14. From Lemma 5.15 we have that the mapping $j\colon \mathsf{INC}(X,\varphi) \to \mathsf{IC}(X,\varphi)$ given by

$$[U] \mapsto [\Lambda_\tau(U), \Lambda_\tau^-(U)],$$

for $\tau > 0$ sufficiently large, is also well-defined.

5.16 Proposition The mapping $\imath$ and j are bijections and $\imath \circ j = j \circ \imath = \mathrm{id}$.

Proof. We show that $j \circ \imath = \mathrm{id}$ as the other identity follows along the same lines. We have that $\imath([N_1, N_0)] = [N_1 \setminus N_0]$ and let $U \in [N_1 \setminus N_0]$. Then, by Theorem 5.11 $(\Lambda_\tau(U), \Lambda_\tau^-(U))$ is an index pair for U. Since $U \sim N_1 \setminus N_0$, Lemma 5.15 implies that $(\Lambda_\tau(U), \Lambda_\tau^-(U)) \sim (N_1, N_0)$, which show that the composition

$$[N_1, N_0] \mapsto [N_1 \setminus N_0] \mapsto [(\Lambda_\tau(N_1 \setminus N_0), \Lambda_\tau^-(N_1 \setminus N_0))],$$

is the identity. ■

■ **5.17 Remark** From the above equivalences we have the mapping $\mathsf{INbhd}(X, \varphi) \to \mathsf{INC}(X, \varphi)$ defined by $U \mapsto [U]$. If we define the following binary operation on $\mathsf{INC}(X, \varphi)$: $[U] \wedge [U'] := [U \cap U']$, then this map is a semi-lattice homomorphism. ■

Before we introduce index theory in the next section it is worthwhile to recognize the importance of index classes and isolating neighborhood classes. As a matter of fact index theory will be a way to organize or label these equivalence class but for every index theory we introduce one does loose information, i.e. the index will not be able to characterize the classes. Different classes may have the same index. To avoid this, or to use the best index is to consider the equivalence classes itself — the equivalence classes are the ultimate index theory. This is too much information however and is nearly impossible to compute or use in a practical way. The equivalence classes do however satisfy a fundamental validity principle.

5.18 Proposition Let $U \subset X$ be an isolating neighborhood. If $\Lambda_\tau(U) = \varnothing$ for some $\tau > 0$, then $U \in [\varnothing]$.

Proof. We show that $U \sim \varnothing$. By assumption $\Lambda_\tau(U) = \varnothing \subset \mathrm{cl}(\varnothing)$ and $\Lambda_\tau(\varnothing) = \varnothing \subset \mathrm{cl}(U)$, which proves that U and $\varnothing$ are equivalent. ■

5.19 Corollary Let $U \subset X$ be an isolating neighborhood. If $U \notin [\varnothing]$, then $\Lambda_\tau(U) \neq \varnothing$ for all $\tau > 0$.

A suitable index theory for index and isolating neighborhood classes will be able to determine when $U \notin [\varnothing]$ and thus give information about $\Lambda_\tau(U)$.

5.3 Invariants for index pairs

Let (N_1, N_0) be an index pair, then the following topological space can be assigned to it. Define the pointed space $(N_1/N_0, *)$. For $[x] \in N_1/N_0$ we have $[x] = x$ if $x \in N_1 \setminus N_0$ and $[x] = *$ if $x \in N_0$.

5.20 Proposition Let $(N_1, N_0) \sim (N_1', N_0')$ be equivalent index pairs. Then, the spaces $(N_1/N_0, *)$ and $(N_1'/N_0', *)$ are homotopy equivalent.

Proof. By assumption

$$\Lambda_\tau(N_1 \setminus N_0) \subset \mathrm{cl}(N_1' \setminus N_0'), \quad \text{and} \quad \Lambda_\tau(N_1' \setminus N_0') \subset \mathrm{cl}(N_1 \setminus N_0).$$

Also $\Lambda_\tau(N_1 \setminus N_0) \subset \mathrm{int}(N_1 \setminus N_0)$ and $\Lambda_\tau(N_1' \setminus N_0') \subset \mathrm{int}(N_1' \setminus N_0')$. From Lemma 5.2(iv) we then derive that

$$\Lambda_{2\tau}(N_1 \setminus N_0) \subset \Lambda_\tau(N_1' \setminus N_0') \subset \mathrm{int}(N_1' \setminus N_0'),$$

and similarly $\Lambda_{2\tau}(N_1' \setminus N_0') \subset \mathrm{int}(N_1 \setminus N_0)$. Define the homotopy $\psi\colon [2\tau, \infty) \times N_1/N_0 \to N_1'/N_0'$

$$\psi(t, [x]) = \begin{cases} [\varphi(6t, x)] & \text{if} \,, \varphi([0, 4t], x) \subset N_1 \setminus N_0 \\ & \qquad \text{and} \;\; \varphi([2t, 6t], x) \subset N_1' \setminus N_0'; \\ * & \text{otherwise.} \end{cases}$$

The above defined homotopy is continuous. Similarly, we define a continuous homotopy $\psi'\colon [2\tau, \infty) \times N_1'/N_0' \to N_1/N_0$. Yet another continuous homotopy is given by

$$h(t, [x]) = \begin{cases} [\varphi(12t, x)] & \text{if} \,, \varphi([0, 12t], x) \subset N_1 \setminus N_0; \\ * & \text{otherwise.} \end{cases}$$

Observe that $\psi' \circ \psi = h(\tau, [x])$ and $h(0, [x]) = \mathrm{id}$. By the same token we construct a homotopy h' such that $\psi \circ \psi' = h'(\tau, [x])$ and $h'(0, [x]) = \mathrm{id}$, which proves that $(N_1/N_0, *)$ and $(N_1'/N_0', *)$ are homotopy equivalent. ∎

▪ **5.21 Remark** It follows from the above proof that there exist flow induced homotopies. This is important for defining suitable invariants for index pair classes. ▪

Instead of using homotopy type to define invariant for index classes we choose to use homology theory of index pairs. However, relative homology of pairs $(N_1/N_0, *)$ and (N_1, N_0) is not necessarily isomorphic in all homology theories. To guarantee an isomorphism we construct special index pairs.

5.3.a Regular index pairs

5.22 Definition An index pair (N_1, N_0) is said to be *regular* if the inclusion N_0 is the sub-level set of a Lyapunov function.

5.23 Proposition For every index pair (N_1, N_0) there exists a regular index pair (N_1, N_0') with $N_0 \subset N_0'$.

Proof. To prove the existence of regular index pair we mimic the construction of Lyapunov functions as discussed in Sect. 4.3.c. By assumption $\Lambda_\tau(N_1 \setminus N_0) \subset \mathrm{int}(N_1 \setminus N_0)$. Define the closed sets $A = N_0 \subset N_1$ and $B = \Lambda_{2\tau}^-(N_1 \setminus N_0) \subset \mathrm{cl}(N_1 \setminus N_0) \subset N_1$. The set $\Lambda_{2\tau}^-(N_1 \setminus N_0)$ is characterized by the property that $\gamma_x([0, 2\tau]) \subset \mathrm{cl}(N_1 \setminus N_0)$ for all $x \in \Lambda_{2\tau}^-(N_1 \setminus N_0)$. By construction $N_0 \cap \Lambda_{2\tau}^-(N_1 \setminus N_0) = \varnothing$. Indeed, if $x \in N_0 \cap \Lambda_{2\tau}^-(N_1 \setminus N_0)$, then $\gamma_y([-\tau, \tau]) \subset N_0 \cap \mathrm{cl}(N_1 \setminus N_0)$, with $y = \varphi(\tau, x)$, since N_0 is forward invariant relative to N_1. This contracts the fact that $N_1 \setminus N_0$ is an isolating neighborhood with $t = \tau$.

The above properties imply that for every $x \in N_1 \setminus (A \cup B)$, $\varphi(t, x) \in A$ for some $0 < t \leq 2\tau$. As before we define

$$\delta(x) = \frac{d(x, A)}{d(x, A) + d(x, B)},$$

and

$$\triangle(x) = \begin{cases} \max_{t \in [0, 2\tau]} \delta(\varphi(t, x)) & \text{for } , \varphi([0, 2\tau], x) \subset N_1 \\ 0 & \text{otherwise.} \end{cases}$$

As before the function $\triangle$ is continuous on N_1.

Observe that $\triangle^{-1}(0) = A = N_0$ and $\triangle^{-1}(1) = B = \Lambda_{2\tau}^-(N_1 \setminus N_0)$. This follow along the same lines as in Sect. 4.3.c. Another property that follows from the above construction is that $\triangle(\varphi(t, x)) \leq \triangle(x)$ for all $t \geq 0$ such that $\varphi(t, x) \subset N_1$.

The following function

$$J(x) := \int_0^{\tau(x)} e^{-s} \triangle(\varphi(s, x)) ds, \quad \tau(x) = \sup_{t \geq 0} \{t \ : \ \varphi([0, t], x) \subset N_1\},$$

is continuous on N_1 and is a Lyapunov function with respect to (A, B).

Having established J we now choose $\epsilon > 0$ and define $N_0' := J^\epsilon \supset N_0$. By the above construction (N_1, N_0') is an index pair, which completes the proof. ∎

5.3.b Ważewski's Principle

Recall the definition of deformation retract.

5.24 Definition Let $A \subset X$. A *deformation retraction* of X onto A is a continuous map $h : X \times [0,1] \to X$ such that

$$\begin{aligned} h(x,0) &= x \quad \text{for all} \quad x \in X, \\ h(x,1) &\in A \quad \text{for all} \quad x \in X, \\ h(a,1) &= a \quad \text{for all} \quad a \in A. \end{aligned}$$

If such an h exists, then A is called a *deformation retract* of X. The map h is called a *strong deformation retraction*. If the third identity is reinforced as follows:

$$h(a,s) = a \text{ for all } a \in A \text{ and all } s \in [0,1],$$

then the set A is a *strong deformation retract* of X.

Note that the map $r : X \to A$ defined by $r(x) = h(x,1)$ has the property $r_{|A} = \mathrm{id}_A$. Any continuous map with this property is called a *retraction* and its image A is called a *retract*. Thus a deformation retract is a special case of a retract.

In order to show that in the above example N' is a strong deformation retract we formulate the WażewskiPrinciple. Let $W \subset X$ be any subset. Define

$$W^0 = \{x \in W \mid \text{there exists } t > 0 \text{ such that } \varphi(t,x) \notin W\}$$

and

$$W^- = \{x \in W \mid \varphi([0,t),x) \not\subset W \text{for all } t > 0\}. \tag{5.3.5}$$

Then W^0 and W^- are the sets of all points which eventually leave W and which immediately leave W in forward time respectively. Note that $W^- \subset W^0$, and both sets could be empty.

5.25 Definition A set W is a *Ważewski set* if the following conditions are satisfied.

1. If $x \in W$ and $\varphi([0,t],x) \subset \mathrm{cl}(W)$, then $\varphi([0,t],x) \subset W$.
2. W^- is closed relative to W^0.

5.26 Theorem If W is a Ważewski set, then W^- is a strong deformation retract of W^0 and W^0 is open relative to W.

Proof. The first step is to construct the strong deformation retraction

$$r : W^0 \times [0,1] \to W^0.$$

Define $\tau : W^0 \to \mathbb{R}$ by

$$\tau(x) = \sup\{t \geq 0 \mid \varphi([0,t],x) \subset W\}.$$

By the definition of W^0, $\tau(x)$ is finite and by the continuity of the flow $\varphi([0,t],x) \subset \mathrm{cl}(W)$. Since W is a Ważewski set, $\varphi(\tau,x) \in W$, and in fact the definition of τ implies that $\varphi(\tau,x) \in W^-$. Observe that $\tau(x) = 0$ if and only if $x \in W^-$.

Assume for the moment that τ is continuous and define r by $r(x,\sigma) = \varphi(\sigma\tau(x),x)$. Now notice that

$$
\begin{aligned}
r(x,0) &= \varphi(0,x) = x \\
r(x,1) &= \varphi(\tau(x),x) \in W^-
\end{aligned}
$$

and for $y \in W^-$

$$
r(y,\sigma) = \varphi(\sigma\tau(x),y) = \varphi(0,y) = y.
$$

Therefore r is a strong deformation retraction of W^0 to W^-.

Returning now to the question of continuity we first prove that τ is upper semicontinuous. Let $x \in W^0$ and $\epsilon > 0$, then $\varphi([\tau(x),\tau(x)+\epsilon],x) \not\subset W$. By the first condition there exists $t_0 \in [\tau(x),\tau(x)+\epsilon]$ such that $\varphi(t_0,)x) \notin \mathrm{cl}(W)$. Thus we can choose V be a neighborhood of $x \cdot t_0$ such that $V \cap \mathrm{cl}(W) = \varnothing$. Now let U be a neighborhood of x such that $\varphi(t,U) \subset V$. Then for $y \in U \cap W$, $\varphi(t_0,y) \notin W$. Note that this proves that W^0 is open relative to W and that $\tau(y) < \tau(x) + \epsilon$. Hence, τ is upper semi-continuous.

To prove lower semi-continuity of τ, let $x \in W^0/W^-$ and let $0 < \epsilon < \tau(x)$. Then $\varphi([0,\tau(x)-\epsilon],x) \subset W^0$. Since W^- is closed relative to W^0, $\varphi([0,\tau(x)-\epsilon],x) \cap W^- = \varnothing$ and hence for all $s \in [0,\tau(x)-\epsilon]$ there exists a neighborhood U_s of $\varphi(s,x)$ such that $U_s \cap W^- = \varnothing$. Of course $\{U_s\}$ covers $\varphi([0,\tau(x)-\epsilon],x)$ which is compact and hence a finite number $\{U_{s_i} \mid i = 1,\dots I\}$ covers $\varphi([0,\tau(x)-\epsilon],x)$. Let $U = \cup_{i=1}^{I} U_{s_i}$, then U is open which implies there exists V a neighborhood of x such that $\varphi([0,\tau(x)-\epsilon],V) \subset U$. Now $U \cap W^- = \varnothing$ implies that for all $y \in V$, $\varphi([0,\tau(x)-\epsilon],y) \cap W^- = \varnothing$. Thus, $\tau(y) \geq \tau(x) - \epsilon$. This implies that τ is lower semicontinuous and hence continuous. ∎

Let (N_1,N_0) be a regular index pair and consider the set $W = \mathrm{cl}(N_1 \setminus N_0)$. Observe that

$$
W^- = N_0 \cap W,
$$

since N_0 is an immediate exit set and W^- is closed which implies that W is a Ważewski set. By Theorem 5.26 we then derive that W^- is a strong deformation retract of W^0 and W^0 is open neighborhood of W^- in W. These properties imply that the pair (W,W^-) is a *neighborhood deformation retract pair*, i.e. There exists a homotopy $h\colon W \times [0,1] \to W$ and a function $u\colon W \to [0,1]$, such that $W^- = u^{-1}(0)$, $h(x,0) = x$ for all $x \in W$, $h(x,t) = x$ for all $x \in W^-$ and for all $t \in [0,1]$, and $h(x,1) \in W^-$ for all $x \in u^{-1}([0,1))$.

5.27 Exercise Prove the above statement. ■

Now consider then extension $\tilde{h}: N_1 \times [0,1] \to N_1$ be taking $\tilde{h} = \mathrm{id}$ on N_0 and $\tilde{h} = h$ on W. Also $\tilde{u} = u$ on W and $\tilde{u} = 0$ on N_0. This then establishes (N_1, N_0) as a neighborhood deformation retract pair.

5.28 Lemma Let (N_1, N_0) be a regular index pair. Then, $\tilde{H}_k(N_1/N_0) \cong H_k(N_1, N_0)$ for all k, where $H_k(\cdot)$ is singular homology with field coefficients and $\tilde{H}_k$ is reduced singular homology.

Proof. From homology theory we have that $\tilde{H}_k(N_1/N_0) \cong H_k(N_1/N_0, *)$, where $*$ represents a point in N_1/N_0. Theset N_0 is a strong deformation retract of the set $K = \tilde{u}^{-1}([0,1))$ and K/N_0 is contractable via $\tilde{h}$. This yields $H_k(N_1/N_0, *) \cong H_k(N_1/N_0, K/N_0)$. Excise the point $* = [N_0]$ in the quotient space N_1/N_0, then $N_1/N_0 \setminus [N_0] = N_1 \setminus N_0$ and $K/N_0 \setminus [N_0] = K \setminus N_0$, which implies $H_k(N_1/N_0, K/N_0) \cong H_k(N_1 \setminus N_0, K \setminus N_0)$. Combining all isomorphisms we obtain

$$\tilde{H}_k(N_1/N_0) \cong H_k(N_1 \setminus N_0, K \setminus N_0) \cong H_k(N_1, K) \cong H_k(N_1, N_0),$$

which completes the proof. ■

If (N_1, N_0) and (N_1', N_0') are equivalent regular index pairs then $(N_1/N_0, *)$ and $(N_1'/N_0', *)$ are homotopy equivalent by Proposition 5.20. For reduced homology this implies $\tilde{H}_k(N_1/N_0) \cong H_k(N_1', N_0')$ and therefore

$$H_k(N_1, N_0) \cong H_k(N_1', N_0'), \quad \forall k. \tag{5.3.6}$$

Relative homology provides an invariant on index classes.

■ **5.29 Remark** If we use different homology theories the statement of Lemma 5.28 follows for every index pair. For example if we use Cech co-homology then we can use arbitrary index pairs. ■

5.4 The Conley Index

We are now in a position to define invariants for isolating neighborhood classes.

5.4.a Definition of the index

Let $U \subset X$ be a isolating neighborhood. Then, by the previous considerations there exists a regular index pair (N_1, N_0) such that $\Lambda_\tau(U) \subset \mathrm{cl}(N_1 \setminus N_0) \subset \mathrm{cl}(U)$ for some $\tau > 0$. This yields the following definition:

5.30 Definition Let $U \subset X$ be an isolating neighborhood and let (N_1, N_0) be a regular index pair for U. The *Conley index* of U is defined by

$$HC_k([U]) := H_k(N_1, N_0), \quad k \in \mathbb{Z}, \tag{5.4.7}$$

where $[U]$ is the isolating neighborhood class of U.

It remains to verify that the Conley index is well-defined. As explained above every isolating neighborhood U allows a regular index pair (N_1, N_0) and by (5.3.6) we have that different index pairs yield isomorphic homologies, with canonical isomorphisms. The appropriate way to define $HC_k([U])$ would be to consider the inverse limit of the associated inverse system. For purposes in this book it suffices to interpret the Conley index by the definition in (5.4.7).

If $U \sim U'$ are equivalent isolating neighborhoods, then also the homologies are isomorphic. Indeed, let (N_1, N_0) and (N_1', N_0') be regular index pairs for U and U' respectively. Then, by Lemma 5.15 we have that $(N_1, N_0) \sim (N_1', N_0')$. Therefore, by Proposition 5.20 and (5.3.6) we then have $H_k(N_1, N_0) \cong H_k(N_1', N_0')$, which proves that the Conley index is well-defined and is an invariant for the index class $[U]$.

Traditionally the Conley index is defined for isolated invariant sets. However, compactness is assumed in that case. in Benci's approach to the Conley index compactness is not introduced at this point which have great advantages as we will see in the forthcoming sections. The the relation of the Conley in Definition 5.4.7 and the traditional definition is postponed until we discuss the compactness issues. We have also chosen a homological definition of the Conley index instead of a homotopy index.

The Conley index as defined in 5.30 yields an invariants for isolating neighborhood classes $[U]$. It is important to point out that this invariant does not necessarily distinguish between different classes, i.e. $HC_k([U]) \cong HC_k([U'])$ for all k does *not* imply necessarily that $U \sim U'$. However, $U \sim U'$, implies $HC_k([U]) \cong HC_k([U'])$ for all k. As consequence, if $HC_k([U]) \not\cong HC_k([U'])$ for some k, then $U \not\sim U'$.

5.31 Lemma If $\varnothing \in [U]$, then $HC_k([U]) = 0$ for all k.

Proof. Use $\Lambda_\tau(U) = N_1 = \varnothing$ for $\tau > 0$ sufficiently large. This proves the lemma. ∎

If we combine Lemma 5.31 with the previous considerations, then $HC_k([U]) \neq 0$ for some k implies that $\varnothing \notin [U]$. Corollary 5.19 then implies that $\Lambda_\tau(U) \neq \varnothing$ for all $\tau > 0$. With additional compactness assumptions this also contains information about the maximal invariant set inside U.

If the Conley index $HC_k([U]) \neq 0$ the above validity principle implies that $\Lambda_\tau(U) \neq \varnothing$ for all $\tau > 0$, which does not necessarily imply that $\mathrm{Inv}(U, \varphi) = \mathrm{Inv}(\mathrm{cl}(U), \varphi) = \bigcap_{\tau > 0} \Lambda_\tau(U)$ is non-empty. Equivalence of isolating neighborhoods and invariance are related as follows:

5.32 Lemma If $U \sim U'$ are equivalent isolating neighborhoods, then $\mathrm{Inv}(U,\varphi) = \mathrm{Inv}(U',\varphi)$.

Proof. By definition there exists a $\tau > 0$ such $\Lambda_\tau(U) \subset \mathrm{cl}(U')$ and $\Lambda_\tau(U') \subset \mathrm{cl}(U)$. Moreover, $\Lambda_\tau(U) \subset \mathrm{cl}(U)$ and $\Lambda_\tau(U') \subset \mathrm{cl}(U')$, which implies that

$$S = \mathrm{Inv}(U,\varphi) \subset \mathrm{cl}(U) \cap \mathrm{cl}(U'), \quad S' = \mathrm{Inv}(U',\varphi) \subset \mathrm{cl}(U) \cap \mathrm{cl}(U').$$

Since both S and S' are invariant we derive that

$$S \subset \mathrm{Inv}(\mathrm{cl}(U) \cap \mathrm{cl}(U'),\varphi) \subset \mathrm{Inv}(\mathrm{cl}(U'),\varphi) = S',$$

and similarly, $S' \subset S$. ∎

5.4.b Compactness properties

The converse of this statement need not be true. In order to achieve a statement about $\mathrm{Inv}(U,\varphi)$ in relation to equivalence we need to impose certain compactness conditions on the isolating neighborhoods U.

5.33 Definition Let $U \subset X$ be an isolating neighborhood with the following property: for every neighborhood V of $\mathrm{Inv}(U,\varphi)$ there exists a $\tau > 0$ such that $\Lambda_\tau(U) \subset V$. The set of isolating neighborhoods with this property will be denoted by $\mathsf{INbhd}_c(X,\varphi)$ and are called *tight isolating neighboroods*. Isolated invariant sets for which there exists a compactly supported isolating neighborhood $U \subset X$ such that $\mathrm{Inv}(U,\varphi) = S$ are called *tight isolated invariant sets* and are denoted by $\mathsf{Isol}_c(X,\varphi)$.

The notion of tight neighborhood allows us to obtain a validity principle concerning isolated invariant sets.

5.34 Lemma Let $U, U' \in \mathsf{INbhd}_c(X,\varphi)$, then $\mathrm{Inv}(U,\varphi) = \mathrm{Inv}(U',\varphi)$ implies that $U \sim U'$.

Proof. Observe that $V = U \cap U'$ is a neighborhood of $S = \mathrm{Inv}(U,\varphi) = \mathrm{Inv}(U',\varphi)$. By assumption there exist $\tau,\tau' > 0$ such that $\Lambda_\tau(U) \subset V \subset U' \subset \mathrm{cl}(U')$ and $\Lambda_{\tau'}(U') \subset V \subset U \subset \mathrm{cl}(U)$. Set $\tau'' = \max\{\tau,\tau'\}$, then

$$\Lambda_{\tau''}(U) \subset \Lambda_\tau(U) \subset \mathrm{cl}(U'), \quad \Lambda_{\tau''}(U') \subset \Lambda_{\tau'}(U') \subset \mathrm{cl}(U),$$

which proves the lemma. ∎

As consequence of Lemma 5.34 we have that $U \not\sim U'$, with $U, U' \in \mathsf{INbhd}_c(X,\varphi)$, implies that $\mathrm{Inv}(U,\varphi) \neq \mathrm{Inv}(U',\varphi)$, which yields the following proposition.

5.35 Proposition Let $U \in \mathsf{INbhd}_c(X,\varphi)$, then $HC_k([U]) \neq 0$ for some k, implies that $\mathrm{Inv}(U,\varphi) \neq \varnothing$.

Proof. If $HC_k([U]) \neq 0$, then Lemma 5.31 implies that $U \not\sim \varnothing$. Lemma 5.34 implies that $\mathrm{Inv}(U,\varphi) \neq \varnothing$. ■

Important question is when isolating neighborhoods are tight. We now give some criteria.

If X is a compact metric space then $\Lambda_\tau(U)$ is compact for all τ and therefore by Cantor's intersection theorem $\mathrm{Inv}(U,\varphi)$ is non-empty and compact. This implies that if $HC_k([U]) \neq 0$ for some k then $\mathrm{Inv}(U,\varphi) \neq \varnothing$. It also follows in the case that X is a compact metric space that every isolating neighborhood is tight. Indeed, for every neighborhood V of $\mathrm{Inv}(U,\varphi)$ one can choose $\tau > 0$ large enough such that $\Lambda_\tau(U) \subset V$.

In the case that X is compact it also holds that U is an isolating neighborhood for $\mathrm{Inv}(U,\varphi)$ if and only if U is an isolating neighborhood.

If X is a locally compact metric then every bounded isolating neighborhood is tight. Indeed, $\Lambda_\tau(U)$ is a closed and bounded subset of U and therefore compact. Therefore also $\mathrm{Inv}(U,\varphi)$ and tightness follows.

Suppose an isolating neighborhood $U \subset X$ has the following property: for every sequence $\{x^n\} \subset U$ and every sequence $t^n \to \infty$ such that $\varphi([0,t^n],x^n) \subset U$, the sequence $\{\varphi(t^n,x^n)\}$ has a limit point. Isolating neighborhoods with the above property are tight.

In the classical treatment of the Conley index, X is a compact, or locally compact metric and the focal point is *compact* isolated invariant sets S. Compact isolating neighborhoods U are tight. For compact isolating neighborhoods the following equivalence can be proved.

5.36 Proposition Let X be a locally compact metric space and let $U, U' \subset X$ be two compact isolating neighborhoods. Then, $U \sim U'$ if and only if $\mathrm{Inv}(U,\varphi) = \mathrm{Inv}(U',\varphi)$.

Proof. If $U, U' \subset X$ are compact isolating neighborhoods, then U, U' are tight and by Lemma 5.34 equivalence of U and U' follows if $\mathrm{Inv}(U,\varphi) = \mathrm{Inv}(U',\varphi)$. The converse follows from Lemma 5.32. ■

Proposition 5.36 implies that $S = \mathrm{Inv}(U,\varphi)$ is an invariant for $[U]$ if U is a compact isolating neighborhood in a locally compact metric space. This justifies the notation $HC_k(S)$ as the Conley index of $[U]$. This retrieves the classical Conley index as introduced by C.C. Conley.

■ **5.37 Remark** We sometimes write $HC_k([U], \varphi)$ and $HC_k(S, \varphi)$. However, if there is no ambiguity about the dependence on the flow, then φ is omitted from the notation. ■

5.5 Index filtrations and the Morse relations

In order to derive the Morse relations for the Conley index we start with the easiest case: filtrations of isolating blocks.

5.5.a Filtrations of isolating blocks and trapping regions

Consider a filtration of attracting blocks

$$\varnothing \subset N_1 \subset \cdots \subset N_m \subset X,$$

with $N_i \subset \mathsf{ABlock}(X, \varphi)$ for $i = 1, \cdots, m$. A filtration is a sub-lattice of $\mathsf{ABlock}(X, \varphi)$ which is linearly ordered. We denote a filtration by $\mathsf{N} = \{N_i\}$. A filtration of trapping regions is defined in exactly the same way. By construction every nested pair $N_i \subset N_j$, $i \le j$ is an index pair. This holds for both attracting blocks and trapping regions. The following lemma provides a relation between filtrations of isolating blocks and trapping regions.

5.38 Lemma Let $\mathsf{N} \subset \mathsf{TrapR}(X, \varphi)$ be a (finite) filtration. Then, there exists a filtration $\mathsf{N}' \subset \mathsf{ABlock}(X, \varphi)$ such that $(N_j, N_i) \sim (N_j', N_i')$ for all $i \le j$. Two such filtrations are said to be equivalent: $\mathsf{N} \sim \mathsf{N}'$.

Proof. Under construction. ■

It is important to point out that the above notions also apply to arbitrary finite sub-lattices of $\mathsf{ABlock}(X, \varphi)$ and $\mathsf{TrapR}(X, \varphi)$. This will play a role in the applications in Chapter 9. If we have a sub-lattice $\mathsf{N} \subset \mathsf{TrapR}(X, \varphi)$, then the analogue of Lemma 5.38 remains true and there exists an isomorphic sub-lattice $\mathsf{N}' \subset \mathsf{ABlock}(X, \varphi)$ such that $(N_j, N_i) \sim (N_j', N_i')$ for every nested pair. We again write $\mathsf{N} \sim \mathsf{N}'$. The above idea of filtrations and sub-lattices can be extended to index pairs.

5.5.b Filtrations of index pairs

Let (N, L) be a regular index pair. Let L' be a set, with $L \subset L' \subset N$, such that (N, L') is a regular index pair. Such a situation already occurred with regular index pairs. Denote the set of sets L' as described above by $\mathsf{ExitR}(N, L, \varphi)$.

5.39 Proposition Let $L', L'' \in \mathsf{ExitR}(N, L, \varphi)$, then $L' \cup L'' \in \mathsf{ExitR}(N, L, \varphi)$ and $L' \cap L'' \in \mathsf{ExitR}(N, L, \varphi)$.

Proof. Since L', L'' are forward invariant relative to N, so are $L' \cap L''$ and $L' \cup L''$. Similarly, $L' \cap L''$ and $L' \cup L''$ are exit sets since they both contain L. It remains to show that $N \setminus (L' \cap L'')$ and $N \setminus (L' \cup L'')$ are isolating neighborhoods. Firstly, $N \setminus (L' \cup L'') = (N \setminus L') \cap (N \setminus L'')$ and therefore by Proposition 5.5 $N \setminus (L' \cup L'')$ is an isolating neighborhood.

Secondly, for $N \setminus (L' \cap L'')$ we argue as follows. Since L' and L'' are regular they are given as sub-level sets of Lyapunov, i.e. $L' = J'^{\epsilon}$ and $L'' = J''^{\epsilon}$. The function $\min\{J', J''\}$ is a Lyapunov function for $L' \cup L''$ which ensures that $N \setminus (L' \cap L'')$ is isolating. ∎

Proposition 5.39 shows that $\mathsf{ExitR}(N, L, \varphi)$ is a bounded distributive lattice. For the case $(N, L) = (X, \varnothing)$ we obtain $\mathsf{ABlock}(X, \varphi)$. Filtrations will now be discussed in terms of sub-lattices of $\mathsf{ExitR}(N, L, \varphi)$. As before filtrations contain both L and N, i.e. a filtration $\mathsf{L} \subset \mathsf{ExitR}(N, L\varphi)$ is given by

$$L \subset L_1 \subset \cdots \subset L_m \subset L_{m+1} = N.$$

If every index pair in an index filtration L is regular, then L is called an *regular index filtration*.

5.40 Lemma Let $\mathsf{L} \subset \mathsf{ExitR}(N, L, \varphi)$ be a (finite) filtration. Then, there exists a filtration $\mathsf{L}' \subset \mathsf{ExitR}(N, L, \varphi)$ such that $(L_j, L_i) \sim (L'_j, L'_i)$ for all $i \le j$. Two such filtrations are said to be equivalent: $\mathsf{L} \sim \mathsf{L}'$.

Proof. Under construction. ∎

5.5.c The Morse relations

Let $\mathsf{L} \subset \mathsf{ExitR}(N, L, \varphi)$ be a regular index filtration, then the Conley index $HC_k([L_j \setminus L_i])$ is well-defined for all $i \le j$. The fact that L is a filtration yields a relation between the different Conley indices. We start with a general statement about filtrations of subsets and homology. For a topological pair (X, Y) the Poincaré series of homology is given by $P_t(X, Y) := \sum_{k \ge 0} \mathrm{rank} H_k(X, Y)$.

5.41 Proposition Let $X_0 \subset \cdots \subset X_n$ be a filtration (of metric spaces), then

$$\sum_{i=0}^{n-1} P_t(X_{i+1}, X_i) = P_t(X_n, X_0) + (1+t) Q_t, \tag{5.5.8}$$

where Q_t has non-negative coefficients.

Proof. For a triple $X_{p-1} \subset X_p \subset X_{p+1}$ we have the long exact sequence

$$\cdots \xrightarrow{\partial_{k+1}} H_k(X_p, X_{p-1}) \xrightarrow{i_k^*} H_k(X_{p+1}, X_{p-1}) \xrightarrow{j_k^*} H_{k-1}(X_{p+1}, X_p) \xrightarrow{\partial_k} \cdots,$$

with $\ker i_k^* = \mathrm{im}\,\partial_{k+1}$, $\ker j_k^* = \mathrm{im}\, i_k^*$ and $\ker \partial_k = \mathrm{im}\, j_k^*$. For the ranks of the subspaces involved we obtain:

$$\begin{aligned}
\mathrm{rank} H_k(X_p, X_{p-1}) &= \mathrm{rank}\mathrm{im}\, i_k^* + \mathrm{rank}\ker i_k^* = \mathrm{rank}\mathrm{im}\, i_k^* + \mathrm{rank}\partial_{k+1} \\
\mathrm{rank} H_k(X_{p+1}, X_{p-1}) &= \mathrm{rank}\mathrm{im}\, j_k^* + \mathrm{rank}\ker j_k^* = \mathrm{rank}\mathrm{im}\, j_k^* + \mathrm{rank}\mathrm{im}\, i_k^* \\
\mathrm{rank} H_k(X_{p+1}, X_p) &= \mathrm{rank}\partial_k + \mathrm{rank}\ker \partial_k = \mathrm{rank}\partial_k + \mathrm{rank}\mathrm{im}\, j_k^*,
\end{aligned}$$

which gives the relation

$$\begin{aligned}
\mathrm{rank} H_k(X_p, X_{p-1}) &+ \mathrm{rank} H_k(X_{p+1}, X_p) - \mathrm{rank} H_k(X_{p+1}, X_{p-1}) \\
&= \mathrm{rank}\partial_k + \mathrm{rank}\partial_{k+1}, \quad \forall k.
\end{aligned}$$

This yields

$$P_t(X_p, X_{p-1}) + P_t(X_{p+1}, X_p) = P_t(X_{p+1}, X_{p-1}) + (1+t)Q_t^p,$$

where $Q_t^p = \sum_{k\geq 0} \mathrm{rank}\partial_k t^k \geq 0$. This shows that Equation (5.5.8) holds for $n = 2$. Suppose Equation (5.5.8) is true for n, then consider the topological triple $X_0 \subset X_n \subset X_{n+1}$, which yields $P_t(X_n, X_0) + P_t(X_{n+1}, X_n) = P_t(X_{n+1}, X_0) + (1+t)\tilde{Q}_t$. Then,

$$\begin{aligned}
\sum_{i=0}^{n} P_t(X_{i+1}, X_i) &= P_t(X_n, X_0) + P_t(X_{n+1}, X_n) + (1+t)Q_t, \\
&= P_t(X_{n+1}, X_0) + (1+t)Q_t^n + (1+t)\tilde{Q}_t \\
&= P_t(X_{n+1}, X_0) + (1+t)Q_t,
\end{aligned}$$

which proves the proposition. ■

For the Conley index we can define the analogue of the Poincaré series:

5.42 Definition Let (N, L) be a regular index pair. Then the *Conley series* is given by the Poincaré series

$$PC_t([N \setminus L]) := \sum_{k\geq 0} \mathrm{rank} HC_k([N \setminus L]) t^k.$$

Let $\mathsf{L} \subset \mathsf{ExitR}(N, L, \varphi)$ be a (finite) index filtration for an index pair (N, L), then Proposition 5.41 we have that

$$\sum_{i=0}^{m} P_t(L_{i+1}, L_i) = P_t(L_{m+1}, L_0) + (1+t)Q_t,$$

where $Q_t = \sum_{k\geq 0} \mathrm{rank}\partial_k t^k \geq 0$. By definition $P_t(L_{i+1}, L_i) = PC_t([L_{i+1} \setminus L_i])$ and $P_t(L_{m+1}, L_i) = PC_t([N \setminus L])$. This gives the following theorem.

5.43 Theorem Let $\mathsf{L} \subset \mathsf{ExitR}(N,L,\varphi)$ be a (finite) index filtration for an index pair (N,L). Then, the Conley indices of the index pairs (L_{i+1},L_i) satisfy

$$\sum_{i=0}^{m} PC_t([L_{i+1} \setminus L_i]) = PC_t([N \setminus L]) + (1+t)Q_t, \tag{5.5.9}$$

and which are called the *Morse relations* of L.

■ **5.44 Remark** If we consider arbitrary sub-lattices $\mathsf{L} \subset \mathsf{ExitR}(N,L,\varphi)$, then the Morse relations remain unchanged. As a matter of fact a sub-lattice contains information about connecting orbits. Some of this information is lost in the Morse relations. We will come back to this issue in Chapter 9. ■

■ **5.45 Remark** If the index pair (N,L) is tight, i.e. $N \setminus L$ is a tight isolating neighborhood, then the Morse relations provide a relation between invariants for invariant sets:

$$\sum_{i=0}^{m} PC_t(M_i) = PC_t(S) + (1+t)Q_t,$$

where $S = \mathrm{Inv}(N \setminus L$ and $M_i = \mathrm{Inv}(L_{i+1} \setminus L_i)$. The sets $\mathsf{M} = \{M_i\}$ obtained via the index filtration L is referred top as a *Morse decomposition*. ■

5.6 Continuation

Here we discuss the continuation theory for the Conley index at least in the compact case. Further extensions pending.

5.7 Cup-length estimates

In this section we discuss topological invariants that can be derived from the ring structure of the cohomological Conley index. This is used for multiplicity results. Maybe mention Ljusternik-Schnirelmann theory and Ljusternik-Schnirelmann category.

5.8 Problems

Let N be a smooth, closed, orientable manifold of dimension n and $f\colon N \to TN$ be a smooth vector field on N. Generically f has finitely many zeroes and denote the index of a zero by $\iota(x) := \iota(f,N,0)$, see Sect. 1.4.b. Then, the *Poincaré-Hopf Index Theorem* states:

$$\sum_{x \in f^{-1}(0)} \iota(x) = \chi(N), \tag{5.8.10}$$

where $\chi(N)$ is the Euler characteristic of N.

5.46 Problem Let $f, g\colon N \to TN$ be vector fields on N with finitely many zeroes. Show that $\sum_{x\in f^{-1}(0)} \iota(x) = \sum_{x'\in g^{-1}(0)} \iota(x')$.

5.47 Problem Let x be a zero of a (negative) gradient vector field $-\nabla J\colon N \to TN$. Show that if $\mu = \mu(x)$ is the Morse index of x, then $\iota(x) = (-1)^{\mu(x)}$.

5.48 Problem Use Problems 5.46 and 5.47 to prove the Poincaré-Hopf Index Theorem in (5.8.10).

The following problem gives a generalization of the Hairy Ball Theorem, cf. Theorem 2.33.

5.49 Problem A smooth, closed, connected, orientable manifold N admits a non-vanishing vector field $f\colon N \to TN$ if and only if $\chi(N) \neq 0$.

6 — Morse Theory

Variational methods are used for finding critical points of differentiable functions. In general the functions are defined on an infinite dimensional space. Such problems typically occur for large classes of ordinary and partial differential equations and are called *variational problems*. In classical mechanics the functions, or functionals are called Lagrangians and the critical point equations are referred to as the Euler-Lagrange equations. In this Chapter we discuss a class of methods, also referred Morse Theory, that is used to find critical points of functions on finite and infinite dimensional spaces. The main characteristic is to link topological properties of the space to the set of critical points of the function in question. Certain aspects of Morse Theory can regarded as a special case of Conley Theory in the case of gradient and gradient-like flows.

6.1 Gradient-like flows on compact spaces

An important class of flows that mimic gradient flows in the continuous case are called gradient-like flows and are defined as follows. Assume that X is a compact metric space.

6.1 Definition A dynamical system $\varphi : \mathbb{R} \times X \to X$ is called *gradient-like* if there exists a continuous function $V : X \to \mathbb{R}$ such that $V(\varphi(t,x))$ is strictly decreasing in t for all $x \notin \mathsf{E} = \{x \in X \mid \varphi(t,x) = x, \ \forall t \in \mathbb{R}\}$, where E is the set of equilibria of φ.

Every gradient flow is trivially gradient-like. Compact invariant sets S have a special structure.

6.2 Lemma If S is a compact invariant set of a gradient-like dynamical system, then S consists of equilibrium points and heteroclinic connecting orbits, i.e. bounded orbits γ_x with $\alpha(x), \omega(x) \subset \mathsf{E}$.

Proof. Since $V(\varphi(t,x))$ is non-increasing, if $\gamma_x = \varphi(\mathbb{R}, x)$ is a bounded orbit, then $V(\varphi(t,x))$ is a decreasing, bounded function which has limits $V(\varphi(t,x)) \to c_\pm$ as $t \to \pm\infty$. Suppose $y \in \omega(x)$. Then by the definition of omega limit set there exist times $t_n \to \infty$ such that $\varphi(t_n, x) = y$, and hence $V(y) = c_+$. Suppose $y \notin \mathsf{E}$. Then for any $\tau > 0$, we have $V(\varphi(\tau, y)) < c_+$. However, $\varphi(t_n + \tau, x) \to \varphi(\tau, y)$ so that $V(\varphi(t_n + \tau, x)) \to c_+$ implying $V(\varphi(\tau, y)) = c_+$, a contradiction. ∎

In particular, when $\mathsf{E} \cap S$ is a finite set, then every orbit $\gamma_x \subset S$ converges to equilibrium points in E. Indeed, since $\alpha(x)$ and $\omega(x)$ are connected sets, it follows that if E is finite, then $\alpha(x)$ and $\omega(x)$ consist of single points.

Let us continue with the case where E is a finite set. Then Lemma 6.2 implies that E is a Morse decomposition for X. Note that the same holds in general (E not necessarily finite) if we take finitely many connected components of E. Let $S \subset X$ be an isolated invariant set (for example when $S = X$), then the Morse relations in Theorem **??** yield:

$$\sum_{x \in \mathsf{E} \cap S} CP_t(\{x\}, \varphi) = CP_t(S, \varphi) + (1+t)Q_t, \tag{6.1.1}$$

where $CP_t(\{x\}, \varphi)$ are the Poincaré polynomials of $HC_*(\{x\}, \varphi)$. The latter is defined via an isolating block for $x \in \mathsf{E} \cap S$.

There is a way to compute the Conley index of an equilibrium point by using the function V. In the case that $V(x) = c$ is a different for every $x \in \mathsf{E} \cap S$, then $V_{c-\epsilon}^{c+\epsilon} = \{x \in X \mid c - \epsilon \le V(x) \le c + \epsilon\}$, with $\epsilon > 0$ sufficiently small, is an isolating block. Then, $(V^{c+\epsilon}, V^{c-\epsilon})$ is an index pair of attracting blocks, where $V^{c\pm\epsilon} = \{x \mid V(x) \le c \pm \epsilon\}$ and $HC_*(\{x\}, \varphi) \cong H_*(V^{c+\epsilon}, V^{c-\epsilon})$. Since the Conley index of $x \in \mathsf{E}$ does not depend on ϵ, the limit $\epsilon \to 0$ formally implies that $HC_*(\{x\}, \varphi) \cong H_*(V^{c+\epsilon}, V^{c-\epsilon}) \cong H_*(V^c, V^c \setminus \{x\})$. We will give a complete proof of this fact in Section 6.4 via the deformation lemma. The latter isomorphism also gives an idea on how to compute $CP_t(\{x\}, \varphi)$ when the values of V are *not* necessarily different on $\mathsf{E} \cap S$. In that case the deformation lemma gives $HC_*(\{x\}, \varphi) \cong H_*\big(V^c \cap N, (V^c \setminus \{x\}) \cap N\big)$, where N is an isolating neighborhood for x.

The Morse relations for gradient-like systems described above can be applied to the special case of gradient flows. Let X be a smooth, closed (compact, no boundary) manifold of dimension $n < \infty$, and $f : X \to \mathbb{R}$ is a smooth function. Then the differential equation

$$x' = -\nabla_g f(x),$$

defines a smooth flow $\varphi : \mathbb{R} \times X \to X$, where $\nabla_g f(x)$ is the gradient of f with respect to a chosen Riemannian metric g on X. The equilibrium points of φ are exactly the critical points of f. Equation 6.1.1 gives the classical Morse relations for functions f with finitely many critical points. In case the critical points are all non-degenerate, i.e. $f''(x)$ is invertible for all critical points x, then the Conley index is given by $CP_t(\{x\}, \varphi) = t^{\mu(x)}$, where $\mu(x) = \#\{\text{negative eigenvalues of } f''(x)\}$. The latter is called the *Morse index* and the function f is a Morse function. These consideration remain valid for open manifolds (finite dimensional) if we consider compact isolated invariant sets S. If φ is a gradient flow on a compact, finite dimensional manifold with boundary, then the same Morse relations apply.

When X is not compact, or locally compact, then finite sets of equilibrium points of a gradient, or gradient-like flow do not necessarily define Morse decompositions. However, one does find Morse tiling for which the Morse relations hold. For gradient systems we use a compactness condition, called the Palais-Smale condition, that replaces compactness of X and makes Conley Theory applicable to gradient flows on non-compact spaces.

6.2 Palais-Smale functions and compactness

Let X be a real Hilbert space with inner product $(\cdot, \cdot)$ and $f : X \to \mathbb{R}$ is a continuously differentiable function. We are interested in the equation

$$f'(x) = 0, \quad x \in X. \tag{6.2.2}$$

Equation (6.2.2) is a special case of the equations studied in Part **??**. Indeed, the gradient of f at x is the unique vector $\nabla f(x)$ given by the Riezs Representation Theorem: $(\nabla f(x), y) = f'(x)y$ for all $y \in X$. The gradient define a continuous mapping $\nabla f : X \to X$ and Equation (6.2.2) is equivalent to $\nabla f(x) = 0$. Since the mapping in question is a gradient the problem has additional structure and is called a variational problem. The additional structure allow different techniques for finding zeroes of $\nabla f(x) = 0$.

Variational problems, especially those coming from differential equations are stated on infinite dimensional spaces and therefore compactness issues may occur when searching for critical points. For function f on a smooth, closed manifold (compact, no boundary) it holds that if $f'(x^n) \to 0$, then there exists a subsequence $x^{n_k} \to x$, and $f'(x) = 0$. Such a sequence is called a *Palais-Smale* sequence and they play an important role in variational methods. Consider the function $f(x) = \arctan(x)$. Clearly, f has no critical values, and thus no critical points on $\mathbb{R}$. The values $c = \pm\pi/2$ are special however. One can for example take sequences $\{x^n\}$, $x^n = \pm n$, such that $f(x^n) \to c$ and $f'(x^n) \to 0$. Regardless of the fact that f' goes to zero along such sequences there are no critical points.

One could argue that there exist critical points at 'infinity'. This would require compactifying our setting.

This simple example already goes to show that due to the non-compactness of $\mathbb{R}$, the domain of definition of f, the notion of critical value and critical point lacks unform estimates. For this very reason Palais and Smale, in their work on Morse theory in infinite dimensions, introduced the following compactness condition.

6.3 Definition A function $f \in C^1(X;\mathbb{R})$ is saidf to satisfy the Palais-Smale condition at c — (PS) for short —, if any sequence $\{x^n\} \subset X$ for which

$$f(x^n) \to c, \qquad f'(x^n) \to 0,$$

has a convergent subsequence.

This condition was referred to as 'Condition (C)' in the original work of Palais and Smale. Functions may also satisfy (PS) for an interval of values c, i.e. a function satisfies (PS) on an interval I if it satisfies (PS) for every $c \in I$. The same holds for the Palais-Smale condtion on X. In that case we do not specify c beforehand.

The (PS) condition has various consequences for Palais-Smale functions. Denote by $\mathrm{Crit}(f,I)$ the set of critical points of f with critical values restricted to the interval I.

6.4 Lemma Let $f \in C^1(X)$ satsify (PS) for some c, then the set $\mathrm{Crit}(f,c)$ is compact. Moreover, $\mathrm{Crit}(f,I)$ is compact whenever I is compact.

Proof. Compactness is established by pointing out that compactness for a metric space is equivalent to sequential compactness. The space $(\mathrm{Crit}(f,c),d)$ with the induced metric is a metric space itself. For any sequence $\{x^n\}$ we have that $f(x^n) = c$, and $f'(x^n) = 0$. The (PS)-condition then implies that $x^{n_k} \to x \in X$. Consequently, $f(x) = c$ and $f'(x) = 0$, and thus $x \in C_f(c)$, which establishes sequential compactness. The same holds for $\mathrm{Crit}(f,I)$. ∎

Another important consequence of the (PS)-condition is uniformity on lower bounds for f' at regular values.

6.5 Lemma Let c be a regular value for f. Then there exists an $\epsilon > 0$, such that $\|f'(x)\|_{X^*} \geq \delta > 0$ for all $x \in f^{-1}[c-\epsilon, c+\epsilon]$.

Proof. The fact that c is regular implies that a neighborhood $[c-\epsilon, c+\epsilon]$, for some $\epsilon > 0$, consists of regular values. If not, one can choose $c^n \to c$, and x^n, with $f(x^n) = c^n$, and $f'(x^n) = 0$. By (PS) we have that $x^{n_k} \to x$, with $f(x) = c$ and $f'(x) = 0$, a contradiction.

For any $c_* \in [c-\epsilon, c+\epsilon]$ one can find a $\delta_{c_*} > 0$ such that $\|f'(x)\|_{X^*} \geq \delta_{c_*} > 0$ for all $x \in f^{-1}(c_*)$. Indeed, otherwise one can find sequences $\{x^n\}$ such that $f'(x^n) \to 0$, which, by (PS), have convergent subsequences converging to a critical point at level c_*, a contradiction.

Finally, $\delta_{c_*} \geq \delta > 0$ for all $c_* \in [c-\epsilon, c+\epsilon]$, from the above arguments. ∎

6.3 The Morse relations for critical points

In order to study zeroes of f' we use the canonical dynamical system generated by the gradient flow equation

$$x' = -\nabla f(x). \tag{6.3.3}$$

From this point on we will assume that f is twice continuously differentiable. In that case Equation (6.3.3) generates a local C^1-flow φ on the Hilbert space X. An important property, intrinsic to gradient flows, involves the following identity:

$$(f \circ \varphi)' = f'(\varphi)\varphi' = -f'(\varphi)\nabla f(\varphi) = -\|\nabla f(\varphi)\|^2 \leq 0,$$

which implies that f is a Lyapunov function on X. Indeed, f is constant at $\mathrm{Crit}(f, \mathbb{R})$ and $f \circ \varphi$ is strictly decreasing outside $\mathrm{Crit}(f, \mathbb{R})$.

6.3.a Gradient flows and Morse decompositions

Let $a < b$ be regular values of f. Note that for smooth functions the regular values form a dense subset of $\mathbb{R}$. Consider the set $f_a^b = \{x \in X \mid a \leq f(x) \leq b\}$. We also use the convention that $f_{-\infty}^{\infty} = X$, $f_{-\infty}^b = f^b = \{x \in X \mid -\infty < f(x) \leq b\}$ and $f_a^{\infty} = f_a = \{x \in X \mid a \leq f(x) < \infty\}$.

6.6 Lemma Let $f \in C^2(X)$ satisfy (PS) and let $a < b$ be regular values of f. Then the set $B = f_a^b$ is an isolating block for the gradient flow φ.

Proof. By Lemma 6.5 we have an $\epsilon > 0$ such that $[a-\epsilon, a+\epsilon]$ and $[b-\epsilon, b+\epsilon]$ are intervals of regular values and $\|\nabla f(x)\| \geq \delta > 0$ for all $x \in f_{a-\epsilon}^{a+\epsilon} \cup f_{b-\epsilon}^{b+\epsilon}$. In order to show that B is a block we start with identifying sets B^- and B^+ as subsets of $\partial B = \{x \in X \mid f(x) = a\} \cup \{x \in X \mid f(x) = b\}$. Define $B^- = \{x \in X \mid f(x) = a\}$ and $B^+ = \{x \in X \mid f(x) = b\}$. We note that $B^- \cap B^+ = \varnothing$. If $x \in B^+$, i.e. $f(x) = b$, then for $t < 0$ it holds that

$$f(\varphi(0,x)) - f(\varphi(t,x)) = -\int_t^0 \|\nabla f(\varphi)\|^2 \leq t\delta^2,$$

which implies that $f(\varphi(t,x)) \geq b - t\delta^2 > b$ for all $t < 0$. In same why if follows that $f(\varphi(t,x)) \leq a - t\delta^2 < a$ for all $t > 0$. Summaring, $\varphi((-\infty,0),x) \cap B = \varnothing$ for all $x \in B^+$ and $\varphi((0,\infty),x) \cap B = \varnothing$ for all $x \in B^-$, which proves that $B = f_a^b$ is an isolating block. ∎

In the same way it follows that sets f^a are attracting blocks and set f_a are repelling blocks (a a regular value). By the theory in Section **??** we have that the Conley index of f_a^b is well-defined and is given by $HC_*(f_a^b,\varphi) = H_*(B,B^-)$. Since X is a non-compact space (not even locally compact in general), the interpretation of the Conley index needs to be discussed in this case. The block $B = f_a^b$ is even a Morse block via the observation that

$$\varnothing \subset f^a \subset f^b \subset X,$$

gives a lattice of attracting neighborhoods and a Morse tiling $f^a < f_a^b < f_b$. From the Lemmas **??** and **??** and the integral estimates in Lemma 6.6 it follows that $HC_*(f_a^b,\varphi) = H_*(B,B^-) \cong H_*(f^b,f^a)$.

We can find finer Morse tilings by considering sequences of regular values. Let

$$a = a_0 < a_1 < \cdots < a_{n-1} < a_n = b,$$

be regular values of f. Then,

$$\varnothing \subset f^{a_0} \subset f^{a_1} \subset \cdots \subset f^{a_{n-1}} \subset f^{a_n} \subset X,$$

is a lattice of attracting neighborhoods (filtration), denoted by $\mathsf{B}(\bar{a})$ with $\bar{a} = (a_0,\cdots,a_n)$. The associated Morse tiling $\mathsf{T}(\mathsf{B})$ of X is given by $f^{a_0} < f_{a_0}^{a_1} < \cdots < f_{a_{n-1}}^{a_n} < f_{a_n}$. A Morse tiling of f_a^b is given by

$$f_{a_0}^{a_1} < \cdots < f_{a_{n-1}}^{a_n}.$$

The Morse relations of Theorem **??** for the Morse tiling $\mathsf{T}(\mathsf{B})$ are

$$\sum_{i=0}^{n} CP_t(f_{a_{i-1}}^{a^i},\varphi) = CP_t(f_a^b,\varphi) + (1+t)Q_t, \tag{6.3.4}$$

where $CP_t(f_{a_{i-1}}^{a^i},\varphi) = P_t(f^{a_i},f^{a_{i-1}})$. Whether a Morse tiling yields a real Morse decomposition depends on compactness properties of $\mathrm{Inv}(f_a^b)$. Morse functions on X and the chosen gradient flow in Equation (6.3.3) this need not be true. However, for interpreting the Morse relations for critical points this does not play a role.

6.3.b Morse relations and critical points

The interpretation of the Conley index $HC_*(f_a^b,\varphi)$ as an invariant to detect critical points in $B = f_a^b$ only uses the (PS) condition.

6.7 Theorem Let $f \in C^2(X)$ satisfy (PS) and let $a < b$ be regular values of f. Then, $HC_*(f_a^b, \varphi) \neq 0$, implies that $\mathrm{Crit}(f, [a,b]) \neq \varnothing$.

Proof. If $\mathrm{Crit}(f, [a,b]) = \varnothing$, then $[a,b]$ consists of regular values and by Lemma 6.5 we have that $\|\nabla f(x)\| \geq \delta$ for all $x \in f_a^b$. Let $x \in f^b$, then, as before, for $t > 0$,

$$f(\varphi(t,x)) - f(\varphi(0,x)) = -\int_0^t \|\nabla f(\varphi)\|^2 \leq -t\delta^2,$$

and therefore $f(\varphi(t,x)) \leq b - t\delta^2$. This implies that for $t \geq \tau > 0$, $\varphi(t, f^b) \subset f^a$, and thus by the Ważewski Principle (Theorem 5.26), f^a is a strong deformation retract of f^b, see also the proof of Lemma **??**. We conclude that $HC_*(f_a^b, \varphi) \cong H_*(f^b, f^a) \cong 0$, which is a contradiction. ∎

The above theorem also also for $a = -\infty$, i.e. $f_{-\infty}^b = f^b$ and for $b = \infty$, i.e. $f_a^\infty = f_a$, or both, i.e. $f_{-\infty}^\infty = X$.

6.8 Theorem Let $f \in C^2(X)$ satisfy (PS) and let a be a regular values of f. Then, $HC_*(f_a, \varphi) \cong H_*(X, f^a) \neq 0$, implies that $\mathrm{Crit}(f, [a, \infty)) \neq \varnothing$.

Proof. Suppose $[a, \infty)$ consists of regular values, i.e. $\mathrm{Crit}(f, [a\infty)) = \varnothing$. Define $W = f_a$, $W^- = f_a^a$ and $W^0 = W$. Then by the Ważewski Principle, f^a is a strong deformation retract of X and therefore $H_*(X, f^a) \cong 0$, which is a contradiction. ∎

Theorem 6.7 provides very basic information about the critical point of f in the set $B = f_a^b$. Consider the case that $\mathrm{Crit}(f, [a,b])$ is a finite set. If $\mathrm{Inv}(f_a^b)$ is compact, then $\mathrm{Crit}(f, [a,b])$ provides a Morse decomposition. In general, this need not be the case for φ. Let $a < c_0 < \cdots c_n < b$ be the critical values. As a consequence of the above Morse relations we obtain for any sufficiently small $\epsilon > 0$ that

$$\sum_{i=0}^{n} CP_t(f_{c_i-\epsilon}^{c_i+\epsilon}, \varphi) = CP_t(f_a^b, \varphi) + (1+t)Q_t. \tag{6.3.5}$$

The fact that we can take $f_{c-\epsilon}^{c+\epsilon}$, for any sufficiently small $\epsilon > 0$, as a block follows from the fact that the index doesn't depend on the particular choice of the block. If we further analyze φ at critical levels we will be able to further compute $CP_t(f_{c_i-\epsilon}^{c_i+\epsilon}, \varphi)$. This will lead to the more traditional Morse relations as we will explain in the next section.

6.4 The deformation lemma

In this section we study the flow φ near an isolated critical level.

6.9 Lemma Suppose $c \in \mathbb{R}$ is an isolated critical value, then $HC_*(f_{c-\epsilon}^{c+\epsilon}, \varphi) \cong H_*\big(f^c \cap N, (f^c \setminus \mathrm{Crit}(f,c)) \cap N\big)$, where N is a sufficiently small neighborhood of $\mathrm{Crit}(f,c)$. In particular, if $\mathrm{Crit}(f,c)$ is a finite set, then

$$HC_*(f_{c-\epsilon}^{c+\epsilon}, \varphi) \cong \bigoplus_{j=1}^{k} H_*\big(f^c \cap N^j, (f^c \setminus x^j) \cap N^j\big),$$

where $\{x^j\} = \mathrm{Crit}(f,c)$ and N^j are sufficiently small disjoint neighborhoods of the critical points x^j.

Proof. We start with proving that f^c is a strong deformation retract of $f^{c+\epsilon}$. Define $W = f_c^{c+\epsilon}$, $W^- = f_c^c \setminus \mathrm{Crit}(f,c)$ and $W^0 = f_c^{c+\epsilon} \setminus W^s(\mathrm{Crit}(f,c))$. It follows that W^- is relatively closed in W^0 (give more details) and W is closed and therefore a Ważewski set. From the Ważewski Principle (Theorem 5.26) it follows that $W^- = f_c^c \setminus \mathrm{Crit}(f,c)$ is a strong deformation retract of $W^0 = f_c^{c+\epsilon} \setminus W^s(\mathrm{Crit}(f,c))$. This also implies that f^c is a strong deformation retract of $f^{c+\epsilon}$. This implies that

N is an attractor, then A is a deformation retract of N.

$$H_*(f^{c+\epsilon}, f^{c-\epsilon}) \cong H_*(f^c, f^{c-\epsilon}).$$

Secondly, we show that $f^{c-\epsilon}$ is a strong deformation retract of $f^c \setminus \mathrm{Crit}(f,c)$. As before let $W = f_{c-\epsilon}^c \setminus \mathrm{Crit}(f,c)$, $W^- = f_{c-\epsilon}^{c-\epsilon}$ and $W^0 = W$. The set W is a Ważewski set and the Ważewski Principle gives that $W^- = f_{c-\epsilon}^{c-\epsilon}$ is a strong deformation retract of $W^0 = f_{c-\epsilon}^c \setminus \mathrm{Crit}(f,c)$, which proves that

$$H_*(f^c, f^{c-\epsilon}) \cong H_*(f^c, f^c \setminus \mathrm{Crit}(f,c)).$$

Combining the isomorphisms gives $H_*(f^{c+\epsilon}, f^{c-\epsilon}) \cong H_*(f^c, f^c \setminus \mathrm{Crit}(f,c))$.

Let N be a neighborhoods of $\mathrm{Crit}(f,c)$ such that $N \subset f_{c-\epsilon}^{c+\epsilon}$ and by $U = \bigcup_{i=1}^m U_i \subset f_{c-\epsilon}^{c+\epsilon}$. excision

$$H_*(f^c, f^c \setminus \mathrm{Crit}(f,c)) \cong H_*(f^c \cap N, (f^c \setminus \mathrm{Crit}(f,c)) \cap N).$$

If $\mathrm{Crit}(f,c)$ is a finite set we can choose N as a disjoint union of isolating neighborhoods of $x^j \in \mathrm{Crit}(f,c)$. ∎

6.10 Exercise Give an alternative proof using the initial value problem $x' = -(f(x_0) - a)\nabla f(x)/\|\nabla f(x)\|^2$, with $x(0) = x_0 \in f_a^b$. ▪

▪ **6.11 Remark** If $f_{c-\epsilon}^{c+\epsilon}$ is locally compact, then follows from the theory of isolated invariant sets and Morse decompositions. Clearly, every $x^j \in \mathrm{Crit}(f,c)$ is an isolated invariant set and B^j and isolating neighborhood and $\mathrm{Crit}(f,c)$ is a Morse decomposition with isolating block $f_{c-\epsilon}^{c+\epsilon}$. By Theorem **??** there exist isolating blocks for x^j. See Section 6.1 for more details. ▪

For a critical value $c \in \mathbb{R}$ we can define the *critical groups* as

$$C_*(\mathrm{Crit}(f,c)) = H_*(f^c \cap N, (f^c \setminus \mathrm{Crit}(f,c)) \cap N),$$

where N is a neighborhood of $\mathrm{Crit}(f,c)$. The same can be defined for any isolated connected component of $\mathrm{Crit}(f,c)$. The definition of critical group is independent of the choice of neighborhood N. For an isolated critical point $x^j \in \mathrm{Crit}(f,\mathbb{R})$ we define $C_*(x^j) = H_*(f^c \cap N^j, (f^c \setminus x^j) \cap N^j)$, where N^j is a neighborhood of x^j.

From the deformation lemma we derive the following version of the Morse relations for critical points.

6.12 Theorem Assume $f \in C^2(X)$ satisfies the (PS) condition and let $a < b$ be regular values such that $\mathrm{Crit}(f,[a,b])$ is a finite set. Then,

$$\sum_{x^j \in \mathrm{Crit}(f,[a,b])} i_t(x^j) = P_t(f^b, f^a) + (1+t)Q_t,$$

where $i_t(x^j) = P_t(f^c \cap N^j, f^c \setminus x^j \cap N^j)$, the Poincaré polynomial of $C_*(x^j)$.

In the case that 0 is an isolated element of the spectrum of $f''(x^j)$ the Gromoll-Meyer Theorem implies that $i_t(x^j)$ is a polynomial and contains index information about the spectrum. This property is satisfied by Fredholm functionals on X.

6.5 Homotopy types and the Morse index

A Morse function satisfies the property that all elements in $\mathrm{Crit}(f,\mathbb{R})$ are non-degenerate, i.e. the operator $f''(x) : X \to X^*$ is invertible for all $x \in \mathrm{Crit}(f,\mathbb{R})$. For any critical point $x \in \mathrm{Crit}(f,\mathbb{R})$ we may linearize f, i.e. compute its Taylor expansion up to order 2:

$$f(x) = f(x_0) + \frac{1}{2}(f''(x_0)(x - x_0), x - x_0) + o(\|x - x_0\|^2).$$

The Morse Lemma provides a local change of coordinates to the quadratic form given by $f''(x)$.

6.13 Lemma Let x_0 be a non-degenerate critical point. Then there exists an neighborhood N of x_0, and a diffeomorphism $h : N \to h(N) \subset X$, with $h(x_0) = 0$, such that

$$f(h^{-1}(y)) = f(x_0) + \frac{1}{2}(f''(x_0)y, y),$$

for all $y \in h(N)$.

A function f is a Morse function on f_a^b if all critical points in $\mathrm{Crit}(f,[a,b])$ are non-degenerate. Morse functions are prevalent.

6.14 Proposition Let $f \in C^2(X)$ satisfying (PS) on the strip f_a^b for regular values $\infty < a < b < \infty$. Then for any $\epsilon > 0$ there exists a Morse function $\widehat{f}$ such that $\|f - \widehat{f}\|_{C^2} < \epsilon$, and which satisfies (PS) on the strip $\widehat{f}_a^b$.

The next step is to investigate the homology $H_*(f^c, f^c \backslash C_f(c))$. For that we invoke the Morse lemma. Before we state the main result of this section we first introduce a characterization of the critical points.

6.15 Definition For a critical point $x \in \mathrm{Crit}(f, \mathbb{R})$ of a C^2 Morse function the Morse index $\mu(x)$ is defined as $\dim E^-$, where E^- is the negative eigenspace in the decomposition $X = E^+ \oplus E^-$.

The Morse co-index $\mu^c(x)$ is defined as the dimension of E^+. In the infinite dimensional case both indices can be infinite, and Morse theory, as we present it here, will be empty. Extensions such as Floer homology can be a way to obtain a working Morse theory in that case and will be discussed in Section 6.6. In the infinite dimensional examples that we will see in the next chapter the Morse index of a critical point is always finite. Such problems can be characterized as semi-definite. Of course in the finite dimensional both the Morse index and co-index are always finite.

Before we use the Morse lemma we recall that for a Morse function it holds that

$$H_*(f^c, f^c \setminus \mathrm{Crit}(f, c)) \cong \bigoplus_j H_*(f^c \cap N^j, (f^c \backslash \{x^j\}) \cap N^j).$$

6.16 Lemma Let $x \in \mathrm{Crit}(f, \mathbb{R})$ be a non-degenerate critical point and N a sufficiently small neighborhood of x. Then

$$H_k(f^c \cap N, f^c \backslash \{x\} \cap N) = \begin{cases} \mathbb{F} & \text{for} \quad k = \mu(x), \\ 0 & \text{otherwise}, \end{cases}$$

where $\mathbb{F}$ is the coefficient field of H_*.

In the next section we will use this information to count critical points.

6.6 Other homology invariants and the Morse inequalities

Here also discuss other (co)-homology theories in order to do strongly indefinite problems.

In the previous section we have established a link between the topology of the pair (f^b, f^a) and the critical points at the (only) critical level c. We recall

$$H_k(f^b, f^a) \simeq \bigoplus_{i\,:\,\mu(x_i)=k} \mathbb{Z} = \mathbb{Z}^{m_k(a,b)}, \tag{6.6.6}$$

where $m_k(a,b) = \#\{i \,:\, \mu(x_i) = k,\ x_i \in C_f(f_a^b)\}$. The next question is, what happens the Morse index and co-index are infinite.

7 — Morse Theory for Elliptic Equations

7.1 Variational principles and critical points

In this chapter Morse theory will be applied to a model class of nonlinear elliptic differential equations. For our purposes here we are concerned with elliptic problems of the form

$$
\begin{aligned}
-\Delta u = g(x,u), &\qquad x \in \Omega,\\
u = 0, &\qquad x \in \partial\Omega,
\end{aligned}
$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary $\partial\Omega$, and $g(x,u)$ is a C^∞-nonlinearity that satisfies the growth estimate

$$
|g_u(x,u)| \le C + C|u|^{p-1}, \quad p > 1,
$$

uniformly in $x \in \overline{\Omega}$, for all $u \in \mathbb{R}$. We will study solutions of this problem using minimax and Morse theory. Standard regularity theory for this equation reveals that solutions are $C^\infty(\Omega)$. Regularity issues will be postponed till later.

As opposed to applying degree theory and fixed point arguments the equation above possesses a alternative formulation for finding solutions; variational principle. Consider the integral

$$
\int_\Omega \left[\frac{1}{2}|\nabla u(x)|^2 - G(x,u(x))\right] dx,
$$

where $G(x,u) = \int_0^u g(x,s)ds$. Clearly, the integral is well-defined for all $u \in C^\infty(\Omega) \cap C_0^1(\overline{\Omega})$. Denote the integral as functional on functions $u(x)$ by f. Let us consider the first variation of the integral with respect to test functions $\varphi \in C_0^\infty(\Omega)$.

This yields

$$\begin{aligned} f(u+\varphi) - f(u) &= \int_\Omega \Big[\nabla u \cdot \nabla \varphi - g(x,u)\varphi\Big] dx \\ &+ \int_\Omega \Big[\frac{1}{2}|\nabla \varphi|^2 - g_u(x, u+\theta\varphi)\varphi^2\Big] dx. \end{aligned}$$

As explained before, under the assumption that $p < \frac{n+2}{n-2}$, when $n \geq 3$, the function f extends to the Sobolev space $H^1_0(\Omega) = \mathrm{clos}_{H^1}(C_0^\infty(\Omega))$, with equivalent norm

$$\|u\|_{H^1_0} := \sqrt{\int_\Omega |\nabla u(x)|^2 dx}.$$

This uses the compact embeddings

$$H^1_0(\Omega) \hookrightarrow \begin{cases} C^0(\overline{\Omega}), & n = 1, \\ L^{p+1}(\Omega), & n = 2, \quad 0 \leq p < \infty, \\ L^{p+1}(\Omega), & n \geq 3, \quad 0 \leq p < \frac{n+2}{n-2}. \end{cases}$$

If we use the above variation formula we obtain that for fixed $u \in H^1_0(\Omega)$ it holds that

$$\Big| f(u+\varphi) - f(u) - \int_\Omega \Big[\nabla u \cdot \nabla \varphi - g(x,u)\varphi\Big] dx \Big| = o(\|\varphi\|_{H^1_0}),$$

which proves that f is differentiable on $H^1_0(\Omega)$. Notation:

$$f'(u)\varphi = \int_\Omega \Big[\nabla u \cdot \nabla \varphi - g(x,u)\varphi\Big] dx.$$

> **7.1 Exercise** Prove the above identity for the Fréchet derivative in the case $g(x,u) = \lambda u + |u|^{p-1}u$, using the first variation and the Sobolev emebeddings. ■

> **7.2 Exercise** †† In Section 6.4 we introduced the notion of gradient. Compute the gradient $\nabla f(u)$ in $H^1_0(\Omega)$. ■

Similarly, the second variation yields

$$\Big| f'(u+\psi)\varphi - f'(u)\varphi - \int_\Omega \Big[\nabla \psi \cdot \nabla \varphi - g_u(x,u)\psi\varphi\Big] dx \Big| = o(\|\psi\|_{H^1_0})\|\varphi\|_{H^1_0},$$

which proves that f twice continuously differentiable on $H^1_0(\Omega)$. Notation: Notation:

$$f''(u)\psi\varphi = \int_\Omega \Big[\nabla \psi \cdot \nabla \varphi - g_u(x,u)\psi\varphi\Big] dx.$$

7.3 Exercise Establish the expression for the second derivative in the case $g(x,u) = \lambda u + |u|^{p-1}u$, by proving the identity for the second Fréchet derivative. ■

The expression for the first derivative explains that the elliptic equation is satisfied in a 'weak' sense, i.e. weak solution $u \in H_0^1(\Omega)$. If additional regularity is known then a simple integration by part provides the identity

$$\int_\Omega \Big[\Big(-\Delta u - g(x,u)\Big)\varphi\Big] dx = 0,$$

for all $\varphi \in H_0^1(\Omega$, which reveals the equation again and u is a 'strong' solution. This identity, without a priori regularity, can also be interpreted in distributional sense, i.e. $-\Delta$ is regarded as a map from $H_0^1(\Omega)$ to its dual Sobolev space $H^{-1}(\Omega)$.

7.4 Exercise Interpret the above identity in the dual space $H^{-1}(\Omega)$. ■

Having established all these preliminary differentiability properties we conclude that solutions of the elliptic equation can be regarded as critical points of the function f. This variational principle allows us to attack the elliptic problem via critical point theory.

Before going to the actual application in the next section we first prove a result concerning the Palais-Smale condition.

7.5 Lemma Let g and Ω be as above and let $1 < p < \infty$, for $n \leq 2$, and $1 < p < \frac{n+2}{n-2}$, for $n \geq 3$. In addition assume that for some $\gamma > 2$,

$$0 < \gamma G(x,u) \leq u g(x,u), \quad \text{for} \quad |u| \geq r > 0.$$

Then, the function f satisfies the Palais-Smale condition on $H_0^1(\Omega)$.

Proof. The requirements on p are needed in order for f to be well-defined and differentiable. Let $\{u^n\}$ be a sequence satisfying

$$f(u^n) \to c \in \mathbb{R}, \quad \text{and} \quad f'(u^n) \to 0,$$

as $n \to \infty$ — a Palais-Smale sequence. In terms of the above integrals this reads:

$$\int_\Omega \Big[\frac{1}{2}|\nabla u^n|^2 - G(x,u^n))\Big] dx \to c, \quad \text{and}$$

$$\Big|\int_\Omega \Big[\nabla u^n \cdot \nabla\varphi - g(x,u^n)\varphi\Big] dx\Big| \leq \epsilon_n \|\varphi\|_{H_0^1}, \quad \epsilon_n \to 0, \quad \forall \varphi \in H_0^1(\Omega).$$

7.6 Exercise Derive the above inequalities from the definitions of f and f'. ■

The first step is to show that a Palais-Smale sequence $\{u^n\}$ is uniformly bounded in $H_0^1(\Omega)$, with the bound only depending on c. In the expression for the derivative we choose $\varphi = \gamma^{-1}u^n$. This gives, upon substitution, that

$$-\epsilon_n\|u^n\|_{H_0^1} \int_\Omega \Big[\frac{1}{2}|\nabla u^n|^2 - g(x,u^n)u^n\Big]dx \le \epsilon_n\|u^n\|_{H_0^1}.$$

Combining this inequality with the expression for $f(u^n)$ we obtain:

$$\begin{aligned}\Big(\frac{1}{2}-\frac{1}{\gamma}\Big)\int_\Omega |\nabla u^n|^2 dx &= \int_\Omega \Big[G(x,u^n) - \gamma^{-1}u^n g(x,u^n)\Big]dx \\ &+ \ c + \epsilon_n + \epsilon_n\gamma^{-1}\|u^n\|_{H_0^1} \le C + \gamma^{-1}\epsilon_n\|u^n\|_{H_0^1}.\end{aligned}$$

This inequality yields the estimate $\|u^n\|_{H_0^1} \le C$.

Since $H_0^1(\Omega)$ is a Hilbert space the boundedness of $\{u^n\}$ implies that $u^{n_k} \rightharpoonup u$ in $H_0^1(\Omega)$. Since the embeddings of $H_0^1(\Omega)$ into $L^{p+1}(\Omega)$ are all compact, provided $p < \frac{n+2}{n-2}$, $n \ge 3$, it holds that $u^{n_k} \to u$ in L^{p+1}. Consequently, $\int_\Omega G(x,u^{n_k})dx \to \int_\Omega G(x,u)dx$. If we combine this with the convergence of f we obtain:

$$\frac{1}{2}\int_\Omega |\nabla u^n|^2 dx = f(u^n) + \int_\Omega G(x,u^{n_k})dx \to c + \int_\Omega G(x,u)dx,$$

which proves that $\|u^n\|_{H_0^1} \to \|u\|_{H_0^1}$, and convergence of $\{u^{n_k}\}$ in $H_0^1(\Omega)$, completing the proof. ∎

Having establish the Palais-Smale condition for f allows us now to apply critical point methods.

7.2 Solutions via Morse Theory

Knowing now that our function f is a proper C^2-function on H_0^1, which satisfies also the other conditions of (H_f'), we can use the Morse relations of Section 6 (Chapter II). We recall

$$i_t(C_f \cap f_a^b) = P_t(f^b, f^a) + (1+t)Q_t,$$

where $a < b$ are regular values of f and

$$f(u) = \frac{1}{2}\int_D |\nabla u|^2 - \frac{\lambda}{2}\int_D u^2 - \frac{1}{p+1}\int_D |u|^{p+1}, \ \ p > 1.$$

By studying the geometry of the sets f^a, for different values of a, we shall try to compute the dimension of certain homology groups $H_n(f^b, f^a)$, for certain values of $a < b$. Using the Morse relations as mentioned above we can find parts of the Morse series $i_t(C_f \cap f_a^b)$. By means of Theorem 7.3 (Chapter II) we can obtain information about the existence of critical points of certain index.

7.7 Theorem Let $D \subset \mathbb{R}^n$ be a smooth bounded domain. Assume that $\lambda < \lambda_1$, $1 < p < \infty$ if $n \leq 2$ and $1 < p < \frac{n+2}{n-2}$ if $n \geq 3$. Then $f(u)$ has at least one non-trivial critical point $u \in H_0^1(D)$, with

$$\mu(u) \leq 1 \leq \mu^*(u).$$

Proof. From Section 2 we already that $f(u)$ satisfies Hypotheses (H_f'). Because $\lambda < \lambda_1$ we can define

$$\|u\|_*^2 = \int_D |\nabla u|^2 - \lambda \int_D u^2,$$

as an equivalent norm on $H_0^1(D)$. We then have

$$f(u) = \frac{1}{2}\|u\|_*^2 - \frac{1}{p+1}\|u\|_{p+1}^{p+1}.$$

Using the sobolev-inequality (Lemma 1.2) we obtain the estimate

$$f(u) = \frac{1}{2}\|u\|_*^2 - \frac{1}{p+1}\|u\|_{p+1}^{p+1} \geq \frac{1}{2}\|u\|_*^2 - \frac{C}{p+1}\|u\|_*^{p+1} \geq \alpha > 0,$$

provided $u \in \partial B_r(0) \in H_0^1(D)$ and $r > 0$ sufficiently small. Computing $f''(0)$ one can easily see that 0 is a non-degenerate minimum (thus isolated), with $i_t(\{0\}) = 1$. For any other critical point u of $f(u)$ one can compute the critical values. We have

$$c = \frac{1}{2}\frac{p-1}{p+1}\|u\|_{p+1}^{p+1}, \quad \forall u \in C_f.$$

From (3.3) one easily sees that 0 is the only critical point at 'energy-level' 0. The levels below 0 ($c < 0$) are regular values of f, which is clear by (3.3) and the (PS)-condition. The level $c = 0$ is also an isolated energy-level, because suppose not, then there exists a sequence of positive critical values $c_n \to 0$, critical points $\{u^n\}$ with $f(u^n) = c_n$ and $f'(u^n) = 0$. By the (PS)-condition one deduces that $u_{n_k} \to 0$ in $H_0^1(D)$, which contradicts the isolatedness of 0.

One can choose $\epsilon > 0$ sufficiently small, such that $c = \epsilon$ is a regular value of $f(u)$. From (3.2) we have that an annulus $\{u;\ r_1 \leq \|u\|_* \leq r_2\}$ is not cantained in f^ϵ, provided $\epsilon > 0$ is small enough. Therefore f^ϵ is not path-connected and f^ϵ has at least two path-connected components, i.e. a small neighbourhood of 0 (use the Morse Lemma) and the set $\{u;\ \|u\|_* \geq R\}$, R large enough. This yields

$$\dim H_0(f^\epsilon) \geq 2.$$

In order to find critical points of $f(u)$ now we consider the pair $(f^\infty, f^\epsilon) = (H_0^1, f^\epsilon)$. Let us consider the exact sequence

$$\longrightarrow H_1(H_0^1, f^\epsilon) \xrightarrow{\partial_1} H_0(f^\epsilon, \emptyset) \xrightarrow{i_0} H_0(H_0^1, \emptyset) \longrightarrow .$$

Clearly dim $H_0(H_0^1,\emptyset) = 1$ and dim $H_0(f^\epsilon,\emptyset) =$ dim $H_0(f^\epsilon) \geq 2$. Using the exactness of the above sequence we deduce that

$$\dim H_1(H_0^1, f^\epsilon) \geq 1.$$

From Theorem **??** we conclude that $\mathrm{Crit}(f,[a,\infty)) \neq \emptyset$. ∎

▪ **7.8 Remark** In Theorem 3.1 the existence of a second critical point is given, with additional information on the Morse-index of the critical point. If one is only interested in the existence of a second critical point one can also do this using the Morse relations of Theorem 4.6. One argues as follows. Suppose 0 is the only critical point of $f(u)$. Clearly f satisfies (H_f) in that case. The Morse relations then give

$$1 = t + p(t) + (1+t)Q_t,$$

which cannot be true for any t, therefore yielding a contradiction. So the assertion that $C_f = \{0\}$ was false and so f must have at least one additional critical point. The details of this reasoning are left to the reader as an exercise. ▪

7.9 Theorem Let $D \subset \mathbb{R}^N$ be a smooth bounded domain. Assume that $\lambda \notin \sigma(-\Delta)$, $1 < p < \infty$ if $N \leq 2$ and $1 < p < \frac{N+2}{N-2}$ if $N \geq 3$. Then $f(u)$ has at least one non-trivial critical point $u \in H_0^1(D)$, with

$$\mu(u) \leq \left\{ \begin{array}{c} n+1 \\ 0 \end{array} \right. \leq \mu^*(u),$$

where $n = m(0)$.

Proof. From the previous we know that if $u \in C_f$ its critical value c is non-negative. Thus every negative level is therefore regular. In order to prove this theorem we shall therefore compute the relative homology groups of the pair $(H_0^1(D), f^{-a})$, with $a > 0$. We prove that $H_k(H_0^1(D), f^{-a}) = 0$ for all $k \geq 0$. To do so we proceed as follows; Let $\mathbf{u} \in \partial B_1(0) = S^\infty$, then

$$f(t\mathbf{u}) = \frac{t^2}{2} - \frac{\lambda}{2} t^2 \|\mathbf{u}\|_{L^2}^2 - \frac{t^{p+1}}{p+1} \|\mathbf{u}\|_{p+1}^{p+1}.$$

It is clear that $\frac{d}{dt} f(t\mathbf{u}) < 0$, whenever $f(t\mathbf{u}) \leq -a$, $a > 0$. Using the Implicite Function Theorem (see e.g. **??**HV]) one concludes that there is an unique function $T(\mathbf{u})$, i.e. $T \in C(S^\infty, \mathbb{R}^+)$, such that

$$f(T(\mathbf{u})\mathbf{u}) = -a, \quad \forall \mathbf{u} \in S^\infty.$$

Using the Sobolev embeddings and the fact $\lambda \notin \sigma(-\Delta)$ we obtain;

$$f(t\mathbf{u}) \geq \frac{t^2}{2} C(\lambda) - \frac{t^{p+1}}{p+1} S^{-\frac{p+1}{2}},$$

where $C(\lambda)$ is nonzero, which implies

$$|T(\mathbf{u})| \geq \delta(\lambda) > 0.$$

We have that $B_\delta(0) \not\subset f^{-a}$, so we define

$$\eta(s,u) = \begin{cases} (1-s)u + sT(\frac{u}{\|u\|_{H_0^1}})\frac{u}{\|u\|_{H_0^1}}, & \|u\|_{H_0^1} \geq \delta,\ f(u) \geq -a, \\ u, & f(u) \leq -a. \end{cases}$$

It is clear that $\eta \in C([0,1] \times H_0^1(D)\backslash B_\delta(0), H_0^1(D)\backslash B_\delta(0))$. One observes now that f^{-a} is a strong deformation retract of $H_0^1(D)\backslash B_\delta(0)$ and thus

$$H_k(H_0^1(D), f^{-a}) = H_k(H_0^1(D), H_0^1(D)\backslash B_\delta(0)), \quad \forall k.$$

Furthermore we define $\xi(s,u) = \frac{u}{\|u\|_{H_0^1}}$, $u \in H_0^1(D)\backslash B_\delta(0)$. Using the map $\xi(s,u)$ one easily proves that

$$H_k(H_0^1(D), H_0^1(D)\backslash B_\delta(0)) = H_k(B^\infty, S^\infty), \quad \forall k,$$

which, using Remark 3.2 (Chapter II), proves our assertion. We have

$$P_t(H_0^1(D), f^{-a}) = 0.$$

Let us continue now with the proof. Whenever $\lambda \notin \sigma(-\Delta)$, the map $f''(0) = -\Delta - \lambda$, seen as bounded map from $H_0^1(D)$ to $H^{-1}(D)$, is invertible. In that case 0 is a non-degenerate critical point with Morse-index $m(0) = \#$ negative eigenvalues of $f''(0)$ (seen now a unbounded map in $L^2(D)$). From Section 3 (Chapter II) we repeat

$$C_n(f,0) = \begin{cases} \mathbb{R} & \text{if}\ , n = m(0) \\ 0 & \text{otherwise.} \end{cases}$$

and thus $i_t(\{0\}) = t^{m(0)}$. For the Morse-index of the set $C_f \cap f_{-a}^\infty$ this yields

$$i_t(C_f \cap f_{-a}^\infty) = t^{m(0)} + p(t),$$

where $p(t)$ is some positive formal series. From the Morse relations of Theorem 6.7 we have

$$t^{m(0)} + p(t) = (1+t)Q_t,$$

which indicates that Q_t either contains the monomials $t^{m(0)}$ or $t^{m(0)-1}$. For that reason $p(t)$ must contain either $t^{m(0)+1}$ or $t^{m(0)-1}$. This yields;

$$i_t(C_f \cap f_{-a}^\infty) = \begin{cases} t^{\mu(0)} + t^{\mu(0)+1} + z_1(t), \\ t^{\mu(0)} + t^{\mu(0)-1} + z_2(t). \end{cases}$$

By Theorem 7.3 we then deduce the existence of a non-trivial critical point $u \in C_f \cap f_{-a}^\infty$ with the additional property as stated in this theorem. ∎

7.3 Multiplicity results for critical points

If one take a closer look at the function f of our example, one observes that the function is even, i.e.

$$f(u) = f(-u)$$

Exploring this symmetry property together with the Morse relations we can find infinitely many solutions of Problem (I).

7.10 Theorem Let $D \subset \mathbb{R}^n$ be a smooth bounded domain. Assume that $\lambda \notin \sigma(-\Delta)$, $1 < p < \infty$ if $n \leq 2$ and $1 < p < \frac{n+2}{n-2}$ if $n \geq 3$. Then $f(u)$ has infinitely many critical points $u \in H_0^1(D)$.

Proof. The conditions in order to apply Theorems 6.6 and 6.7 are satisfied by our previous considerations. For the proof of this theorem we need some refinement of the Morse relations in that case that f possesses certain symmetry properties. Denote by $\mathcal{G}$ the class of functions satisfying (H_f') and in addition possess an symmetry property. The approximation procedure we carried out in section 6 can now be performed using approximations having the same symmetry properties. We obtain

$$i_t(C_f \cap f_a^b; \mathcal{G}) = P_t(f^b, f^a; \mathcal{G}) + (1+t)Q_t(\mathcal{G}).$$

We argue now by contradiction (see Remark 3.2). Assume now that the number of critical points of f is finite. Again 0 is a non-degenerate critical point and all other critical points are of course isolated. By the symmetry in the function f we see that if $u \in C_f$ also $-u \in C_f$. For $i_t(C_f - \{0\})$ this yields, using Theorem 7.1;

$$i_t(C_f - \{0\}) = 2\sum_{k \geq 0} a_k t^k,$$

where the sum is finite. The latter observation can be justified as follows. The Morse-index of critical points of f is finite. This can be seen by analyzing $f''(u)$ (see e.g. **??**). Nevertheless if critical points would have infinite index, they would not appear in the Morse series i_t and therefore we can restrict ourselves to critical points with finite index $\mu(u)$. By Theorem 5.2 (Chapter II) it follows then that the dimensions of $C_n(f, u)$ are finite whenever $n \in [\mu(u), \mu^*(u)]$ and the dimensions are zero if $n \notin [\mu(u), \mu^*(u)]$. This finally yields that the sum in (4.3) consists of only finitely many terms.

Take $a > 0$, then

$$i_t(C_f) = i_t(C_f \cap f_{-a}^{\infty}) = t^{m(0)} + 2\sum_{k \geq 0} a_k t^k.$$

Now we use the Morse relations (4.2) together with (4.4);

$$t^{\mu(0)} + 2\sum_{k \geq 0} a_k t^k = P_t(H_0^1(D), f^{-a}) + (1+t)Q_t = (1+t)Q_t,$$

where Q_t is a proper polynomial. Therefore we can evaluate (4.5) at $t = 1$. We obtain

$$1 + 2M = 2N, \quad N, M < \infty,$$

which is a conttradiction unless the number of critical points is infinite. ∎

7.4 Functions lacking compactness

So far we have seen that if $p < \frac{N+2}{N-2}$, the (PS)-condition is satisfied. In this section we shall take a closer look at the case $p = \frac{N+2}{N-2}$. For example, if $\lambda = 0$ and D is starshaped, Problem (I) does not have a solution except from the trivial one ($u \equiv 0$) and for this reason f has no critical points different from $u = 0$ (see Theorem 1.4, Chapter III). This yields that $C_f = \{0\}$ and $i_t(C_f) = 1$. If the Morse relations of Chapter II were to hold we would have

$$1 = P_t(f^b, f^a), \quad a < b \leq \infty \ \text{(regular)}, \tag{5.1}$$

where $Q_t = 0$. Choosing $b = \infty$ and $a < 0$, we immediately obtain contradiction because $P_t(f^\infty, f^a) = 0$ in that case (choosing different values of a and b will also lead to a contradiction). What is exactly the reason the Morse relations do not hold? This question can be answered as follows. In the proof of Lemma 4.2 and 4.3 one uses the (PS)-condition in an essential way. If the condition is not satisfied strange things can happen and the Lemmas 4.2 and 4.3 are no longer valid. This can be seen by going through the motions of the proofs of these lemmas. To make this more clear we shall illustrate this by means of the following example.

Consider the function

$$g(u) = \frac{u}{1 + u^2} \in C^\infty(\mathbb{R}, \mathbb{R}). \tag{5.2}$$

Clearly $c = \{1/2, -1/2\}$ are the only critical values of f. The value $c = 0$ is exceptional, because there are sequences of points $\{u^n\}$ such that $g(u^n) \to 0$, $g'(u^n) \to 0$ and $u^n \to \infty$. This implies that g can never satisfy (PS) in the strip g_a^b, with $a < b < 0$ or $0 < a < b$. One can easily picture what happens if one tries to deform g^b onto g^a;

Let us consider the following function

$$f(u) = \frac{1}{2} \int_D |\nabla u|^2 - \frac{N-2}{2N} \int_D |u|^{\frac{2N}{N-2}}. \tag{5.3}$$

This function does not satisfy the (PS)-condition in the strip f_a^∞, $a < 0$. We have in fact the following lemma;

7.11 Proposition Let $D \subset \mathbb{R}^N$ be a smooth bounded domain and $N \geq 3$. Then function f, defined in (5.3), satisfies the (PS)-condition in f_a^b if and only if

$$b < \frac{1}{N} S^{\frac{N}{2}}, \tag{5.4}$$

where S is the largest constant such that

$$S\|u\|^2_{L^{\frac{2N}{N-2}}} \leq \|\nabla u\|^2_{L^2}. \tag{5.5}$$

The constant is given by

$$S = N(N-2)\pi \Big(\frac{\Gamma(\frac{N}{2})}{\Gamma(N)}\Big)^{\frac{2}{N}}. \tag{5.6}$$

Lemma 5.1 gives that (PS) is not satisfied in f_a^∞. One can get more precise statements about the behaviour of f as its 'energy' increases. For this we refer to **??** concerning the so-called concertration compactness method. Using Lemma 5.1 one can compute the Poincare-series of the pair (f^b, f^a) for the function f given in (5.3).

7.12 Corollary Let $D \subset \mathbb{R}^N$ be a smooth bounded domain and $N \geq 3$. Then for the function f, defined in (5.3), we have

$$P_t(f^b, f^a) = \begin{cases} 0 & \text{if } , a > 0 \\ 1 & \text{if} a < 0. \end{cases}$$

provided $b < \frac{1}{N} S^{\frac{N}{2}}$.

Proof. From Lemma 5.1 we have that f satisfies (PS) in f_a^b, there $b < \frac{1}{N} S^{\frac{2}{N}}$. The point 0 is a non-degenerate local minimum of f. Let us assume there are more critical points in f_a^b. Let u be a non-trivial critical point. Then u satisfies the following Euler equation

$$-\Delta u = u^{\frac{N+2}{N-2}}, \quad \text{in } H^{-1}(D) \tag{5.7}$$

and

$$\begin{aligned} \int_D |\nabla u|^2 &= \int_D |u|^{\frac{2N}{N-2}}, && (7.4.1) \\ \int_D |\nabla u|^2 - \frac{N-2}{2N} \int_D |u|^{\frac{2N}{N-2}} &= c < \frac{1}{N} S^{\frac{N}{2}}. && (7.4.2) \end{aligned}$$

combining (5.5) and (5.7) we obtain

$$\int_D |\nabla u|^2 \geq S^{\frac{N}{2}}. \tag{5.10}$$

Substituting (5.8) into (5.9) we get

$$c = \int_D |\nabla u|^2 - \frac{N-2}{2N}\int_D |u|^{\frac{2N}{N-2}} = \frac{1}{N}\int_D |\nabla u|^2 \geq S^{\frac{N}{2}},$$

using (5.10), which leads to a contradiction. Therefore 0 is the only critical point in f_a^b. Thus $i_t(C_f \cap f_a^b) = 1$, when $a < 0$ and $i_t(C_f \cap f_a^b) = 0$, when $a > 0$. Using the Morse relations we conclude the proof. ∎

Now we perturb the above problem slightly, i.e. we consider the function;

$$f(u) = \frac{1}{2}\int_D |\nabla u|^2 - \frac{\lambda}{2}\int_D u^2 + \frac{N-2}{2N}\int_D |u|^{\frac{2N}{N-2}}. \tag{5.11}$$

As before f satisfies the (PS)-condition beneath some fixed energy-level. We have the following lemma;

7.13 Lemma (**??**) Let $D \subset \mathbb{R}^N$ be a smooth bounded domain and $N \geq 3$. Then function f, defined in (5.11), satisfies the (PS)-condition in f_a^b if and only if

$$b < \frac{1}{N}S^{\frac{N}{2}}.$$

Proof. The prove of this lemma is similar to the proof of Lemma 5.1. Because the pertubation $\lambda \int_D u^2$ has compact derivative (compactness of the embedding of $H_0^1(D)$ into $L^2(D)$), the critical energy-level is the same as in Lemma 5.1. ∎

Using the Morse relations again we shall prove existence of critical points of f.

7.14 Theorem Let $D \subset \mathbb{R}^N$ be a smooth bounded domain and let $N \geq 4$. If furthermore $0 < \lambda < \lambda_1$ (first eigenvalue of $-\Delta$), f has at least one non-trivial (positive) critical point and consequently a solution (weak) to equation (0.1) with $p = \frac{N+2}{N-2}$. If the above condition λ is not satisfied there are no (positive) critical points and no solutions to (0.1).

Proof. From Lemma 5.3 we have that f satisfies (PS) in the strip f_a^b when $b < \frac{1}{N}S^{\frac{N}{2}}$. In order to prove the existence of critical points we argue by contradiction. Suppose f has no critical points besides 0 in f^c, $c = \frac{1}{N}S^{\frac{N}{2}}$. Clearly then $c - \delta$, $\delta > 0$ small, is a regular value of f. We shall apply the Morse relations now in the strip $f_a^{c-\delta}$, for suitably choosen $a, \delta > 0$. From the proof of Theorem 3.1 (Chapter III) we already know that $\dim H_0^(f^a) \geq 2$, provided $a > 0$ is sufficiently small. Next we want to be able to choose a $\delta > 0$ such that $f^{c-\delta}$ contains a path connecting 0 and an arbitrary point $u_0 \in f^a$, with the additional property that $f(u_0) < 0$. In order to realize the latter one needs to make some technical estimates.

Consider the following function

$$v_\epsilon(x) = \frac{\phi_\epsilon(x)}{(\epsilon^2 + |x|^2)^{\frac{N-2}{2}}}, \tag{5.12}$$

where $\phi_\epsilon(x) \in C_0^\infty(D)$ is a smooth cutt-off function. Let us now study f on the half-line $\{tv_\epsilon(x)\}_{t\geq 0}$. We then have

$$f(tv_\epsilon) = \frac{t^2}{2}\int_D |\nabla v_\epsilon|^2 - \frac{\lambda}{2}t^2\int_D v_\epsilon^2 - \frac{N-2}{2N}t^{\frac{2N}{N-2}}\int_D |v_\epsilon|^{\frac{2N}{N-2}}. \tag{5.13}$$

The strategy now is to choose ϵ such that $\max f(tv_\epsilon) < \frac{1}{N}S^{\frac{2}{N}}$. It is clear that one can restrict t to the interval $[0, t^*]$ if t^* is sufficiently large, i.e. one chooses t^* so that $f(t^*v_\epsilon) < 0$ (see proof Theorem 3.1). Striaghtforward computation shows that

$$\max_{t\in[0,t^*]} f(tv_\epsilon) = \frac{1}{N}\left(\frac{\int_D |\nabla v_\epsilon|^2 - \lambda\int_D v_\epsilon^2}{\left(\int_D |v_\epsilon|^{\frac{2N}{N-2}}\right)^{\frac{N-2}{N}}}\right)^{\frac{N}{2}}. \tag{5.14}$$

Estimating (5.14) one obtains (see **??** for details);

$$\max_{t\in[0,t^*]} f(tv_\epsilon) = \begin{cases} \frac{1}{N}\left(S - \lambda C\epsilon^2 + O(\epsilon^{N-2})\right)^{\frac{N}{2}} & \text{for }, N \geq 5 \\ \frac{1}{N}\left(S - \lambda C\epsilon^2\log(|\epsilon|) + O(\epsilon^2)\right)^{\frac{N}{2}} & \text{for} N = 4. \end{cases}$$

Clearly if $\lambda > 0$ and $\epsilon > 0$ (sufficiently small) we see that

$$\max_{t\in[0,t^*]} f(tv_\epsilon) < \frac{1}{N}S^{\frac{N}{2}}. \tag{5.15}$$

From (5.15) one deduces that if $\epsilon > 0$ is small one can pick a $\delta > 0$ small so that the line-segment $\{tv_\epsilon\}_{t\in[0,t^*]}$, which connects 0 and t^*v_ϵ, is contained in $f_a^{c-\delta}$.

We conclude the proof by showing that $H_1(f^{c-\delta}, f^a)$ is nontrivial. Because $\dim H_0(f^a) \geq 2$, $H_0(f^a)$ has at least two generators, say $[0]$ and $[tv_\epsilon]$ (see e.g. **??**). Consider the map

$$i_0 : \ H_0(f^a) \longrightarrow H_0(f^{c-\delta}),$$

induceded by the natural embedding $f^a \hookrightarrow f^{c-\delta}$. From the previous it follows that in the set $f^{c-\delta}$ the elements 0 and tv_ϵ can be connected by a path, thus $i_0([0] - [tv_\epsilon]) = 0$. Using the exactness of the sequence

$$\longrightarrow H_1(f^{c-\delta}, f^a) \xrightarrow{\partial_1} H_0(f^a) \xrightarrow{i_0} H_0(f^{c-\delta}) \longrightarrow,$$

it follows that $[0] - [tv_\epsilon] \in Im(\delta_1)$. This yields $H_1(f^{c-\delta}, f^a) \neq 0$. For the Poincare-polynomial (because $c - \delta < \frac{1}{N}S^{\frac{N}{2}}$ in the strip $f_a^{c-\delta}$, (PS) is satisfied together with the other hypotheses of (H_f') and therefore we know from the previous that $P_t(f^{c-\delta}, f^a)$ is finite) this gives

$$P_t(f^{c-\delta}, f^a) = t + p(t).$$

By assumption the Morse-polynomial is

$$i_t(C_f \cap f_a^{c-\delta}) = 1.$$

From the Morse relations we then obtain

$$1 = t + p(t) + (1+t)Q_t, \quad Q_t \geq 0.$$

This is clearly a contradiction there t does not appear in the left hand side and the t in right hand side is inadmissible. We conclude therefore that there is at least one non-trivial critical point u of f in the strip $f_a^{c-\delta}$. This completes the proof. ■

8 — Strongly Indefinite Elliptic Systems

In the previous chapter we studied the existence of periodic solutions in conservative ordinary differential equations. The variational structure made it possible to formulate a finite dimensional reduction of the problem for which the Brouwer degree can be utilized. In this chapter we will again consider a class of variational problems for which a mixture of variational techniques and the Leray-Schauder degree can be used.

8.1 Elliptic Systems

We study the existence of non-trivial solutions for a class of variational problems where a first problem is to find the appropriate functional analytic setting. Consider the following system of two coupled semilinear Poisson equations:

$$\left[\begin{array}{l} -\Delta v = \lambda u + u^p; \\ -\Delta u = \mu v + v^q, \end{array}\right. \tag{8.1.1}$$

with the Dirichlet boundary conditions $u = v = 0$ on ∂D. Here D is a bounded domain in $\mathbb{R}^n$ with a smooth boundary, and Δ is the Laplace operator. We use the convention $u^p := |u|^{p-1}u$ and $v^q := |v|^{q-1}v$. Problem (8.1.1) allows for a variational formulation, i.e. solutions arise as critical points of the Lagrangian

$$J(\mathbf{u}) = \int_D \nabla u \nabla v \, dx - \int_D F(u) dx - \int_D G(v) dx, \tag{8.1.2}$$

where $\mathbf{u} = (u, v)$ and the functions $F(u) = \frac{\lambda}{2}|u|^2 + \frac{1}{p+1}|u|^{p+1}$ and $G(v) = \frac{\mu}{2}|v|^2 + \frac{1}{q+1}|v|^{q+1}$ are the primitives of of the right hand sides in (8.1.1). The objective of this paper is to establish a natural functional analytic frame work for the study of

J, and to obtain existence results for critical points of J by means of the topological min-max approach due to Benci and Rabinowitz, cf. [6]

8.1 Exercise Show, using a formal calculation, that the Euler-Lagrange equations of J in (8.1.2) yield the elliptic system in (8.1.1). (Hint: Assume that J is defined for sufficiently smooth functions u and v and use perturbations in $C_0^\infty(D)$.) ▪

The quadratic part of J, is strongly indefinite. Indeed, if we replace (u,v) by $(u,-v)$, this results in a sign change of A, so that $\mathbf{u} = (u,v) = (0,0)$ is a saddle point for A having an infinite Morse index: any decomposition of the $\mathbf{u}$-space into two subspaces H_1 and H_2 such that A restricted to H_1 has a minimum in $(0,0)$, and A restricted to H_2 has a maximum in $(0,0)$, will necessarily imply that both H_1 are infinite dimensional.

To get some feeling for the subtleties involved we first consider the case that $p = q$, $\lambda = \mu$ and $u \equiv v$, when Problem (8.1.1) reduces to the well known Dirichlet problem for the semilinear Poisson equation,

$$-\Delta u = \lambda u + u^p, \quad u = 0 \text{ on } \partial D.$$

For $1 < p < \frac{n+2}{n-2}$ a (positive) solution is obtained from a mountain pass argument[3] applied to the associated Lagrangian. The variable function u is taken in the Sobolev space $H_0^1(D)$. This space embeds compactly into $L^{p+1}(D)$ if p satisfies $1 < p < \frac{n+2}{n-2}$, i.e. if p is subcritical. On the other hand, if $p \geq \frac{n+2}{n-2}$, the Pohozaev identity[1] excludes the existence of positive (classical) solutions when D is starshaped.[27]

For Problem (8.1.1) we ask the same questions. At first glance, it seems natural to look for a critical point of $J(\mathbf{u})$, varying both u and v over $H_0^1(D)$. For $J(\mathbf{u})$ to be well defined one is then however forced to assume that $1 < p < \frac{n+2}{n-2}$ and $1 < q < \frac{n+2}{n-2}$. This is to restrictive and therefore *not* the right approach, because the natural assumption on p and q is

$$\frac{1}{p+1} + \frac{1}{q+1} > \frac{n-2}{n}, \ \ p,q > 1, \ \ n > 2. \tag{8.1.3}$$

This is in agreement with a generalization of the Pohozaev identity which shows that positive classical solutions on star shaped domains can only exist if (8.1.3) holds. For solutions of Problem (8.1.1) this generalized Pohozaev identity reads

$$\Big(\frac{n}{p+1} - \alpha\Big) \int_D |u|^{p+1} dx + \Big(\frac{n}{q+1} - (n-2-\alpha)\Big) \int_D |u|^{q+1} dx + \lambda \int_D |u|^2 + \mu \int_D |v|^2 = \oint_{\partial D} \frac{\partial u}{\partial \nu}\frac{\partial v}{\partial \nu}(x,\nu),$$

[1] Solutions satisfy the identity

$$\oint_{\partial D} \Big|\frac{\partial u}{\partial \nu}\Big|^2 (x,\nu) = \lambda \int_D |u|^2 dx + \Big(\frac{n}{p+1} - \frac{n-2}{n}\Big) \int_D |u|^{p+1} dx,$$

called the Pohozaev identity.

where ν is the outward unit normal on ∂D. We note that this identity follows from a variant of Noether's first theorem.[**?**, 31, 15] By choosing $\alpha = \frac{n}{p+1}$ it follows that the existence of a positive solution pair, combined with the boundary point lemma,[28] implies that (8.1.3) must hold.

We must find the appropriate functional analytic setting in order to accommodate the hyperbola of critical exponents. With the right functional analytic setting we can use appropriate variational techniques to obtain solutions to (8.1.1). The main result of this chapter is:

8.2 Theorem Assme p and q satisfy Condition (8.1.3) and $\lambda \geq 0$ and $\mu \geq 0$ satisfy $\lambda\mu < \lambda_1^2$, where λ_1 is the first eigenvalue of $-\Delta$ with zero boundary conditions. Then, the elliptic system in (8.1.1) has a non-trivial (classical) solution (u,v) with positive components.

The remainder of this chapter is devoted to proving Theorem 8.2 and explaining the most important tools needed in the proof.

8.2 The functional analytic frame work.

In this section we discuss appropriate function spaces such that the functional $J(\mathbf{u})$ well-defined and sufficiently smooth on these spaces.

8.2.a Function spaces

Consider the strictly positive selfadjoint operator $-\Delta$ with domain $H^2(D) \cap H_0^1(D) \subset L^2(D)$. For simplicity we use the Fourier coefficients of u and v with respect to a fixed orthonormal basis of $L^2(D)$, consisting of eigenfunctions $\phi_1, \phi_2, \phi_3, \dots$ of $-\Delta$, $\phi_1 > 0$, corresponding to positive eigenvalues $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \ldots \uparrow \infty$, counted with multiplicity. Throughout this paper these eigenfunctions are normalized in $L^2(D)$: $\int_D \phi_i\phi_j = \delta_{ij}$, and $L^2(D) = \{u = \sum_{k=1}^{\infty} \xi_k\phi_k : \sum_{k=1}^{\infty} \xi_k^2 < \infty\}$, with inner product $(u,v)_{L^2} = \int_D uv = \sum_{k=1}^{\infty} \xi_k\eta_k$. The operators $(-\Delta)^{r/2}$ are defined by

$$(-\Delta)^{r/2}u = \sum_{k=1}^{\infty} \lambda_k^{r/2}\xi_k\phi_k, \tag{8.2.4}$$

with domain

$$D((-\Delta)^{r/2}) = \Theta^r(D) = \{\sum_{k=1}^{\infty} \xi_k\phi_k \in L^2(D) : \sum_{k=1}^{\infty} \lambda_k^r\xi_k^2 < \infty\}, \tag{8.2.5}$$

if $r \geq 0$. We note that $\Theta^r(D) = H_0^r(D) = H^r(D)$ for $0 < r < 1/2$, $\Theta^{1/2}(D) = H_{00}^{1/2}(D)$, cf.[20], $\Theta^r(D) = H_0^r(D)$ for $1/2 < r \leq 1$, and that $\Theta^r(D) = H^r(D) \cap H_0^1(D)$ for $1 < r \leq 2$. If $r < 0$ the domain is equal to the whole of $L^2(D)$, and extends in a natural way to a larger space which we shall define for $-2 < r < 0$ in a moment.

For $r \geq 0$ the space $\Theta^r(D)$ is a Hilbert space with inner product

$$((u,v))_{\Theta^r} = (u,v)_{L^2} + ((-\Delta)^{r/2}u, (-\Delta)^{r/2}v)_{L^2}.$$

The corresponding norm is the graph norm of the operator $(-\Delta)^{r/2}$. We can identify $\Theta^r(D)$ with the space

$$\varpi^r = \{\xi = (\xi_1, \xi_2, \dots) : \sum_{k=1}^{\infty} \lambda_k^r \xi_k^2 < \infty\},$$

which is a Hilbert space with inner product $(\xi, \eta)_r = \sum_{k=1}^{\infty} \lambda_k^r \xi_k \eta_k$. If $r = 0$ this is just the standard separable Hilbert space l^2. The standard way to define $\Theta^{-r}(D)$, $r > 0$, is as a representation of the dual space $\Theta^r(D)'$. Since $\Theta^r(D)$ can be identified with ϖ^r, $\Theta^r(D)'$ can be identified with the dual of ϖ^r, which is represented as ϖ^{-r} setting

$$\langle \xi, \eta \rangle = \sum_{k=1}^{\infty} \xi_k \eta_k, \quad \xi \in \varpi^r, \quad \eta \in \varpi^{-r}.$$

In this way the space $\Theta^r(D)'$ is isomorphic to

$$\Theta^{-r}(D) = \{\sum_{k=1}^{\infty} \xi_k \phi_k \in (H^2(D) \cap H_0^1(D))' : \sum_{k=1}^{\infty} \lambda_k^{-r} \xi_k^2 < \infty\}.$$

Since u and v in Θ^r are represented by ξ and η in ϖ^r we obtain: $((u,v))_{\Theta^r} = (\xi, \eta)_0 + (\xi, \eta)_r$. We shall write

$$(u,v)_{\Theta^r} = ((-\Delta)^{r/2}u, (-\Delta)^{r/2}v)_{L^2(D)} = (\xi, \eta)_r, \tag{8.2.6}$$

and clearly this defines an equivalent inner product on Θ^r. In fact we have, denoting the norms corresponding to (1.10) by $\|u\|_{\Theta^r} = |\xi|_r$, that

$$\|u\|_{L^2(D)} = \|\xi\|_{l^2} = |\xi|_0 \leq \lambda_1^{-r/2} |\xi|_r = \lambda_1^{-r/2} \|u\|_{\Theta^r}, \tag{8.2.7}$$

which can be viewed as a generalized Poincaré inequality. Observe that $(-\Delta)^s : \Theta^r(D) \to \Theta^{r-2s}(D)$ is an isomorphism.

8.2.b The quadratic form

The motivation to introduce above spaces is to extend the quadratic form $\int_D \nabla u \nabla v \, dx$ to functions u and v with different regularity properties. Since

$$A(\mathbf{u}) = \int_D \nabla u \nabla v \, dx = \sum_{k=1}^{\infty} \lambda_k \xi_k \eta_k = \sum_{k=1}^{\infty} \lambda_k^{r/2} \xi_k \lambda_k^{1-r/2} \eta_k,$$

we have

$$\left| \int_D \nabla u \nabla v \, dx \right| \leq \{\sum_{k=1}^{\infty} \lambda_k^r \xi_k^2\}^{\frac{1}{2}} \{\sum_{k=1}^{\infty} \lambda_k^{2-r} \eta_k^2\}^{\frac{1}{2}} = \|u\|_{\Theta^r} \|v\|_{\Theta^{2-r}}.$$

By defining the product Hilbert spaces $E^r(D) = \Theta^r(D) \times \Theta^{2-r}(D)$ for $0 \le r \le 2$, the quadratic form A extends uniquely to $E^r(D)$ for all r. Observe that $-\Delta$ is an isometric map from the first (second) component of $E^r(D)$ onto the dual of the second (first) component. The selfadjoint operator defined by $-\Delta_S \mathbf{u} = (-\Delta v, -\Delta u)$, with $\mathbf{u} = (u,v) \in E^r(D)$, is an isometry and there exists a unique selfadjoint isometry $L\colon E^r(D) \to E^r(D)$, such that $\frac{1}{2}\langle -\Delta_S \mathbf{u}, \mathbf{u}\rangle = \frac{1}{2}(L\mathbf{u},\mathbf{u})_{E^r}$, for $\mathbf{u} = (u,v) \in E^r(D)$. The operator L can be expressed in terms of fractional powers of $-\Delta$:

$$L = \begin{pmatrix} 0 & (-\Delta)^{1-r} \\ (-\Delta)^{r-1} & 0 \end{pmatrix}. \tag{8.2.8}$$

> **8.3 Exercise** Derive the expression for L in (8.2.8) and show that L is an isometry. ■

We now continue with the properties of the isometry L. Note that

$$L\mathbf{u}^{\pm} = L(u, \pm(-\Delta)^{r-1}u) = (\pm u, (-\Delta)^{r-1}u) = \pm \mathbf{u}^{\pm},$$

are the mutually orthogonal eigenspaces of the eigenvalues 1 and -1 of L. Orthonormal bases consisting of eigenvectors of $E^{\pm}$ are given by

$$\Big\{ \frac{1}{\sqrt{2}}(\lambda_k^{-\frac{r}{2}}\phi_k, \pm\lambda_k^{\frac{r}{2}-1}\phi_k) : k = 1,2,\dots \Big\},$$

and

$$E^r(D) = E^+ \oplus E^- = \{\mathbf{u} = \mathbf{u}^+ + \mathbf{u}^-,\ \mathbf{u}^{\pm} \in E^{\pm}\}.$$

This yields the decomposition $A(\mathbf{u}) = A(\mathbf{u}^+) + A(\mathbf{u}^-)$, whereas

$$A(\mathbf{u}^+) - A(\mathbf{u}^-) = \frac{1}{2}\|\mathbf{u}\|^2_{E^r(D)},$$

The derivative of $A(\mathbf{u})$ defines a bilinear form $B(\mathbf{u},\mathbf{p}) = A'(\mathbf{u})\mathbf{p}$ with $A(\mathbf{u}) = \frac{1}{2}B(\mathbf{u},\mathbf{u})$ and $B(\mathbf{u}^+,\mathbf{u}^-) = 0$. As a matter of fact A is infinitely many times continuously differentiable on $E^r(D)$ for all r.

8.2.c The functional J is well-defined

Next we examine for which values of r the nonlinearities in (8.1.1) are well-defined and sufficiently differentiable on $E^r(D)$.

From interpolation theory[20] it follows that the injection $\Theta^r(D) \to H^r(D)$ is continuous. By the Sobolev embeddings,

$$\Theta^r(D) \to H^r(D) \to L^p(D), \text{ if } 1 \le p \le \frac{2n}{n-2r} < \infty \tag{1.29}$$

is bounded for $0 < r < 2n$. The second injection is compact if $1 \le p < \frac{2n}{n-2r}$. If $2r \ge n$, these statements hold for any $1 \le p < \infty$. Also $\Theta^{r_1}(D) \to \Theta^{r_2}(D)$ is compact if $r_1 > r_2$ (for the corresponding spaces ϖ^r this can be proved directly). As an immediate consequence we have that for $n > 2r$ and $n > 4 - 2r$,

$$E^r(D) \to L^{p+1}(D) \times L^{q+1}(D), \tag{8.2.9}$$

is a continuous embedding whenever

$$1 \le p+1 \le \frac{2n}{n-2r}, \quad 1 \le q+1 \le \frac{2n}{n+2r-4}.$$

This embedding is compact if both inequalities bounding p and q from above are strict. If $r \ge 2n$, there is no restriction on p, and if $n \le 4 - 2r$, there is no restriction on q.

The non-quadratic term in J is given by

$$b_0(\mathbf{u}) = \frac{1}{p+1}\int_D |u|^{p+1}dx + \frac{1}{q+1}\int_D |v|^{q+1}dx,$$

with p, q satisfying (8.1.3). The latter ensures that for every p and q there exists an $r \in (0,2)$ such that b is well-defined on $E^r(D)$. The first and second derivatives are given by

$$b_0'(\mathbf{u})\mathbf{p} = \int_D u^p \phi dx + \int_D v^q \psi dx;$$
$$\langle b_0''(\mathbf{u})\mathbf{p},\mathbf{q}\rangle = p\int_D |u|^{p-1}\phi\eta dx + q\int_D |v|^{q-1}\psi\zeta dx.$$

The above derivatives exist for appropriate choice of r and are continuous which implies that J is a C^2-functional on $E^r(D)$. The functional We restrict to this range of $r \in (0,2)$ because we need the compactness of the embedding $E^r(D) \to L^2(D) \times L^2(D)$.

8.4 Exercise Prove the expressions for b' and b''. ▪

8.3 Compactness and Geometry

In order to find solutions of the elliptic system given in (8.1.1) we employ the variational structure, i.e. we establish solutions are critical points of J. In this section we use an characterization of critical values and a deformation argument to find critical values. Crucial in all the arguments is compactness. A critical compactness condition the ensures that the set critical points is locally compact is called the Palais-Smale condition which can be satisfied for J provided the appropriate growth conditions of b. In this section we restrict ourselves to the case $\lambda = \mu = 0$.

8.3.a The Palais-Smale condition

We have to show that every sequence $\{\mathbf{u}_n\}$ in $E^r(D)$ satisfying

$$J(\mathbf{u}_n) \text{ is bounded in } E^r(D), \quad J'(\mathbf{u}_n) \to 0, \text{ as } n \to \infty,$$

has a convergent subsequence. The key point here is to prove that such a sequence is necessarily bounded in $E^r(D)$. For then the compactness of b' implies, since $J'(\mathbf{u}_n)$ converges in $(E^r(D))'$, that a subsequence of $A'(\mathbf{u}_n)$ also converges. Since L is an isometry we also have that $L\mathbf{u}_n$ and $\mathbf{u}_n$ converge in $E^r(D)$.

To prove that $\mathbf{u}_n$ is bounded we proceed as follows. For some $M > 0$ and arbitrarily small $\epsilon_n > 0$ we have, omitting the subscripts,

$$\begin{aligned} M + \epsilon\|\mathbf{u}\|_{E^r(D)} &\geq J(\mathbf{u}) - \frac{1}{2}\langle J'(\mathbf{u}), \mathbf{u}\rangle \\ &\geq \left(\frac{1}{2} - \frac{1}{p+1}\right)\int_D |u|^{p+1}dx + \left(\frac{1}{2} - \frac{1}{q+1}\right)\int_D |v|^{q+1}dx. \end{aligned}$$

Hence,

$$\|u\|^{p+1}_{L^{p+1}} + \|v\|^{q+1}_{L^{q+1}} \leq C + \epsilon\|\mathbf{u}\|_{E^r}. \tag{8.3.10}$$

For $\mathbf{u}^{\pm} = (u^{\pm}, v^{\pm})$, we also have

$$\begin{aligned} \|\mathbf{u}^{\pm}\|^2_{E^r} - \epsilon\|\mathbf{u}^{\pm}\|_{E^r} &\leq \left|(L\mathbf{u}, \mathbf{u}^{\pm})_{E^r} - \langle J'(\mathbf{u}), \mathbf{u}^{\pm}\rangle\right| \\ &= \left|\langle b'(\mathbf{u}), \mathbf{u}^{\pm}\rangle\right| = \left|\int_D u^p u^{\pm} dx + \int_D v^q v^{\pm} dx\right| \\ &\leq \|u\|^p_{L^{p+1}}\|u^{\pm}\|_{L^{p+1}} + \|v\|^q_{L^{q+1}}\|v^{\pm}\|_{L^{q+1}} \\ &\leq \left\{\|u\|^p_{L^{p+1}} + \|v\|^q_{L^{q+1}}\right\}\|\mathbf{u}^{\pm}\|_{E^r}. \end{aligned}$$

Dividing the first and the last expression by $\|\mathbf{u}^{\pm}\|_{E^r}$ we obtain

$$\|\mathbf{u}^{\pm}\|_{E^r} - \epsilon \leq \|u\|^p_{L^{p+1}} + \|v\|^q_{L^{q+1}}. \tag{8.3.11}$$

Combining (8.3.10) and (8.3.11) for $\mathbf{u} = \mathbf{u}^+ + \mathbf{u}^-$, we obtain

$$\|\mathbf{u}\|_{E^r} \leq C\left\{1 + \{C + \epsilon\|\mathbf{u}\|_{E^r}\}^{\frac{p}{p+1}} + \{C + \epsilon\|\mathbf{u}\|_{E^r}\}^{\frac{q}{q+1}}\right\},$$

which keeps $\|\mathbf{u}\|_{E^r}$ away from infinity. This implies that the Palais-Smale condition is satisfied, and thereby concludes the proof of the theorem.

An important consequence of the Palais-Smale condition are lower bounds on ∇J with respect to intervals of regular values.

8.5 Lemma Let $-\infty < a < b < \infty$ and let $[a, b]$ be an interval of regular values of J. Then, $\|\nabla J(\mathbf{u})\|_{E^r(D)} \geq \delta > 0$, for all $\mathbf{u} \in E^r(D)$ that satisfy $a \leq J(\mathbf{u}) \leq b$.

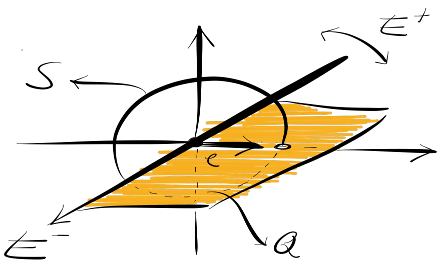


Figure 8.1: The linking sets S and ∂Q.

Proof. Suppose not, then there exists a sequence $\mathbf{u}^n \in J_a^b = \{\mathbf{u} \in E^r(D) : a \le J(\mathbf{u}) \le b\}$, such that $J'(\mathbf{u}^n) \to 0$. By assumption $J(\mathbf{u}^n)$ is bounded. Since the Palais-Smale condition holds there exists a subsequence $\mathbf{u}^{n_k} \to \mathbf{u}$ and $\mathbf{u}$ is a critical point with $c = J(\mathbf{u}) \in [a,b]$, which contradicts the fact that $[a,b]$ is an interval of regular values. Therefore there exists a $\delta > 0$ such that $\|\nabla J(\mathbf{u})\|_{E^r(D)} \ge \delta > 0$ for all $\mathbf{u} \in J_a^b$. ■

8.6 Lemma Let $c \in \mathbb{R}$ be a regular value of J. Then there exists an $\epsilon > 0$ such that $[c-\epsilon, c+\epsilon]$ is an interval of regular values.

Proof. Suppose such an $\epsilon > 0$ does not exist, then there exists a sequence $\mathbf{u}^n \in E^r(D)$ such that $J(\mathbf{u}^n) \to c$ and $J'(\mathbf{u}^n) = 0$. Since J satisfies the Palais-Smale condition there exists a subsequence $\mathbf{u}^{n_k} \to \mathbf{u}$ and $\mathbf{u}$ is a critical point with $c = J(\mathbf{u})$ which contradicts the fact that c is a regular value. Therefore, there exists an $\epsilon > 0$ such that $[c-\epsilon, c+\epsilon]$ is an interval of regular values. ■

8.3.b Linking sets

In order to find a critical point we define a value c which is a critical value for J. The idea of this method is find set S and Q in $E^r(D)$ which satisfy an infinite dimensional linking condition. The sets S and Q have to be chosen such S and Q 'link' and and there exists numbers $\alpha > \omega$ such that $J|_S \ge \alpha$ and $J|_{\partial Q} \le \omega$. We use the sets S and Q to define a critical value. This type of method is called a minimax method and the procedure we describe here is an adaptation of a general theorem due to Benci and Rabinowitz.[6]

Let ρ, $s_1 > \rho$ and s_2 be positive numbers to be specified later on, and let $\mathbf{e}^\pm$ be the first vectors in the basis of $E^\pm$. We set $[0, s_1\mathbf{e}^+] = \{s\mathbf{e}^+;\ \ 0 \le s \le s_1\}$, $\tilde{H} = \text{span}\,[\mathbf{e}^+] \oplus E^-$, and

$$Q = [0, s_1\mathbf{e}^+] \oplus \left(\overline{B}_{s_2} \cap E^-\right), \quad S = \partial B_\rho \cap E^+,$$

where B_R denotes an open ball with radius R centered at the origin, see Figure 8.1

8.7 Lemma There exist $\rho > 0$ and $\alpha > 0$ such that $J|_S \geq \alpha$.

Proof. On E^+ the quadratic part $A(\mathbf{u})$ of J reduces to $(1/2)\|\mathbf{u}\|^2_{E^r}$, so that A has a strict local minimum on E^+ at $\mathbf{u}^+ = 0$. For J we have, using the Sobolev inequalities for $\Theta^r(D)$, that

$$\begin{aligned} J(\mathbf{u}^+) &= (1/2)\|\mathbf{u}\|^2_{E^r} - \frac{1}{p+1}\int_D |u^+|^{p+1}dx - \frac{1}{q+1}\int_D |v^+|^{q+1}dx \\ &\geq \frac{1}{2}\|\mathbf{u}^+\|^2_{E^r} - C\|\mathbf{u}^+\|^{p+1}_{E^r} - C\|\mathbf{u}^+\|^{q+1}_{E^r}, \end{aligned}$$

Thus we can fix $\rho > 0$ and $\alpha > 0$ such that $J(\mathbf{u}) \geq \alpha > 0$ on S. ■

In the next estimate we choose $\omega = 0$.

8.8 Lemma Let $\rho > 0$. There exist $s_1 > \rho$ and s_2 such that $J|_{\partial Q} \leq 0$.

Proof. Next we show that for suitable choices of s_1 and s_2 the function $J(\mathbf{u})$ is nonpositive on ∂Q. Note that the boundary ∂Q of the cylinder Q is taken in the space $\tilde{H}$, and consists of three parts, namely the bottom $Q \cap \{s = 0\}$, the lid $Q \cap \{s = s_1\}$, and the 'lateral' boundary $[0, s_1\mathbf{e}^+] \oplus (\partial B_{s_2} \cap E^-)$. Clearly $J(\mathbf{u}) \leq 0$ on the bottom because $A(\mathbf{u}) \leq 0$ in E^- and functional $b_0(\mathbf{u})$ is nonnegative. For the remaining two parts of the boundary we first observe that, for $\mathbf{u} = \mathbf{u}^- + s\mathbf{e}^+ \in \tilde{H}$,

$$J(\mathbf{u}^- + s\mathbf{e}^+) = \frac{1}{2}s^2 - \frac{1}{2}\|\mathbf{u}^-\|^2_{E^r(D)} - b_0(\mathbf{u}^- + s\mathbf{e}^+). \tag{8.3.12}$$

Then, for $\gamma = \min\{p+1, q+1\} > 2$,

$$b_0(\mathbf{u}^- + s\mathbf{e}^+) \geq B_1 \int_D |\mathbf{u}^- + s\mathbf{e}^+|^\gamma - B_2|D|,$$

where B_1, B_2 are constants. Thus, writing $\mathbf{u}^- = t\mathbf{e}^- + \mathbf{u}_2^-$, where t is a real number, and $\mathbf{u}_2^- \in E^-$ is perpendicular to $\mathbf{e}^-$ in $E^r(D)$, $\mathbf{u}_2^-$ is also perpendicular to $\mathbf{e}^-$ and $\mathbf{e}^+$ in $L^2 \times L^2$, and conclude

$$\begin{aligned} b_0(\mathbf{u}^- + s\mathbf{e}^+) &\geq B_1 \int_D |\mathbf{u}_2^- + t\mathbf{e}^- + s\mathbf{e}^+|^\gamma - B_2|D| \\ &\geq B_3\Big(\int_D |\mathbf{u}_2^- + t\mathbf{e}^- + s\mathbf{e}^+|^2\Big)^{\gamma/2} - B_4 \\ &= B_3\Big(\int_D |\mathbf{u}_2^-|^2 + \int_D |t\mathbf{e}^- + s\mathbf{e}^+|^2\Big)^{\gamma/2} - B_4 \\ &\geq B_3\Big(\int_D |t\mathbf{e}^- + s\mathbf{e}^+|^2\Big)^{\gamma/2} - B_4 \\ &\geq B_3\Big(s^2 \sin^2\chi \int_D |\mathbf{e}^+|^2\Big)^{\gamma/2} - B_4 = B_5 s^\gamma - B_4. \end{aligned}$$

Here χ is the (positive) angle between $\mathbf{e}^+$ and $\mathbf{e}^-$ with respect to the inner product in $L^2 \times L^2$, see Exercise 2 in Sect. 8.7.

For J this yields

$$J(\mathbf{u}^- + s\mathbf{e}^+) \leq \frac{1}{2}s^2 - B_5 s^\gamma + B_4 - \frac{1}{2}\|\mathbf{u}^-\|^2_{E^r(D)}. \tag{8.3.13}$$

Now choose s_1, s_2 such that

$$\psi(s) = \frac{1}{2}s^2 - B_5 s^\gamma + B_4 \leq 0 \quad \forall s \geq s_1, \quad s_2^2 > 2\max_{s\geq 0}\psi(s),$$

to make J negative on the lid and on the lateral boundary respectively. ■

8.4 Existence of critical points.

In this section we shall prove that J has a critical point $\mathbf{u} = (u,v) \in E^r(D)$, with $r \in (0,2)$ appropriately chosen.

8.4.a Deformation of S and Q

In the usual setting of linking theory the sets S and ∂Q link in the following sense. Let Φ be the set of mappings $\varphi\colon E^r(D) \to E^r(D)$ of the form $\varphi(\mathbf{u}) = \mathbf{u} - k(\mathbf{u})$, with k compact. The sets S and ∂Q are then said to link if for all $\varphi \in \Phi$ with $\varphi|_{\partial Q} = \mathrm{Id}$ we have $\partial Q \cap S = \emptyset$ and

$$\varphi(Q) \cap S \neq \emptyset.$$

8.9 Exercise Show that with the choice of ρ, s_1 and s_2 the sets S and ∂Q link. ■

If we alter the notion of linking slightly by choosing a different set of mappings Φ we can still have a meaningful notion of linking. Therefore consider the normalized gradient flow:

$$\dot{\mathbf{u}} = -\omega(\mathbf{u})\nabla J(\mathbf{u}) = -\omega(\mathbf{u})\left[L\mathbf{u} + \nabla b(\mathbf{u})\right], \tag{8.4.14}$$

where ∇b is the gradient of b with respect to $\|\cdot\|_{E^r}$. For ω we choose $\omega(\mathbf{u}) = [1 + \|\nabla J(\mathbf{u})\|_{E^r}]^{-1}$ which satisfies $0 \leq \omega(\mathbf{u}) \leq 1$. This make the right hand side of (8.4.14) a C^1 bounded vector field on $E^r(D)$ and the initial value problem in (8.4.14) defines a global flow $\varphi\colon \mathbb{R} \times E^r(D) \to E^r(D)$.

If we use the canonical splitting $E^r(D) = E^- \oplus E^+$ the equations become

$$\begin{aligned}
\dot{\mathbf{u}}^- &= \omega(\mathbf{u})\mathbf{u}^- - \omega(\mathbf{u})P^-\nabla b(\mathbf{u});\\
\dot{\mathbf{u}}^+ &= -\omega(\mathbf{u})\mathbf{u}^+ - \omega(\mathbf{u})P^+\nabla b(\mathbf{u}),
\end{aligned}$$

where $P^\pm$ are the orthogonal projections onto $E^\pm$. The variation of constants formula yields the representation:

$$\varphi(t,\mathbf{u}) = e^{\theta(t,\mathbf{u})}\mathbf{u}^- + e^{-\theta(t,\mathbf{u})}\mathbf{u}^+ - k(t,\mathbf{u}), \tag{8.4.15}$$

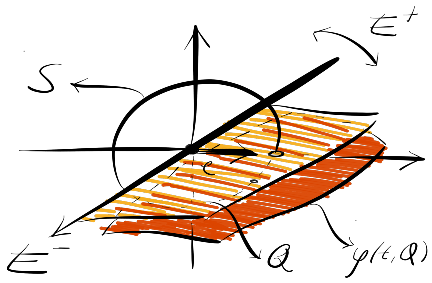


Figure 8.2: The sets S and $\varphi(t,Q)$ intersect for all deformations.

where $\theta(t,\mathbf{u}) = \int_0^t \omega\big(\varphi(s,\mathbf{u})\big)ds$ and

$$k(t,\mathbf{u}) = \int_0^t \Big[e^{\theta(t,\mathbf{u})-\theta(s,\mathbf{u})}\omega\big(\varphi(s,\mathbf{u})\big)P^-\nabla b\big(\varphi(s,\mathbf{u})\big)\Big]ds$$
$$+ \int_0^t \Big[e^{-\theta(t,\mathbf{u})+\theta(s,\mathbf{u})}\omega\big(\varphi(s,\mathbf{u})\big)P^+\nabla b\big(\varphi(s,\mathbf{u})\big)\Big]ds.$$

Via this representation of φ we can derive various properties.

8.10 Lemma The continuous mapping $k\colon \mathbb{R}\times E^r(D) \to E^r(D)$ is compact.

Proof. Under construction. ∎

The choice of the sets Q and S above implies the follow non-intersection property:

8.11 Lemma Let Q and S be given by Lemmas 8.7 and 8.8. Then, $\varphi(t,\partial Q)\cap S = \emptyset$, for all $t \geq 0$.

Proof. The functional J is a Lyapunov function for φ, i.e. $\dot{J}(\varphi(t,\mathbf{u})) = -\omega(\varphi(t,\mathbf{u}))\|\nabla J(\varphi(t,\mathbf{u}))\|^2_{E^r(D)} \leq 0$. This implies, since $J|_{\partial Q} \leq 0$, that $\varphi(t,\partial Q) \leq 0$, for $t \geq 0$. On the other hand $J|_S > 0$, which implies $\varphi(t,\partial Q)\cap S = \emptyset$, for all $t \geq 0$. ∎

The compactness of the operator k yields the following intersection property, see Fig. 8.2:

8.12 Lemma Let Q and S be given by Lemmas 8.7 and 8.8. Then, $\varphi(t,Q)\cap S \neq \emptyset$, for all $t \geq 0$.

Proof. The condition $\varphi(t,Q)\cap S \neq \emptyset$, for all $t \geq 0$, is equivalent to the existence of $\mathbf{u} = \mathbf{u}^- + s\mathbf{e}^+ \in Q$ with $\varphi(t,\mathbf{u}) \in S$. We need to solve the equations

$$P^-\varphi(t,\mathbf{u}) = 0, \quad \|\varphi(t,\mathbf{u})\|_{E^r(D)} = \rho, \ \forall t \geq 0.$$

From the variation of constants formula in (8.4.15) we derive

$$\mathbf{u}^- = e^{-\theta(t,\mathbf{u}^- + s\mathbf{e}^+)}P^-k(t,\mathbf{u}^- + s\mathbf{e}^+).$$

In the coordinates $v = (\mathbf{u}^-, s) \in E^- \times \mathbb{R}$ we have the following equation

$$\begin{pmatrix} \mathbf{u}^- \\ s \end{pmatrix} - \begin{pmatrix} e^{-\theta(t,\mathbf{u}^- + s\mathbf{e}^+)} P^- k(t, \mathbf{u}^- + s\mathbf{e}^+) \\ s + \rho - \|\varphi(t,\mathbf{u})\|_{E^r(D)} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

which is of the form $v - K_t(v) = 0$ on $E^r(D)^- \times \mathbb{R}$ and where $K_t \colon E^- \times \mathbb{R} \to E^- \times \mathbb{R}$ is compact for all $t \in \mathbb{R}$. The domain of definition is $\overline{\Omega} = \{v = (\mathbf{u}^-, s) \in E^- \times \mathbb{R} : \mathbf{u}^- + s\mathbf{e}^+ \in Q\} = \big(B_{s_2}(0) \cap E^-\big) \times [0, s_1]$.

8.13 Exercise Show that $K_t \colon E^- \times \mathbb{R} \to E^- \times \mathbb{R}$ is compact for all $t \in \mathbb{R}$. ▪

Lemma 8.11 implies that $\varphi(t, \partial Q) \cap S = \emptyset$ for all $t \geq 0$, and thus the equation $v - K_t(v) = 0$ has no solutions $v \in \partial\Omega$, for all $t \geq 0$. Therefore the Leray-Schauder degree is well-defined for $t \geq 0$ and the homotopy invariance of the Leray-Schauder degree implies that $\deg_{LS}(\mathrm{Id} - K_t, \Omega, 0) = \deg_{LS}(\mathrm{Id} - K_0, \Omega, 0)$, where $K_0(v) = (0, s + \rho - \|\mathbf{u}^- + s\mathbf{e}^+\|_{E^r(D)}) = \Big(0, s + \rho - \sqrt{\|\mathbf{u}^-\|^2_{E^r(D)} + s^2}\Big)$.

Next we compute the degree $\deg_{LS}(\mathrm{Id} - K_0, \Omega, 0)$. Consider the homotopy $\mathrm{Id} - L_\tau$, where $L_\tau(v) = \Big(0, s + \rho - \sqrt{\tau^2\|\mathbf{u}^-\|^2_{E^r(D)} + s^2}\Big)$, which is a compact perturbation of Id for all $\tau \in [0,1]$.

8.14 Exercise Prove the above statement. ▪

Since the equation $v - L_\tau(v) = 0$ is equivalent to $\mathbf{u}^- = 0$ and $s = \rho$ for all τ, it follows that there are no solutions $v \in \partial\Omega$ because we chose $s_1 > \rho$. Consequently $\mathrm{Id} - L_\tau$ is a legitimate homotopy and $\deg_{LS}(\mathrm{Id} - K_0, \Omega, 0) = \deg_{LS}(\mathrm{Id} - L_1, \Omega, 0) = \deg_{LS}(\mathrm{Id} - L_0, \Omega, 0)$, and $\deg_{LS}(\mathrm{Id} - L_0, \Omega, 0) = \deg_{LS}(\mathrm{Id}, \Omega, \rho) = 1$.

We now conclude that $\deg_{LS}(\mathrm{Id} - K_t, \Omega, 0) = 1$ and therefore $v - K_t(v) = 0$ has a solution for all $t \geq 0$, which is equivalent to $\varphi(t, Q) \cap S \neq \emptyset$, for all $t \geq 0$. ■

8.4.b Minimax values

Following Benci and Rabinowitz[6, **?**] we now a value c for J which is based on the choices of S and Q. Define

$$c = \inf_{t \geq 0} \sup_{\mathbf{u} \in Q} J\big(\varphi(t, \mathbf{u})\big). \tag{8.4.16}$$

8.15 Lemma The value c defined in (8.4.16) is a critical value for J and $c > 0$.

Proof. The set Q is bounded in $E^r(D)$ and by (8.3.13) we derive that $\sup_{\mathbf{u} \in Q} J(\mathbf{u}) < \infty$. Furthermore, since φ is gradient-like, i.e. J is a Lyapunov function for φ, we obtain that $\sup_{\mathbf{u} \in Q} J\big(\varphi(t, \mathbf{u})\big) \leq \sup_{\mathbf{u} \in Q} J(\mathbf{u}) < \infty$. By lemma 8.7 $\inf_{\mathbf{u} \in S} J(\mathbf{u}) \geq \alpha > 0$ and since $\varphi(t, Q) \cap S \neq \emptyset$ for all $t \geq 0$, we conclude that

$$\infty > \sup_{\mathbf{u} \in Q} J\big(\varphi(t, \mathbf{u})\big) \geq \inf_{\mathbf{u} \in S} J(\mathbf{u}) \geq \alpha > 0.$$

Now argue by contradiction and assume that c, as defined in (8.4.16), is a regular value. By Lemma 8.6 there exists an $\epsilon > 0$ such that $[c-\epsilon, c+\epsilon]$ is an interval of regular values. By Lemma 8.5 there exists a $\delta > 0$ such that $\|\nabla J(\mathbf{u})\|_{E^r(D)} \geq \delta > 0$. Consequently,

$$\dot{J}(\varphi(t,\mathbf{u})) = -\omega(\varphi(t,\mathbf{u}))\frac{\|\nabla J(\varphi(t,\mathbf{u}))\|^2_{E^r(D)}}{1+\|\nabla J(\varphi(t,\mathbf{u}))\|_{E^r(D)}} \leq -\frac{\delta^2}{1+\delta} =: \delta',$$

which implies the estimate $J\big(\varphi(t,\mathbf{u})\big) \leq J(\mathbf{u}) - \delta' t$, $t \geq 0$.

By definition of c we can choose t_0 such that $\sup_{\mathbf{u}\in Q} J\big(\varphi(t_0,\mathbf{u})\big) < c+\epsilon$. For $t_1 = 2\epsilon/\delta'$ we have

$$\begin{aligned} J\big(\varphi(t_0+t_1,\mathbf{u})\big) &= J\big(\varphi(t_1,\varphi(t_0,\mathbf{u}))\big) \\ &\leq J\big(\varphi(t_0,\mathbf{u})\big) - \delta' t_1^2 \leq c+\epsilon-\delta' t_1 < c-\epsilon. \end{aligned}$$

In particular, if $\mathbf{u} \in Q$ the value of J can drop below $c-\epsilon$ under the deformation of φ. This implies that $c \leq c-\epsilon$, which is a contradiction.

By definition $0 < \alpha \leq \sup_{\mathbf{u}\in Q} J\big(\varphi(t,\mathbf{u})\big) < \infty$, which proves that $c = \inf_{t\geq 0}\sup_{\mathbf{u}\in Q} J\big(\varphi(t,\mathbf{u})\big) \geq \alpha > 0$. ■

The conclusion of this subsection is that J has critical values and therefore critical points.

8.16 Proposition Under Condition (8.1.3) for p and q and $\lambda = \mu = 0$. Then J has a non-trivial critical point $\mathbf{u} \in E^r(D)$.

Proof. It remains to show that the critical point established above are non-trivial, i.e. $\mathbf{u} \neq 0$. Note that $J(0) = 0$. However, $J(\mathbf{u}) = c > 0$, and therefore

$$\begin{aligned} 0 < c &= J(\mathbf{u}) - \frac{1}{2}\langle J'(\mathbf{u}), \mathbf{u}\rangle \\ &= \Big(\frac{1}{2} - \frac{1}{p+1}\Big)\int_D |u|^{p+1}dx + \Big(\frac{1}{2} - \frac{1}{q+1}\Big)\int_D |v|^{q+1}dx. \end{aligned}$$

which proves that the critical points found above are non-trivial. ■

8.4.c Weak solutions

In this subsection we explain how critical points yield weak solutions to (8.1.1).

8.17 Proposition Under Condition (8.1.3) for p and q and $\lambda = \mu = 0$. Then a critical point $\mathbf{u} = (u,v)$ of J is a solution of the Euler-Lagrange equations

$$-\Delta v = u^p \quad \text{in } \Theta^{-r}(D), \qquad -\Delta u = v^q \quad \text{in } \Theta^{r-2}(D),$$

which can be regarded as the appropriate weak formulation of (8.1.1). Moreover, both $u \neq 0$ and $v \neq 0$.

Proof. Let $\mathbf{u}$ be such a critical point. Then, for all $\mathbf{p} = (\phi, \psi) \in E^r(D)$,

$$B(\mathbf{u}, \mathbf{p}) = (L\mathbf{u}, \mathbf{p})_{E^r} = \int_D u^p \phi dx + \int_D v^q \psi dx,$$

so that, substituting $\psi = 0$, for all $\phi \in \Theta^r(D)$,

$$\int_D u^p \phi dx = ((-\Delta)^{1-r} v, \phi)_{\Theta^r} =$$

$$((-\Delta)^{r/2}(-\Delta)^{1-r} v, (-\Delta)^{r/2}\phi)_{L^2} = \langle -\Delta v, \phi \rangle.$$

Thus $-\Delta v = u^p$ in $\Theta^{-r}(D)$, and likewise, $-\Delta u = v^q$ in $\Theta^{r-2}(D)$.

From Proposition 8.16 we have that at least $u \neq 0$, or $v \neq 0$. If, say $u = 0$, then $0 = -\Delta u = v^q \neq 0$, a contraction. ∎

8.4.d Positive solutions

The consideration in the subsection above yield non-trivial weak solutions to (8.1.1). In order to prove Theorem 8.2 we need to select critical points $\mathbf{u}$ of J for which $u(x) > 0$ and $v(x) > 0$ for $x \in \Omega$. In order to find such critical points we redefine f and g to be zero on $(-\infty, 0]$, i.e.

$$f(u) = (u_+)^p, \quad g(v) = (v_+)^q,$$

where the subscript refers to taking the positive part, and is not be confused with the superscripts referring to elements in $E^{\pm}$. Applying Proposition 8.17 combined with the positivity properties of $(-\Delta)^{-1}$, the components u and v of any critical point have to be nonnegative. Thus it suffices to adapt the arguments above and in particular the construction of the set Q.

As before we take $s_1 = s_2$. This immediately implies that the quadratic part of $J(\mathbf{u})$ is nonpositive not only on the bottom, but also on the lateral boundary of $Q = Q_s$. Hence the same holds for $J(\mathbf{u})$ because $b(\mathbf{u})$ is nonnegative.

It remains to establish the nonpositivity of J on the lid of Q_s, which is the disk $D_s = \{(s\mathbf{e}^+, \mathbf{u}^-) : \|\mathbf{u}^-\|_{E^r} \leq s\}$, and as before this will be done by estimating b from below on D_s. We have the estimate

$$F(u) \geq C(u_+^\gamma - 1), \;\; G(v) \geq C(v_+^\gamma - 1),$$

with $\gamma = \min\{\frac{1}{p+1}, \frac{1}{q+1}\} > 2$. Fixing $s = 1$ we claim that there exists a constant $\delta > 0$, such that

$$\int_D (u_+^\gamma + v_+^\gamma) > \delta \quad \forall \mathbf{u} = (u, v) \in D_1. \tag{8.4.17}$$

Assuming (8.4.17) for the moment, we conclude, using homogeneity, that $b(\mathbf{u}) \geq \delta s^{\gamma} - C$ on D_s. Combining with (8.3.12), J must be negative on D_s if s is large because $\gamma > 2$. Thus the proof is complete if we prove (8.4.17).

Every $\mathbf{u}$ in D_1 is of the form $\mathbf{u} = \mathbf{e}^+ + t\mathbf{e}^- + \mathbf{u}_2^-$, with t real and $\mathbf{u}_2^- \in E^-$ perpendicular in $E^r(D)$ to $\mathbf{e}^-$, as well as in $L^2 \times L^2$. Note that $\mathbf{e}^-$ has one positive and one negative component, and that the components of $\mathbf{u}_2^-$ both change sign (unless they are identical to zero), because they are perpendicular in $L^2(D)$ to ϕ_1. This implies that the integral in (8.4.17) is always positive. Moreover, γ is subcritical, so the integral is easily seen to be continuous with respect to the weak topology in $E^r(D)$, and D_1 is weakly compact. Hence the existence of a $\delta > 0$ such that (8.4.17) holds follows.

8.5 Regularity of solutions

In this section we prove that critical points of J are classical solutions of Problem (8.1.1), cf.[32] We establish the main regularity statement for general non-linearities f and g which asymptotically grow like u^p and v^q respectively, and with p, q satisfying

$$\frac{1}{p+1} + \frac{1}{q+1} \geq \frac{n-2}{n}, \quad p,q > 1, \quad n > 2. \tag{8.5.18}$$

8.18 Proposition Suppose the functions f and g be as described above and let $0 < r < 2$ be such that (8.5.18) holds. Then every critical point of J has the property that $u, v \in L^{\alpha}(D)$ for all $1 \leq \alpha < \infty$.

Proof. We assume that $n \geq 2$. Because of (h3) we can rewrite the conclusion of Proposition 8.17

$$-\Delta v = a(x)u \ \text{ in } \ \Theta^{-r}(D), \quad -\Delta u = b(x)v \ \text{ in } \ \Theta^{r-2}(D), \tag{8.5.19}$$

where

$$a(x) = \frac{f(u(x))}{u(x)}, \quad b(x) = \frac{g(v(x))}{v(x)}.$$

By (8.5.18) $a(x) \in L^{\frac{p+1}{p-1}}$ and $b(x) \in L^{\frac{q+1}{q-1}}(D)$. Now let $\epsilon > 0$, then there exists[32] functions $q_\epsilon \in L^{\frac{p+1}{p-1}}(D)$, and $f_\epsilon \in L^{\infty}(D)$, such that

$$a(x)u(x) = q_\epsilon(x)u(x) + f_\epsilon(x), \ \text{ a.e. in } \ D, \quad \|q_\epsilon\|_{L^{\frac{p+1}{p-1}}} < \epsilon.$$

Inserting the latter into (8.5.19) we obtain

$$-\Delta v = q_\epsilon(x)u + f_\epsilon(x) \ \text{ in } \ \Theta^{-r}(D), \quad -\Delta u = b(x)v \ \text{ in } \ \Theta^{r-2}(D).$$

We denote the multiplication operators $u(x) \mapsto b(x)u(x)$ and $u(x) \mapsto q_\epsilon(x)u(x)$ by $\mathcal{B}$ and $\mathcal{Q}^\epsilon$. Then, inverting the Laplacians in and eliminating v yields

$$u = (-\Delta)^{-1}\mathcal{B}(-\Delta)^{-1}\mathcal{Q}^\epsilon u + (-\Delta)^{-1}\mathcal{B}(-\Delta)^{-1} f_\epsilon,$$

or, equivalently, $(\mathrm{Id} - K^\epsilon)u = h_\epsilon$, where

$$K^\epsilon = (-\Delta)^{-1}\mathcal{B}(-\Delta)^{-1}\mathcal{Q}^\epsilon, \quad h_\epsilon = (-\Delta)^{-1}\mathcal{B}(-\Delta)^{-1} f_\epsilon. \tag{8.5.20}$$

Thus the proof will be complete if we show for all large $\alpha < \infty$ that h_ϵ is in L^α, and that the operator $\mathrm{Id} - K^\epsilon : L^\alpha \to L^\alpha$ is invertible. The latter will be achieved by a bound of order ϵ on the operator norm of K^ϵ.

We observe that, as a consequence of the Hardy-Littlewood-Sobolev inequality, the operator $(-\Delta)^{-1}$ is bounded from L^{α_1} to L^{α_2} with

$$\frac{1}{\alpha_2} = \frac{1}{\alpha_1} - \frac{2}{n}, \quad 1 < \alpha_1 < \alpha_2 < \infty.$$

Also, because of Hölder's inequality, the multiplication operator $\mathcal{C}$ corresponding to a function $c(x) \in L^\sigma$ is bounded from L^{β_1} to L^{β_2} with

$$\frac{1}{\beta_2} = \frac{1}{\beta_1} + \frac{1}{\sigma}, \quad 1 \le s, \beta_1, \beta_2 \le \infty, \tag{8.5.21}$$

and its operator norm is equal to $\|c\|_{L^\sigma}$. Note that s and β_1 have to be sufficiently large in order to keep β_2 larger then one, which is needed, because in (8.5.20) each multiplication operator is succeeded by $(-\Delta)^{-1}$. Combining these two results we find that K^ϵ is bounded from L^α to L^β with

$$\frac{1}{\beta} = \frac{1}{\alpha} + \frac{p-1}{p+1} - \frac{2}{n} + \frac{q-1}{q+1} - \frac{2}{n} = \frac{1}{\alpha} + 2\Big[\frac{n-2}{n} - \frac{1}{p+1} - \frac{1}{q+1}\Big],$$

and that, denoting the two constants appearing in the two Hardy-Littlewood-Sobolev inequalities we apply by C_1, C_2,

$$\begin{aligned} \|K^\epsilon u\|_{L^\beta} &\le C_1 \|b\|_{L^{\frac{q+1}{q-1}}} C_2 \|q_\epsilon(x)\|_{L^{\frac{p+1}{p-1}}} \|u\|_{L^\alpha} \\ &\le C_1 \|b\|_{L^{\frac{q+1}{q-1}}} C_2 \epsilon \|u\|_{L^\alpha}. \end{aligned}$$

Here we have to be careful, because we use Equation (8.5.21) two times, first with $\beta_1 = \alpha$ and $\sigma = \frac{p+1}{p-1}$ for the operator $\mathcal{Q}^\epsilon$, and then for the operator $\mathcal{B}$ with

$$\frac{1}{\beta_1} = \frac{1}{\alpha} + \frac{p-1}{p+1} - \frac{2}{N} \text{ and } s = \frac{q+1}{q-1}.$$

Thus we have to choose α so large that $\frac{1}{\alpha} + \frac{p-1}{p+1} < 1$, and

$$\frac{1}{\alpha} + \frac{p-1}{p+1} - \frac{2}{n} + \frac{q-1}{q+1} = \frac{1}{\alpha} + 2\Big[\frac{n-1}{n} - \frac{1}{p+1} - \frac{1}{q+1}\Big] < 1.$$

This is possible because p and q are (sub)critical in the sense of (8.1.3). It follows that $\beta > \alpha$, and that $\beta = \alpha$ if $n > 2$ and p and q are critical. In the latter case the arguments above are valid for all $p+1 \leq \alpha < \infty$. Moreover, because of (3.12), the norm of K^ϵ can be made arbitrarily small by choosing ϵ small, so that $\mathrm{Id} - K^\epsilon : L^\alpha \to L^\alpha$ is invertible for all large $\alpha < \infty$. Along the same lines one has h_ϵ in every L^α with $1 < \alpha < \infty$. Thus we conclude from $(\mathrm{Id} - K^\epsilon)u = h_\epsilon$ that $u \in L^\alpha$ for all $1 \leq \alpha < \infty$. For v the argument is similar. ∎

8.19 Remark Note that it is only at this stage that we know that both u and v satisfy the boundary conditions in the sense that $u, v \in H_0^1(D)$. ∎

8.20 Corollary Critical points of J are classical solutions of Problem (8.1.1) and in Theorem 8.2 "nonnegative" can be replaced by "strictly positive".

Proof. This is now standard. One has the L^p-estimates for the second derivatives due to Agmon, Douglis & Nirenberg[1], so that Sobolev embeddings and Schauder estimates[12] finish the proof. ∎

Combining Proposition 8.16 and Corollary 8.20 completes the proof of Theorem 8.2 in the case $\lambda = \mu = 0$.

8.6 Nonlinear Eigenvalues problems

In this section we complete the proof of Theorem 8.2 by considering $\lambda \neq$ and $\mu \neq 0$. The quadratic part of J is now given by

$$A_*(\mathbf{u}) = \frac{1}{2}(L_*\mathbf{u}, \mathbf{u})_{E^r(D)} = \int_D \nabla u \nabla v - \frac{\lambda}{2}\int_D u^2 - \frac{\mu}{2}\int_D v^2,$$

where now

$$L_* = \begin{pmatrix} -\lambda(-\Delta)^{-r} & (-\Delta)^{1-r} \\ (-\Delta)^{r-1} & -\mu(-\Delta)^{r-2} \end{pmatrix}$$

is bounded and selfadjoint. Unlike L, L_* is not an isometry.

In order to determine the spectrum of L_*, we note that $E^r(D)$ is the direct Hilbertspace sum of the spaces E_k, $k = 1, 2, \ldots$, where E_k is the two-dimensional subspace of $E^r(D)$, spanned by $(\phi_k, 0)$ and $(0, \phi_k)$. An orthonormal basis of E_k is given by

$$\Big\{\frac{1}{\sqrt{2}}(\lambda_k^{-\frac{r}{2}}\phi_k, 0), \frac{1}{\sqrt{2}}(0, \lambda_k^{\frac{r}{2}-1}\phi_k)\Big\}.$$

Every E_k is invariant under L_*, and in E_k the restriction of L_* is given by the symmetric matrix

$$L^k = \begin{pmatrix} -\lambda\lambda_k^{-r} & 1 \\ 1 & -\mu\lambda_k^{r-2} \end{pmatrix}$$

The eigenvalues of L^k are given by

$$\mu_k^{\pm} = -\frac{\lambda\lambda_k^{-r} + \mu\lambda_k^{r-2}}{2} \pm \sqrt{(\frac{\lambda\lambda_k^{-r} + \mu\lambda_k^{r-2}}{2})^2 + 1 - \frac{\lambda\mu}{\lambda_k^2}},$$

with corresponding eigenvectors

$$\left(1, \frac{\lambda\lambda_k^{-r} - \mu\lambda_k^{r-2}}{2} \pm \sqrt{1 + (\frac{\lambda\lambda_k^{-r} - \mu\lambda_k^{r-2}}{2})^2}\right).$$

We have $\mu_k^- < 0 < \mu_k^+$ if $\lambda\mu < \lambda_k^2$. If $\lambda\mu > \lambda_k^2$ the signs of μ_k^+ and μ_k^- are the same: positive (negative) if λ and μ are negative (positive). If $\lambda\mu = \lambda_k^2$, then $\mu_k^+ = 0$ ($\mu_k^- = 0$) if λ and μ are positive (negative). Also note that $\mu_k^{\pm} \to \pm 1$ as $k \to \infty$.

Let E^+ (E^-) be the subspace spanned by eigenvectors with positive (negative) eigenvalues, and E^0 the nullspace of L_*. Then

$$E^r(D) = E^+ \oplus E^0 \oplus E^- = \{\mathbf{u} = \mathbf{u}^+ + \mathbf{u}^0 + \mathbf{u}^-,\ \mathbf{u}^{\pm} \in E^{\pm},\ \mathbf{u}^0 \in E^0\}.$$

It follows that both E^+ and E^- are infinite dimensional, and that E^0 has finite dimension: $\lambda\mu \neq \lambda_k^2$ implies $\dim E^0 = 0$ while for $\lambda\mu = \lambda_k^2$ the dimension of E^0 is equal to the multiplicity of λ_k.

We introduce a equivalent (inner product) norm $\|\cdot\|_*$ on $E^r(D)$ by

$$(L_*\mathbf{u}^+,\mathbf{u}^+) - (L_*\mathbf{u}^-,\mathbf{u}^-) + \|\mathbf{u}^0\|^2_{L^2\times L^2} = \frac{1}{2}\|\mathbf{u}\|_*^2,$$

The equivalence of $\|\cdot\|_*$ and $\|\cdot\|_{E^r}$ follows from $\mu_k^{\pm} \to \pm 1$ as $k \to \infty$. and the fact that E^0 is finite dimensional.

8.21 Proposition Under Condition (8.1.3) for p and q and $\lambda \geq 0$, $\mu \geq 0$ and $\lambda\mu < \lambda_1^2$. Then J has a non-trivial critical point $\mathbf{u} \in E^r(D)$.

Proof. We adjust the proof of Proposition 8.16. If $\lambda\mu \neq \lambda_k^2$ for all k we have $\dim E^0 = 0$, and (4.9) reduces to

$$(L_*\mathbf{u}^+,\mathbf{u}^+) - (L_*\mathbf{u}^-,\mathbf{u}^-) = \frac{1}{2}\|\mathbf{u}\|_*^2.$$

We take for $\mathbf{e}^+$ an eigenvector in E^+, such that $\mathbf{e}^+$ belongs to some E_k with the other eigenvector $\mathbf{e}^-$ in E_k belonging to E^- ($\mathbf{e}^+$ and $\mathbf{e}^-$ normalized with respect to $\|\cdot\|_*$). Note that $\mathbf{e}^-$ is the only eigenvector of L_* not perpendicular to $\mathbf{e}^+$ in $L^2(D) \times L^2(D)$. Using $\|\cdot\|_*$ instead of $\|\cdot\|_{E^r}$, the proof is then identical to the proof of Proposition 8.16.

If $\lambda\mu = \lambda_k^2$ for some k, the proof is slightly more complicated. We replace again $\|\cdot\|_{E^r}$ by $\|\cdot\|_*$, choose $\mathbf{e}^{\pm}$ as above, and set $H_1 = E^+$, $H_2 = E^0 \oplus E^-$, and

$$Q = [0, s_1\mathbf{e}^+] \oplus \left(\overline{B}_{s_2} \cap H_2\right), \quad \tilde{H} = \text{span}\,[\mathbf{e}^+] \oplus H_2, \quad S = \partial B_\rho \cap H_1.$$

Elements of $H_1 = E^+$ are denoted by $\mathbf{u}^+$, and elements of $H_2 = E^0 \oplus E^-$ by $\mathbf{u}^0 + \mathbf{u}^-$.

To verify the geometric conditions in Step 1, we have to estimate

$$J_*(\mathbf{u}^0 + \mathbf{u}^- + s\mathbf{e}^+) = \frac{1}{2}s^2 - \frac{1}{2}\|\mathbf{u}^-\|_*^2 - b(\mathbf{u}^0 + \mathbf{u}^- + s\mathbf{e}^+)$$

on the boundary of the cylinder Q. The lateral boundary however is no longer given by $\|\mathbf{u}^-\|_* = s_2$ but by $\|\mathbf{u}^0 + \mathbf{u}^-\|_* = s_2$. Thus if s_2 is large, the norm $\|\mathbf{u}^-\|_*$ can still be small on the lateral boundary, provided $\|\mathbf{u}^0\|_*$ is large. Estimate $b(\mathbf{u}^0 + \mathbf{u}^- + s\mathbf{e}^+)$ from below

$$\begin{aligned} b(\mathbf{u}^0 + \mathbf{u}^- + s\mathbf{e}^+) &\geq B_3\Big(\int_D |\mathbf{u}^- + \mathbf{u}^0 + s\mathbf{e}^+|^2\Big)^{\gamma/2} - B_4 \\ &\geq B_3\Big(\sin^2\chi \int_D |\mathbf{u}^0 + s\mathbf{e}^+|^2\Big)^{\gamma/2} - B_4 \\ &= B_3 \sin^\gamma \chi \big(\|\mathbf{u}^0\|_*^2 + s^2\big)^{\frac{\gamma}{2}} - B_4 \\ &\geq B_5\|\mathbf{u}^0\|_*^2 + B_5 s^\gamma - B_6. \end{aligned}$$

Here χ is the (positive) angle between E^- and $E^0 \oplus [\mathbf{e}^+]$ with respect to the inner product in $\mathcal{L} \times \mathcal{L}$. The analogue of (2.9) for J_* is

$$\begin{aligned} J_*(\mathbf{u}^0 + \mathbf{u}^- + s\mathbf{e}^+) &\leq \frac{1}{2}s^2 - B_5 s^\gamma + B_6 - B_7\big(\|\mathbf{u}^-\|_*^2 + \|\mathbf{u}^0\|_*^2\big) \\ &= \frac{1}{2}s^2 - B_5 s^\gamma + B_6 - B_7\|\mathbf{u}^- + \mathbf{u}^0\|_*^2. \end{aligned}$$

The proof now proceeds along the same lines as before.

It remains to show that J_* satisfies the Palais-Smale condition. So let $\{\mathbf{u}_n\}$ in $E^r(D)$ be a sequence with $J_*(\mathbf{u}_n)$ bounded in $E^r(D)$, and $J_*'(\mathbf{u}_n) \to 0$. We have to do some extra work to show that such a sequence is bounded. Using the decomposition $E^r(D) = E^+ \oplus E^0 \oplus E^-$ the estimates (8.3.12) and (8.3.13) $\mathbf{u}^\pm$ remain the same. We have

$$\|\mathbf{u}^\pm\|_* \leq C\Big\{1 + \{C + \epsilon\|\mathbf{u}\|_*\}^{\frac{p}{p+1}} + \{C + \epsilon\|\mathbf{u}\|_*\}^{\frac{q}{q+1}}\Big\}.$$

To controle the component $\mathbf{u}^0$ we modify (2.12) and derive

$$\begin{aligned} M + \epsilon\|\mathbf{u}\|_{E^r(D)} &\geq \int_D \big(\frac{1}{2}uf(u) - F(u) + \frac{1}{2}vg(v) - G(v)\big)dx \\ &\geq \big(\frac{\gamma}{2} - 1\big)b(\mathbf{u}) \geq B_1 \int_D |\mathbf{u}|^\gamma - B_2|D| \\ &\geq B_3\Big(\int_D |\mathbf{u}^- + \mathbf{u}^0 + \mathbf{u}^+|^2\Big)^{\gamma/2} - B_4 \\ &\geq B_3\Big(\sin^2\chi \int_D |\mathbf{u}^0|^2\Big)^{\gamma/2} - B_4 \\ &= B_5\|\mathbf{u}^0\|_*^\gamma - B_4, \end{aligned}$$

where χ is the angle between E^0 and $E^+ \oplus E^-$ in $L^2(D) \times L^2(D)$. Combining (4.15) and (4.16) we obtain

$$\|\mathbf{u}\|_* \le C\Big\{1 + \{C + \epsilon\|\mathbf{u}\|_*\}^{\frac{p}{p+1}} + \{C + \epsilon\|\mathbf{u}\|_*\}^{\frac{q}{q+1}} + \{C + \epsilon\|\mathbf{u}\|_*\}^{\frac{1}{\gamma}}\Big\},$$

and as before this implies that the sequence $\mathbf{u}_n$ is bounded. This completes the proof. ∎

With Proposition 8.21 we can use the adjustment in 8.4.d to obtain positive solutions. This finally allows us to prove Theorem 8.2

Proof of Theorem 8.2. We adjust the arguments in 8.4.d which are based on the maximum principle, and on the existence of an eigenvector $\mathbf{e}^+$ with positive components. For (8.1.1) the maximum principle holds provided λ and μ are nonnegative with $\lambda\mu < \lambda_1^2$.[8] With respect to $\mathbf{e}^+$ we note that in the proofs of Proposition 8.16 and Proposition 8.21 the choice of $\mathbf{e}^+$ in E^+ was rather arbitrary. In 8.4.d however the positivity properties of $\mathbf{e}^\pm$ were crucial, so adjusting this proof to the case of J_* we need an eigenvector with positive components. Such an eigenvector can only exist in E_1, so we look at the eigenvalues and eigenvectors of L_*^1. Thus we must take $k = 1$ and a plus sign in the expression for $\mu_k^\pm$. To make μ_1^+ positive it is necessary and sufficient to assume that $\lambda\mu < \lambda_1^2$. The corresponding eigenvector has positive components, so we can choose $\mathbf{e}^+$ as desired. Since λ and μ are nonnegative we can use $\|\cdot\|_*$. The proof is then a trivial variant of the proof 8.4.d. ∎

8.7 Problems

8.22 Problem Prove the Pohozaev identity given in Sect. 8.1.

8.23 Problem Let $\mathbf{e}^\pm$ as defined in Sect. 8.2. Show that

$$\|t\mathbf{e}^- + s\mathbf{e}^+\|_{L^2\times L^2} \ge |s|\sin(\chi)\|\mathbf{e}^+\|_{L^2\times L^2},$$

where χ is the $L^2 \times L^2$-angle between $\mathbf{e}^-$ and $\mathbf{e}^+$.

9 — Discrete Parabolic Dynamics

This chapter entails an application of Conley Theory to an important class of dynamical systems; parabolic recurrence relations. We construct a Morse-type theory on certain spaces of braid diagrams. A topological invariant of closed positive braids is defined and is correlated with the existence of invariant sets of *parabolic flows* defined on discretized braid spaces via parabolic recurrence relations. Parabolic flows, a type of one-dimensional lattice dynamics, evolve singular braid diagrams in such a way as to decrease their topological complexity; algebraic lengths decrease monotonically. This topological invariant is derived from the homological Conley index. This culminates in very general forcing theorems for the existence of infinitely many braid classes.

9.1 Parabolic recurrence relations

We start with a class of discrete dynamical systems defined by recurrence relations with next-neighbor coupling. These will be referred to as *parabolic recurrence relations*.

9.1 Definition On the space of sequences $\mathbb{R}^{\mathbb{Z}}$, define a sequence of C^∞-functions $\{R_i\}_{i\in\mathbb{Z}}$, $R_i : \mathbb{R}^3 \to \mathbb{R}$, which satisfy

(i) $\partial_1 R_i > 0$ and $\partial_3 R_i > 0$;

(ii) $R_{i+d} = R_i$, for some $d \in \mathbb{N}$.

A parabolic recurrence relation is given by the equation

$$R_i(x_{i-1}, x_i, x_{i+1}) = 0, \tag{9.1.1}$$

with $\{x_i\}_{i\in\mathbb{Z}} \in \mathbb{R}^{\mathbb{Z}}$ and $\{R_i\}$ satisfying (i) and (ii).

A parabolic recurrence relation with the periodicity condition (ii) define finite iterative system of mappings. Let $z = (u,v)$, then solve w_i from $R_i(u_i,v_i,w_i) = 0$, which gives $w_i = g_i(u_i,v_i)$. Define the mappings $F_i(u,v) = \big(v, g_i(u,v)\big)$. The iterative system $z_{i+1} = F_i(z_i)$ defines a discrete dynamical system and since Since $\frac{\partial u_{i+1}}{\partial v_i} > 0$, these are twist maps. Parabolic recurrence relations occurs is various applications in dynamical systems, as well as in physical models. We are interest in fixed points and periodic points of $\{F_i\}$, i.e. $z = \widehat{F}(z)$, where $\widehat{F} = F_{d-1} \circ \cdots \circ F_0$, or $z = \widehat{F}^n(z)$. In terms of the parabolic recurrence relation this means sequence $\{x_i\}_{i\in\mathbb{Z}}$, with $x_{i+d} = x_i$ and $\{x_i\}$ satisfies (9.1.1).

The space of d-periodic sequences in $\mathbb{R}^{\mathbb{Z}}$ will be definoted by Ω_d and the smooth mapping

$$\Omega_d \to \mathbb{R}^d, \qquad \{x_i\} \mapsto \boldsymbol{x} = (x_0, \cdots, x_{d-1}),$$

yields global coordinates on Ω_d, which makes Ω_d a smooth manifold. We may represent d-periodic sequences in Ω_d by piece wise linear interpolations

$$\beta(\boldsymbol{x})(s) = x_{\lfloor d\cdot s\rfloor} + (d\cdot s - \lfloor d\cdot s\rfloor)(x_{\lceil d\cdot s\rceil} - x_{\lfloor d\cdot s\rfloor}),$$

for $s \in [0,1]$. We identify Ω_d with the space of piecewise linear d-functions as defined above. In principle, connecting the *anchor points* x_i with line segment enables us to distinguish between different sequences geometrically, i.e. sequences $\boldsymbol{x}$ and $\boldsymbol{x}'$ with different coordinates represents different piecewise linear functions $\beta(\boldsymbol{x})$ and $\beta(\boldsymbol{y})$. We will show now that periodic sequences $\boldsymbol{x}$ that occur as solutions of parabolic recurrence relations has special geometric properties with respect to their piecewise linear interpolations $\beta(\boldsymbol{x})$.

9.2 Definition Two sequences $\boldsymbol{x},\boldsymbol{y} \in \Omega_d$, with $\boldsymbol{x} \neq \boldsymbol{y}$, are said to be *transverse* if

$$(x_{i-1} - y_{i-1})(x_{i+1} - y_{i+1}) < 0,$$

for every index i for which $x_i = y_i$. We write $\boldsymbol{x} \pitchfork \boldsymbol{y}$. Transversality is a local property and therefore the same applies to sequences $\boldsymbol{x},\boldsymbol{y} \in \mathbb{R}^{\mathbb{Z}}$.

Transversality is a natural property of solutions of parabolic recurrence relations as the following lemma shows.

9.3 Lemma Let $\boldsymbol{x},\boldsymbol{y} \in \Omega_d$, with $\boldsymbol{x} \neq \boldsymbol{y}$, be solutions of Equation (9.1.1), then $\boldsymbol{x} \pitchfork \boldsymbol{y}$.

Proof. Consider the case where $\boldsymbol{x}$ and $\boldsymbol{y}$ coincide for at least one coordinate. Since $\boldsymbol{x} \neq \boldsymbol{y}$, one can choose an index i such that $x_i = y_i$ and $x_{i+1} > y_{i+1}$ (the reverse inequality is the same). Suppose $x_{i+1} \geq y_{i+1}$, then, since $\partial_3 R_i > 0$, it follows that $0 = R_i(x_{i-1}, x_i, x_{i+1}) > R_i(y_{i-1}, y_i, y_{i+1}) = 0$, which is a contradiction. Therefore,

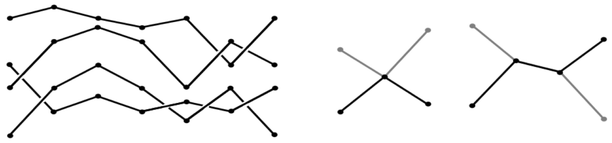


Figure 9.1: Linking sequences [left] and non-transverse intersections, or tangencies [right].

$(x_{i-1}-y_{i-1})(x_{i+1}-y_{i+1})<0$. This implies that indices for which $x_i=y_i$ are isolated. ■

Graphically two sequences $\boldsymbol{x}$ and $\boldsymbol{y}$ are represented by intersecting graphs $\beta(\boldsymbol{x})$ and $\beta(\boldsymbol{y})$ and as such transverse sequences correspond to transverse graphs. Intersections can be counted and leads to the notion of linking, see Fig. 9.1.

9.4 Definition The linking number of two transverse sequences $\boldsymbol{x}\pitchfork\boldsymbol{y}$ the linking number is defined as

$$\operatorname{link}(\boldsymbol{x},\boldsymbol{y})=\#\{\text{intersections of } \beta(\boldsymbol{x}) \text{ and } \beta(\boldsymbol{y})\},$$

which is an even integer.

The linking number $\operatorname{link}(\boldsymbol{x},\boldsymbol{y})$ is also equal to two times the number of connected components of the set $\{s \mid (\beta(\boldsymbol{x})-\beta(\boldsymbol{y})(s)>0\}$. The linking number is a local invariant, i.e. the linking number does not change under small perturbations of the sequences $\boldsymbol{x}$ and $\boldsymbol{y}$.

For two transverse sequences an occurrence $x_i=y_i$ is called an intersection at an anchor point. However, if $\boldsymbol{x}$ and $\boldsymbol{y}$ are non-transverse sequences, then $x_i=y_i$ and $(x_{i-1}-y_{i-1})(x_{i+1}-y_{i+1})\geq 0$ for some i, and such a an occurrence is called a tangency, see Fig. 9.1.

9.2 Parabolic flows and braids

In order to study solutions of Equation (9.1.1) we embed Equation (9.1.1) in a canonical dynamical system. Parabolic recurrence relations may be regarded as vector field in a suitable way. Since R_i is d-periodic, $R=(R_0,\cdots,R_{d-1})\in T_{\boldsymbol{x}}\Omega_d$ defines a vector field on Ω_d. We integrate the vector field via the equation

$$x_i'=R_i(x_{i-1},x_i,x_{i+1}), \tag{9.2.2}$$

which generates a local C^∞-flow φ on Ω_d. Such a flow φ is called a *parabolic flow* on Ω_d. Of course a vector field R only provide a local flow in gerenal and a global flow may be obtained by considering for example $\frac{R}{1+|R|^2}$. However, since we will be studying blocks it does not matter whether φ is a local or global flow. The following lemma motivates why linking of periodic sequences is naturally related to parabolic flows.

9.5 Lemma Let $\boldsymbol{x},\boldsymbol{y} \in \Omega_d$, with $\boldsymbol{x} \neq \boldsymbol{y}$ and $\boldsymbol{x} \not\pitchfork \boldsymbol{y}$, then there exists an $\epsilon > 0$ such that $\varphi(t,\boldsymbol{x}) \pitchfork \varphi(t,\boldsymbol{y})$ for all $t \in [-\epsilon,\epsilon] \setminus \{0\}$ and

$$\mathrm{link}\big(\varphi(-t,\boldsymbol{x}),\varphi(-t,\boldsymbol{y})\big) > \mathrm{link}\big(\varphi(t,\boldsymbol{x}),\varphi(t,\boldsymbol{y})\big),$$

for all $t \in (0,\epsilon]$.

Proof. Under construction. ∎

The behavior of parabolic flows with respect to transversality motivates the following definition.

9.6 Definition An unordered set of sequences $\boldsymbol{x} = \{\boldsymbol{x}^1,\cdots,\boldsymbol{x}^n\}$, is called a *discrete braid* on n strands if the following conditions are satisfied:

(i) $\boldsymbol{x}^\alpha = (x_0^\alpha,\cdots,x_d^\alpha) \in \mathbb{R}^{d+1}$, for all $\alpha = 1,\cdots,n$;

(ii) there exists a permutation $\sigma \in S_n$ such that $x_d^\alpha = x_0^{\sigma(\alpha)}$, for all $\alpha = 1,\cdots,n$;

(iii) $\boldsymbol{x}^\alpha \pitchfork \boldsymbol{x}^{\alpha'}$ for all pairs $\alpha \neq \alpha'$.

The piecewise linear interpolation $\beta(\boldsymbol{x}) = \{\beta(\boldsymbol{x}^\alpha)\}$ defined a *braid diagram* with positive crossings.

Fig. 9.1 gives an example of a positive piecewise linear braid diagram and therefore a discrete braid. Sometimes a braid is denoted by $(\boldsymbol{x},\sigma)$. Two braids (x,σ) and $(\boldsymbol{x}',\sigma')$ are close if there exists a permutation $\pi \in S_n$ such that $\boldsymbol{x}^{\pi(\alpha)}$ and $\boldsymbol{x}'^{\alpha}$ are close in $\mathbb{R}^{d+1}$ for all α and $\pi \circ \sigma' = \sigma \circ \pi$. The space of all discrete braids on n strands, topologized as above, is denoted by Ω_d^n. The space of braids Ω_d^n is a metric space. The permutation is often omitted from the notation. Two braids $\boldsymbol{x}$ and $\boldsymbol{x}'$ are homotopic, or equivalent, $\boldsymbol{x} \sim \boldsymbol{x}'$, iof there exists a continuous path $\boldsymbol{x}(t)$ in Ω_d^n, such that $\boldsymbol{x}(0) = \boldsymbol{x}$ and $\boldsymbol{x}(1) = \boldsymbol{x}'$. The path components of Ω_d^n are called *braid classes* and are denoted by $[\boldsymbol{x}] \subset \Omega_d^n$. Define $\bar{\Omega}_d^n$ as the space of unordered sets $\boldsymbol{x} = \{\boldsymbol{x}^1,\cdots,\boldsymbol{x}^n\}$, satisfying (i) and (ii) of Definition 9.6. The set $\Sigma_d^n := \bar{\Omega}_d^n \setminus \Omega_d^n$ is called the set of *singular braids*. The set Ω_d^n is a complete metric space and $\Omega_d^n \subset \bar{\Omega}_d^n$ is open in $\bar{\Omega}_d^n$. A special subset of singularities is given by

$$\Sigma_d^{n,-} = \big\{\boldsymbol{x} \in \Sigma_d^n \mid x_i^\alpha = x_i^{\alpha'},\ \text{for some } \alpha \neq \alpha'\big\}.$$

For a discrete braid $\boldsymbol{x} \in \Omega_d^n$ we can define the word-length, or word-metric

$$\ell(\boldsymbol{x}) = \#\{\text{intersections in } \beta(\boldsymbol{x})\}.$$

Obviously the word-length is well-defined and can be related to the linking number of two strands. Let $\widehat{\boldsymbol{x}} \in \Omega_{nd}^n$ be the n-fold extension of $\boldsymbol{x}$, i.e. $\widehat{x}_i^\alpha = x_i^\alpha$ for $i = 1,\cdots,d$, $\widehat{x}_i^\alpha = x_{i-d}^{\sigma(\alpha)}$ for $i = d,\cdots,2d$, $\widehat{x}_i^\alpha = x_{i-2d}^{\sigma^2(\alpha)}$ for $i = 2d,\cdots,3d$, etc. Then counting of mutual intersections gives:

$$\ell(\boldsymbol{x}) = \frac{1}{2n} \sum_{\alpha \neq \alpha'} \mathrm{link}(\widehat{\boldsymbol{x}}^\alpha,\widehat{\boldsymbol{x}}^{\alpha'}). \tag{9.2.3}$$

The parabolic equation in (9.2.2) defines a (local) flow Ψ on $\bar{\Omega}_d^n$. Since a parabolic recurrence relation is d-periodic in i we define the flow Ψ as follows: $\widehat{\Psi(t,\boldsymbol{x})} = \Psi(t,\widehat{\boldsymbol{x}})$.

9.7 Exercise Show, using the periodicity of R_i with respect to i, that Ψ is well-defined on the space $\bar{\Omega}_d^n$. ▪

From Lemma 9.5 we derive how the word-length behaves with respect to parabolic flows on $\bar{\Omega}_d^n$, see Fig. 9.2.

9.8 Lemma Let $\boldsymbol{x} \in \Sigma_d^n \setminus \Sigma_d^{n,-}$, then there exists an $\epsilon > 0$ such that $\Psi(t,\boldsymbol{x}) \in \Omega_d^n$ for all $t \in [-\epsilon,\epsilon] \setminus \{0\}$ and

$$\ell\big(\Psi(-t,\boldsymbol{x})\big) > \ell\big(\Psi(t,\boldsymbol{x})\big),$$

for all $t \in (0,\epsilon]$.

Proof. By definition $\widehat{\Psi(t,\boldsymbol{x})} = \Psi(t,\widehat{\boldsymbol{x}})$ and if $\boldsymbol{x} \in \Sigma_d^n \setminus \Sigma_d^{n,-}$, then $\widehat{\boldsymbol{x}} \in \Sigma_{nd}^n \setminus \Sigma_{nd}^{n,-}$. If $\widehat{\boldsymbol{x}} \in \Sigma_{nd}^n \setminus \Sigma_{nd}^{n,-}$, then at least two strands are not transverse and none are collapsed. Then by Equation (9.2.3) and Lemma 9.5 it follows that $\ell\big(\Psi(-t,\boldsymbol{x})\big) > \ell\big(\Psi(t,\boldsymbol{x})\big)$. Since this holds for any non-tranverse strands Lemma 9.5 also implies that $\Psi(t,\boldsymbol{x}) \in \Omega_d^n$ for all $t \in [-\epsilon,\epsilon] \setminus \{0\}$. ∎

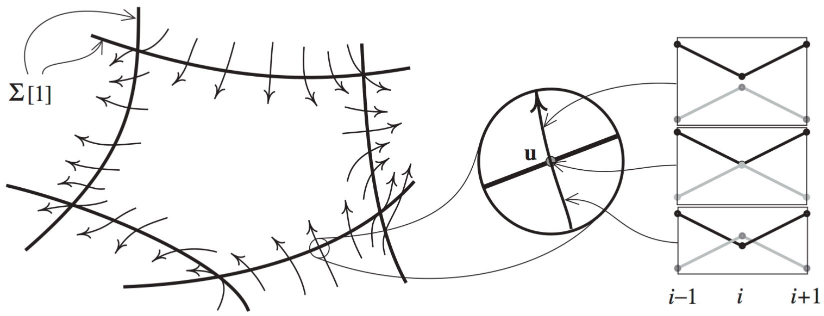


Figure 9.2: The behavior of parabolic flows with respect to intersections and braid classes.

Define Ω_d^n rel Ω_d^m be the space of ordered pairs $(\boldsymbol{x},\boldsymbol{y}) \in \Omega_d^n \times \Omega_d^m$ such that $\{\boldsymbol{x}^1,\cdots \boldsymbol{x}^n,\boldsymbol{y}^1,\cdots,\boldsymbol{y}^m\} \in \Omega_d^{n+m}$. The path components of Ω_d^n rel Ω_d^m are denoted by $[\boldsymbol{x} \text{ rel } \boldsymbol{y}]$ and are called *relative braid classes*. Define the projection $\pi : \Omega_d^n \text{ rel } \Omega_d^m \to \Omega_d^m$ by $(\boldsymbol{x},\boldsymbol{y}) \mapsto \boldsymbol{y}$ and for given $\boldsymbol{y} \in \Omega_d^m$, called the *skeleton*, the set $\pi^{-1}(\boldsymbol{y}) = [\boldsymbol{x}]$ rel $\boldsymbol{y}$ is called a *relative braid class fiber* and can be canonically embedded into Ω_d^n. We will always identify $[\boldsymbol{x}]$ rel $\boldsymbol{y}$ with a subset of Ω_d^n!

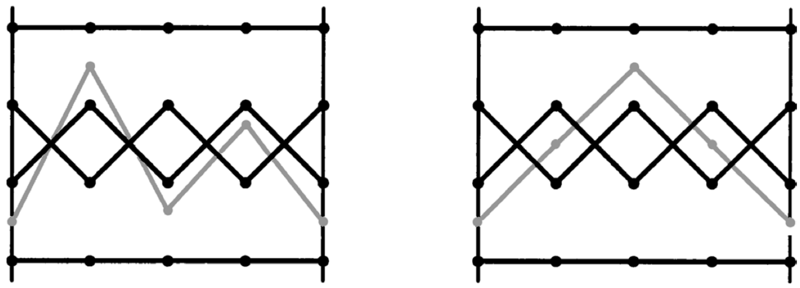


Figure 9.3: An improper class [left] and a proper class [right].

9.9 Definition A relative braid class $[\boldsymbol{x} \text{ rel } \boldsymbol{y}] \in \Omega_d^n \text{ rel } \Omega_d^m$ is *bounded* if every fiber $[\boldsymbol{x}] \text{ rel } \boldsymbol{y}$ is a bounded set in Ω_d^n. A relative braid class is *proper* if

$$\overline{\pi^{-1}(\boldsymbol{y}')} \cap \Sigma_d^{n+m,-} = \varnothing,$$

for every fiber $\pi^{-1}(\boldsymbol{y}') = [\boldsymbol{x}'] \text{ rel } \boldsymbol{y}' \subset [\boldsymbol{x} \text{ rel } \boldsymbol{y}]$. If a relative braid classes is not proper it is called *improper*.

Intuitively, proper classes have the property that strands in $\boldsymbol{x}$ cannot be collapsed onto strands in $\boldsymbol{y}$, nor can strands in $\boldsymbol{x}$ reduce to a coarser braid with fewer than n strands, see Fig. 9.2.

9.3 Isolating blocks and braid classes

We will now relate to concept of relative braid classes to parabolic flows. Suppose that $\boldsymbol{y} \in \Omega_d^m$ is braid consisting of solutions of Equation 9.1.1. To be more precise, $\widehat{\boldsymbol{y}} = \{\widehat{\boldsymbol{y}}^\alpha\}$, with $\widehat{\boldsymbol{y}}^\alpha \in \Omega_{nd}$, is a set of nd-periodic solutions of Equation (9.1.1). Conversely, if we have a k-periodic solutions $\widehat{\boldsymbol{y}} \in \Omega_k$, with $m = k/d \in \mathbb{N}$, then the translate $\{\boldsymbol{y}^\alpha\}$ over the indices $i = 1, \cdots, d$, represents an element $\boldsymbol{y} = \{\boldsymbol{y}^\alpha\} \in \bar{\Omega}_d^m$. If the period k is minimal, then $\boldsymbol{y} \in \Omega_d^m$ due to Lemma 9.3. If we have more than one periodic solution, the union yields a braid $\boldsymbol{y}$ by Lemma 9.3. In terms of parabolic flows we have that $\Psi(t, \boldsymbol{y}) = \boldsymbol{y}$, for all $t \in \mathbb{R}$, which implies that $\boldsymbol{y}$ is stationary for Ψ and is therefore an invariant set of Ψ.

Let $\boldsymbol{y} \in \Omega_d^m$ be stationary for a parabolic flow Ψ and consider a relative braid class fibers $[\boldsymbol{x}] \text{ rel } \boldsymbol{y} \in \Omega_d^n \text{ rel } \boldsymbol{y}$. As explained above braid class fibers are considered as subsets of Ω_d^n. Since $\boldsymbol{y}$ is stationary for Ψ, $\overline{\Omega_d^n \text{ rel } \boldsymbol{y}} \cong \bar{\Omega}_d^n$ is an invariant set for Ψ and we can restrict Ψ to $\bar{\Omega}_d^n$ after identification. In the following we consider the special case that $n = 1$ and $\Psi|_{\bar{\Omega}_d} = \varphi$. The connected components of $\Omega_d \text{ rel } \boldsymbol{y}$ are subsets of Ω_d.

9.10 Lemma Let $\boldsymbol{y} \in \Omega_d^m$ be a stationary braid (skeleton) for Equation (9.1.1) and let $[\boldsymbol{x} \text{ rel } \boldsymbol{y}]$ be a proper and bounded relative braid class. Then any fiber

$$B := [\boldsymbol{x}] \text{ rel } \boldsymbol{y} \subset \Omega_d \subset \bar{\Omega}_d,$$

is an isolating block for the associated parabolic flow φ.

Proof. Since B is bounded, $\mathrm{cl}(B)$ is compact. Let $x \in \partial B$, then, since the braid class is proper, it follows from Lemma 9.8 that there exists an $\epsilon > 0$ such that $\varphi([-\epsilon, 0), x) \not\subset B$, $\varphi((0, \epsilon], x) \not\subset B$, or neither are contained in B. This implies that boundary points are in either B^-, B^+, or in $B^- \cap B^+$. By compactness we can choose a uniform $\epsilon = \tau > 0$, which proves that B is an isolating block. ∎

The Conley index of B is well-defined and will be denoted by $HC_*(B, \varphi) \cong H_*(\mathrm{cl}(B), B^-)$. Lemma 9.8 characterizes B^- as the set of boundary points for which the word-length is decreasing by flowing from B to B^c. To be more precise, for every point $x \in \partial B$ we can choose a neighborhood $W \subset \bar{\Omega}_d$ such that $B \setminus W$ consists of finitely many connected components W_j and $W_0 = B \cap W$. Then,

$$B^- = \mathrm{cl}\big\{x \in \partial B \mid \ell(W_0) > \ell(W_j),\ \forall j\big\}.$$

The set B^+ is defined similarly. Since the sets $\mathrm{int}_{\partial B} B^-$ and $\mathrm{int}_{\partial B} B^+$ are smooth codimension-1 submanifolds in $\bar{\Omega}_d$ a coorientation is defined by the unit normal in the direction of decreasing ℓ. Therefore, B, B^- and B^+ are completely determined by the topological type of the relative braid class. The dynamics of φ always complies with the coorientation of the boundary of a proper braid class.

The next theorem shows that the Conley index of a braid class fiber is topological invariant of a discrete relative braid class.

9.11 Theorem Let $\boldsymbol{y} \in \Omega_d^m$ be a stationary braid (skeleton) for Equation (9.1.1) and let $[\boldsymbol{x} \text{ rel } \boldsymbol{y}]$ be a proper and bounded relative braid class. Then for any fiber $B = [\boldsymbol{x}] \text{ rel } \boldsymbol{y}$ the Conley index $HC_*(B, \varphi)$ is well-defined, and

(i) if Ψ' is any parabolic flow for which $\Psi'(t, \boldsymbol{y}) = \boldsymbol{y}$, for all $t \in \mathbb{R}$, then $HC_*(B, \varphi') \cong HC_*(B, \varphi)$, where $\varphi' = \Psi'|_{\Omega_d}$;

(ii) if $[\boldsymbol{x}'] \text{ rel } \boldsymbol{y}'$ and $[\boldsymbol{x}] \text{ rel } \boldsymbol{y}$ are fibers in $[\boldsymbol{x} \text{ rel } \boldsymbol{y}]$, then $HC_*(B, \varphi) \cong HC_*(B', \varphi')$, where $B = [\boldsymbol{x}] \text{ rel } \boldsymbol{y}$, $B' = [\boldsymbol{x}'] \text{ rel } \boldsymbol{y}'$ and φ and φ' parabolic flows for which $\boldsymbol{y}$ and $\boldsymbol{y}'$ are stationary respectfully.

The Conley index $HC_*(B, \varphi)$ is a *braid class invariant* for $[\boldsymbol{x} \text{ rel } \boldsymbol{y}]$ and is denoted by $h(\boldsymbol{x} \text{ rel } \boldsymbol{y})$.

Proof. Under construction. ∎

Let us now discuss some examples of the proper and bounded braid classes and compute their braid class invariants.

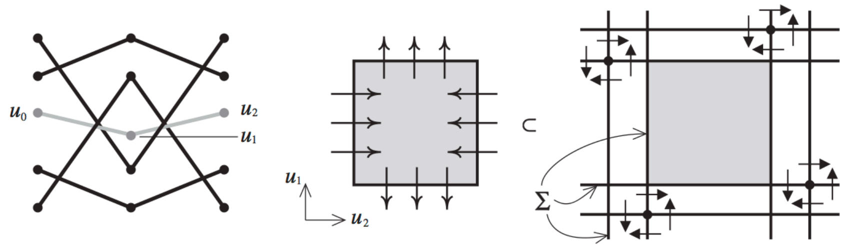


Figure 9.4: The braid of Example 1 [left] and the associated configuration space with parabolic flow [middle]. On the right is an expanded view of Ω_2 rel $\boldsymbol{y}$ where the fixed points of the flow correspond to the four fixed strands in the skeleton $\boldsymbol{v}$. The braid classes adjacent to these fixed points are not proper.

9.12 Example Consider the proper period-2 braid illustrated in Fig. **??**[left]. (Note that deleting any strand in the skeleton yields an improper braid.) There is exactly one free strand with two anchor points (recall that these are *closed* braids and the left and right sides are identified). The anchor point in the middle, x_1, is free to move vertically between the fixed points on the skeleton. At the endpoints, one has a singular braid in Σ which is on the exit set since a slight perturbation sends this singular braid to a different braid class with fewer crossings. The end anchor point, x_2 $(= x_0)$ can freely move vertically in between the two fixed points on the skeleton. The singular boundaries are in this case *not* on the exit set since pushing x_2 across the skeleton increases the number of crossings.

Since the points x_1 and x_2 can be moved independently, the configuration space B in this case is the product of two compact intervals. The exit set B^- consists of those points on ∂B for which x_1 is a boundary point. Thus, the homotopy index of this relative braid is $[B/B^-] \simeq S^1$.

9.13 Example Consider the proper relative braid presented in Fig. **??**[left]. Since there is one free strand of period three, the configuration space B is determined by the vector of positions (x_0, x_1, x_2) of the anchor points. This example differs greatly from the previous example. For instance, the point x_0 (as represented in the figure) may pass through the nearest strand of the skeleton above and below without changing the braid class. The points x_1 and x_2 may not pass through any strands of the skeleton without changing the braid class *unless* x_0 has already passed through. In this case, either x_1 or x_2 (depending on whether the upper or lower strand is crossed) becomes free.

To simplify the analysis, consider (x_0, x_1, x_2) as all of $\mathbb{R}^3$ (allowing for the moment singular braids and other braid classes as well). The position of the skeleton induces a cubical partition of $\mathbb{R}^3$ by planes, the equations being $x_i = y_i^\alpha$

for the various strands v^α of the skeleton $\mathbf{v}$. The braid class B is thus some collection of cubes in $\mathbb{R}^3$. In Fig. **??**[right], we illustrate this cube complex associated to B, claiming that it is homeomorphic to $D^2 \times S^1$. In this case, the exit set B^- happens to be the entire boundary ∂B and the quotient space is homotopic to the wedge-sum $S^2 \vee S^3$.

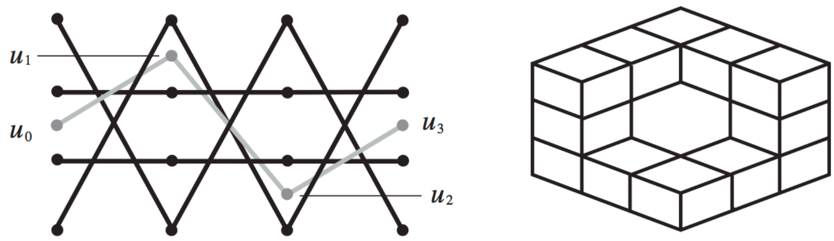


Figure 9.5: The braid of Example 9.13 and the configuration space B.

9.14 Example To introduce the spirit behind the forcing theorems of the latter half of the paper, we reconsider the period two braid of Example 1. Take an n-fold cover of the skeleton as illustrated in Fig. **??**. By weaving a single free strand in and out of the strands as shown, it is possible to generate numerous examples with nontrivial index. A moment's meditation suffices to show that the configuration space B for this lifted braid is a product of $2n$ intervals, the exit set being completely determined by the number of times the free strand is "threaded" through the inner loops of the skeletal braid as shown.

For an n-fold cover with one free strand we can select a family of 3^n possible braid classes describes as follows: the even anchor points of the free strand are always in the middle, while for the odd anchor points there are three possible choices. Two of these braid classes are not proper. All of the remaining $3^n - 2$ braid classes are bounded and have homotopy indices equal to a sphere S^k for some $0 \leq k \leq n$. Several of these strands may be superimposed while maintaining a nontrivial homotopy index for the net braid: we leave it to the reader to consider this interesting situation.

Stronger results follow from projecting these covers back down to the period two setting of Example 1. If the free strand in the cover is chosen not to be isotopic to a periodic braid, then it can be shown via a simple argument that some projection of the free strand down to the period two case has nontrivial homotopy index. Thus, the simple period two skeleton of Example 1 is the seed for an infinite number of braid classes with nontrivial homotopy indices. Using the techniques of [**?**], one can use this fact to show that any parabolic recurrence relation ($R = 0$) admitting this skeleton is forced to have positive topological entropy: cf. the related results from the Nielsen-Thurston theory of disc homeomorphisms [**?**].

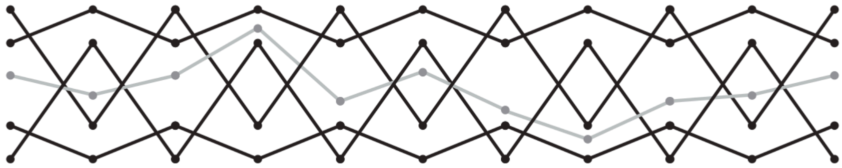


Figure 9.6: The lifted skeleton of Example 1 with one free strand.

9.4 Stabilization and global invariants

9.4.a Free braid classes and the extension operator

Via the results of the previous section, the homotopy index is an invariant of the *discretized* braid class: keeping the period fixed and moving within a connected component of the space of relative discretized braids leaves the index invariant. The *topological* braid class, as defined in §**??**, does not have an implicit notion of period. The effect of refining the discretization of a topological closed braid is not obvious: not only does the dimension of the index pair change, the homotopy types of the isolating neighborhood and the exit set may change as well upon changing the discretization. It is thus perhaps remarkable that any changes are correlated under the quotient operation: the homotopy index is an invariant of the *topological* closed braid class.

On the other hand, given a complicated braid, it is intuitively obvious that a certain number of discretization points are necessary to capture the topology correctly. If the period d is too small $\mathcal{D}_d^n \text{ rel } v$ may contain more than one path component with the same topological braid class:

9.15 Definition A relative braid class $[\mathbf{u} \text{ rel } v]$ in $\mathcal{D}_d^n \text{ rel } v$ is called *free* if

$$(\mathcal{D}_d^n \text{ rel } v) \cap \{\mathbf{u} \text{ rel } v\} = [\mathbf{u} \text{ rel } v]; \tag{9.4.4}$$

that is, if any other discretized braid in $\mathcal{D}_d^n \text{ rel } v$ which has the same topological braid class as $\mathbf{u} \text{ rel } v$ is in the same discretized braid class $[\mathbf{u} \text{ rel } v]$.

A braid class $[\mathbf{u}]$ is free if the above definition is satisfied with $v = \emptyset$. Not all discretized braid classes are free: see Fig. 9.4.a.

Define the *extension map* $\mathbb{E} : \bar{\mathcal{D}}_d^n \to \bar{\mathcal{D}}_{d+1}^n$ via concatenation with the trivial braid of period one (as in Fig. 9.8(a)):

$$(\mathbb{E}\mathbf{u})_i^\alpha := \begin{cases} u_i^\alpha & i = 0, \dots, d \\ u_d^\alpha & i = d+1. \end{cases} \tag{9.4.5}$$

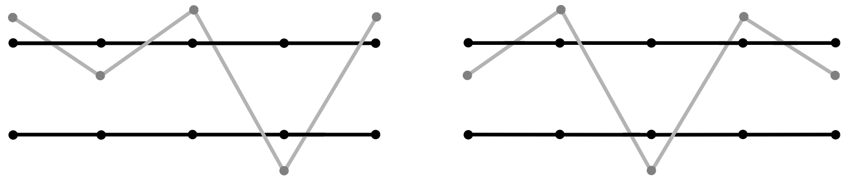


Figure 9.7: An example of two non-free discretized braids which are of the same topological braid class but define disjoint discretized braid classes in $\mathcal{D}_4^1 \text{ rel } v$.

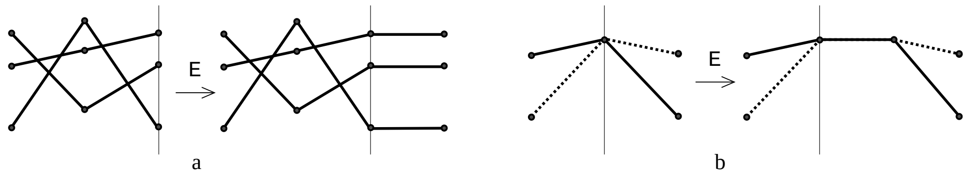


Figure 9.8: (a) The action of $\mathbb{E}$ extends a braid by one period; occasionally, (b), $\mathbb{E}$ produces a singular braid. Vertical lines denote the d^{th} discretization line.

The reader may note (with a little effort) that the non-equivalent braids of Fig. 9.4.a become equivalent under the image of $\mathbb{E}$. There are exceptional cases in which $\mathbb{E}\mathbf{u}$ is a singular braid when $\mathbf{u}$ is not: see Fig. 9.8(b). If the intersections at $i = d$ are generic then $\mathbb{E}\mathbf{u}$ is a nonsingular braid. One can always find such a representative in $[\mathbf{u}]$, again denoted by $\mathbf{u}$. Therefore the notation $[\mathbb{E}\mathbf{u}]$ means that $\mathbf{u}$ is chosen in $[\mathbf{u}]$ with generic intersection at $i = d$. The same holds for relative classes $[\mathbb{E}\mathbf{u} \text{ rel } \mathbb{E}v]$, i.e. choose $\mathbf{u} \text{ rel } v \in [\mathbf{u} \text{ rel } v]$ such that all intersections of $\mathbf{u} \cup v$ at $i = d$ are generic.

Note that under the action of $\mathbb{E}$ boundedness of a braid class is not necessarily preserved, i.e. $[\mathbf{u} \text{ rel } v]$ may be bounded, and $[\mathbb{E}\mathbf{u} \text{ rel } \mathbb{E}v]$ unbounded. For this reason we will prove a stabilization result for *topological* bounded proper braid classes.

9.4.b A topological invariant

Consider a period d discretized relative braid pair $\mathbf{u} \text{ rel } v$ which is not necessarily free. Collect all (a finite number) of the discretized braids $\mathbf{u}(0), \ldots, \mathbf{u}(m)$ such that the pairs $\mathbf{u}(j) \text{ rel } v$ are all topologically isotopic to $\mathbf{u} \text{ rel } v$ but not pairwise discretely isotopic. For the case of a free braid class, $m = 1$.

9.16 Definition Given $\mathbf{u} \text{ rel } v$ and $\mathbf{u}(0), \ldots, \mathbf{u}(m)$ as above, denote by $\mathbf{h}(\mathbf{u} \text{ rel } v)$ the wedge of the homotopy indices of these representatives,

$$\mathbf{h}(\mathbf{u} \text{ rel } v) := \bigvee_{j=0}^{m_d} h(\mathbf{u}(j) \text{ rel } v), \tag{9.4.6}$$

where $\vee$ is the topological wedge which, in this context, identifies all the constituent exit sets to a single point.

This wedge product is well-defined by Theorem **??** by considering the isolating neighborhood $N = \cup_j \mathrm{cl}[\mathbf{u}(j) \ \mathrm{rel} \ v]$. In general a union of isolating neighborhoods is not necessarily an isolating neighborhood again. However, since the word metric strictly decreases at Σ the invariant set decomposes into the union of invariant sets of the individual components of N. Indeed, if an orbit intersects two components it must have passed through Σ: contradiction.

The principal topological result of this paper is that $\mathbf{h}$ is an invariant of the *topological* bounded proper braid class $\{\mathbf{u} \ \mathrm{rel} \ \{v\}\}$.

9.17 Theorem Given $\mathbf{u} \ \mathrm{rel} \ v \in \mathcal{D}^n_d \ \mathrm{rel} \ v$ and $\tilde{\mathbf{u}} \ \mathrm{rel} \ \tilde{v} \in \mathcal{D}^n_{\tilde{d}} \ \mathrm{rel} \ \tilde{v}$ which are topologically isotopic as bounded proper braid pairs, then

$$\mathbf{h}(\mathbf{u} \ \mathrm{rel} \ v) = \mathbf{h}(\tilde{\mathbf{u}} \ \mathrm{rel} \ \tilde{v}). \tag{9.4.7}$$

The key ingredients in this proof are that (1) the homotopy index is invariant under $\mathbb{E}$ (Theorem 9.17); and (2) discretized braids "converge" to topological braids under sufficiently many applications of $\mathbb{E}$ (Proposition 9.25).

9.18 Theorem For $\mathbf{u} \ \mathrm{rel} \ v$ any bounded proper discretized braid pair, the wedged homotopy index of Definition 9.16 is invariant under the extension operator:

$$\mathbf{h}(\mathbb{E}\mathbf{u} \ \mathrm{rel} \ \mathbb{E}v) = \mathbf{h}(\mathbf{u} \ \mathrm{rel} \ v). \tag{9.4.8}$$

Proof. By the invariance of the index with respect to the skeleton v, we may assume that v is chosen to have all intersections generic ($v_i^\alpha \neq v_i^{\alpha'}$ for all strands $\alpha \neq \alpha'$). Thus, from the proof of Lemma **??** in Appendix **??**, we may fix a recurrence relation R having v as fixed point(s) for which $\partial_1 R_0 = 0$.

For $\epsilon > 0$ consider the one-parameter family of augmented recurrence functions[1] $R^\epsilon = (R_i^\epsilon)_{i=0}^d$ on braids of period $d+1$:

$$\begin{aligned} R_i^\epsilon(u_{i-1}^\alpha, u_i^\alpha, u_{i+1}^\alpha) &:= R_i(u_{i-1}^\alpha, u_i^\alpha, u_{i+1}^\alpha), \quad i = 0, .., d-1, \\ \epsilon \cdot R_d^\epsilon(u_{d-1}^\alpha, u_d^\alpha, u_{d+1}^\alpha) &:= u_{d+1}^\alpha - u_d^\alpha. \end{aligned} \tag{9.4.9}$$

Because of our choice of $R_0(r,s,t) = R_0(s,t)$ as being independent of the first variable, R_0^ϵ is decoupled from the extension of the braid as u_{d+1}^α wraps around to $u_0^{\tau(\alpha)}$. By construction the above system satisfies Axioms (A1)-(A2) for all $\epsilon > 0$ with, in particular, the strict monotonicity of (A1) holding only on one side. One therefore has a parabolic flow Ψ_ϵ^t on $\bar{\mathcal{D}}^n_{d+1}$ for all $\epsilon > 0$. In the singular limit $\epsilon = 0$, this forces $u_d^\alpha = u_{d+1}^\alpha$, and one obtains the flow $\Psi_0^t = \mathbb{E} \circ \Psi^t$.

[1] Recall the indexing conventions: for a period $d+1$ braid, $u_0^{\tau(\alpha)} = u_{d+1}^\alpha$, and $R_0 := R_{d+1}$.

Since the skeleton v has only generic intersections, $\mathbb{E}v$ is a nonsingular braid. From Equation (9.4.9), all stationary solutions of Ψ^t are stationary solutions for Ψ^t_ϵ, i.e., $\Psi^t_\epsilon(\mathbb{E}v) = \mathbb{E}v$, for all $\epsilon \geq 0$. Notice that this is not true in general for non-constant solutions.

Denote by $\mathcal{B}_{d+1} \subset \mathcal{D}^n_{d+1}$ rel $\mathbb{E}v$ the subset of relative braids which are topologically isotopic to $\mathbb{E}\mathbf{u}$ rel $\mathbb{E}v$. Likewise, denote by $\mathcal{B}_d \subset \bar{\mathcal{D}}^n_{d+1}$ the image under $\mathbb{E}$ of the subset of braids in $\mathcal{D}^n_d$ rel v which are topologically isotopic to $\mathbf{u}$ rel v. In other words,

$$\mathcal{B}_{d+1} := \{\mathbb{E}\mathbf{u} \text{ rel } \mathbb{E}v\} \cap \mathcal{D}^n_{d+1} \text{ rel } \mathbb{E}v \;\; ; \;\; \mathcal{B}_d := \mathbb{E}\Big(\{\mathbf{u} \text{ rel } v\} \cap \mathcal{D}^n_d \text{ rel } v\Big). \quad (9.4.10)$$

As per the paragraph preceding Definition 9.16, there are a finite number of connected components of each of these sets. Clearly, $\mathcal{B}_d$ is a codimension-n subset of $\mathrm{cl}(\mathcal{B}_{d+1})$. Since not all braids in $\{\mathbf{u} \text{ rel } v\} \cap \mathcal{D}^n_d$ rel v have generic intersections, the set $\mathcal{B}_d$ may tangentially intersect the boundary of $\mathcal{B}_{d+1}$. We will denote this set of $\mathbb{E}$-*singular braids* by $\Sigma_{\mathbb{E}} := \partial\mathcal{B}_{d+1} \cap \mathcal{B}_d$: see Fig. 9.9.

By performing an appropriate change of coordinates (cf. [**?**]), we can recast the parabolic system R^ϵ as a singular perturbation problem. Let $\boldsymbol{x} = (x_j)^{nd}_{j=1}$, with $x_{i+1+(\alpha-1)d} := u^\alpha_i$, and let $\boldsymbol{y} = (y_\alpha)^n_{\alpha=1}$, with $y_\alpha := (u^\alpha_{d+1} - u^\alpha_d)$. Upon rescaling time as $\tau := t/\epsilon$, the vector field induced by our choice of R^ϵ is of the form

$$\begin{aligned} \frac{d\boldsymbol{x}}{d\tau} &= \epsilon X(\boldsymbol{x},\boldsymbol{y}), \\ \frac{d\boldsymbol{y}}{d\tau} &= -\boldsymbol{y} + \epsilon Y(\boldsymbol{x}), \end{aligned} \quad (9.4.11)$$

for some (unspecified) vector fields X and Y with the functional dependence indicated. The product flow of this vector field (9.4.11) in the new coordinates is denoted by Φ^τ_ϵ and is well-defined on $\bar{\mathcal{D}}^n_{d+1}$. In the case $\epsilon = 0$, the set $\mathcal{M} := \{\boldsymbol{y} = 0\} \subset \bar{\mathcal{D}}^n_{d+1}$ is a submanifold of fixed points containing $\mathcal{B}_d$ for which the flow Φ^τ_0 is transversally nondegenerate (since here $\boldsymbol{y}' = -\boldsymbol{y}$). By construction $\mathrm{cl}(\mathcal{B}_d) = \mathrm{cl}(\mathcal{B}_{d+1}) \cap \mathcal{M}$, as illustrated in Fig. 9.9 (in the simple case where all braid classes are free and $\mathcal{B}_{d+1}$ is thus connected).

The remainder of the analysis is a technique in singular perturbation theory following [**?**]: one relates the τ-dynamics of Equation (9.4.11) to those of the t-dynamics on $\mathcal{M}$, whose orbits are of the form $(\boldsymbol{x}(t),0)$, where $\boldsymbol{x}(t)$ satisfies the limiting equation $d\boldsymbol{x}/dt = X(\boldsymbol{x},0)$. The Conley index theory is well-suited to this situation.

For any compact set $D \subset \mathcal{M}$ and $r \in \mathbb{R}$, let $D(r) := \{(\boldsymbol{x},\boldsymbol{y}) \mid (\boldsymbol{x},0) \in D,\ \|\boldsymbol{y}\| \leq r\}$ denote the "product" radius r neighborhood in $\bar{\mathcal{D}}^n_{d+1}$. Denote by $C = C(D)$ the maximal value $C := \max_D \|Y(\boldsymbol{x})\|$. Due to the specific form of (9.4.11), we obtain the following uniform squeezing lemma.

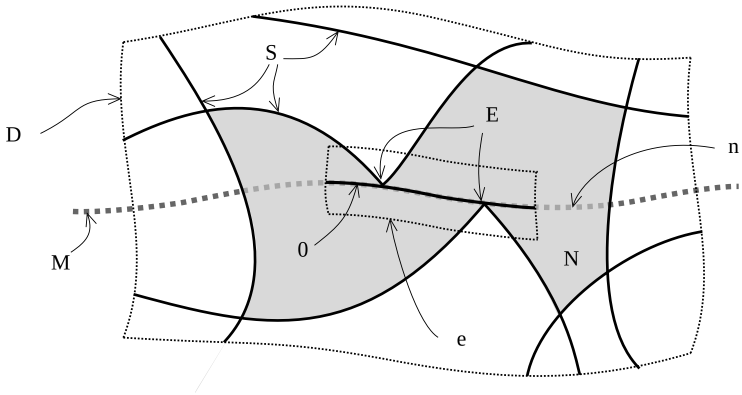


Figure 9.9: The rescaled flow acts on $\mathcal{B}_{d+1}$, the period $d+1$ braid classes. The submanifold $\mathcal{M}$ is a critical manifold of fixed points at $\epsilon = 0$. Any appropriate isolating neighborhood N_0 in $\mathcal{B}_d$ thickens to an isolating neighborhood $N_0(2\epsilon C)$ which is not necessarily contained in $\mathcal{B}_{d+1}$.

9.19 Lemma If S is any invariant set of Φ_ϵ^τ contained in some $D(r)$, then in fact $S \subset D(\epsilon C)$. Moreover, for all points $(\boldsymbol{x}, \boldsymbol{y})$ with $\boldsymbol{x} \in D$ and $\|\boldsymbol{y}\| = 2\epsilon C$ it holds that $\frac{d}{d\tau}\|y\| < 0$.

Proof. Let $(\boldsymbol{x}, \boldsymbol{y})(\tau)$ be an orbit in S contained in some $D(r)$. Take the inner product of the $\boldsymbol{y}$-equation with $\boldsymbol{y}$:

$$\begin{aligned}\langle \frac{d\boldsymbol{y}}{d\tau}, \boldsymbol{y}\rangle(\tau_0) &= -\|\boldsymbol{y}(\tau_0)\|^2 + \epsilon\langle Y(\boldsymbol{x}(\tau_0)), \boldsymbol{y}(\tau_0)\rangle, \\ &\leq -\|\boldsymbol{y}\|^2 + \epsilon C\|\boldsymbol{y}\|.\end{aligned}$$

Hence $\frac{d}{d\tau}\|\boldsymbol{y}\| \leq -\|\boldsymbol{y}\| + \epsilon C$, and we conclude that if $\|\boldsymbol{y}(\tau_0)\| > \epsilon C$ for some $\tau_0 \in \mathbb{R}$, then $\frac{d}{d\tau}\|\boldsymbol{y}\| < 0$. Consequently $\|y(\tau)\|$ grows unbounded for $\tau < \tau_0$ and therefore $(\boldsymbol{x}, \boldsymbol{y}) \notin S$, a contradiction. Thus $\|y(\tau)\| \leq \epsilon C$ for all $\tau \in \mathbb{R}$.

For points $(\boldsymbol{x}, \boldsymbol{y})$ with $\boldsymbol{x} \in D$ and $\|\boldsymbol{y}\| = 2\epsilon C$, the above inequality gives that $\frac{d}{d\tau}\|\boldsymbol{y}\| \leq -\|\boldsymbol{y}\| + \epsilon C < 0$.

By compactness of the proper braid class, it is clear that $\mathcal{B}_{d+1}$, and thus the maximal isolated invariant set of Φ_ϵ^τ given by $S_\epsilon := \mathrm{Inv}(\mathcal{B}_{d+1}, \Phi_\epsilon^\tau)^2$, is strictly contained (and thus isolated) in $D(r)$ for some compact $D \subset \mathcal{M}$ and some r sufficiently large. Fix $C := C(D)$ as above. Lemma 9.19 now implies that as ϵ becomes small, S_ϵ is squeezed into $D(\epsilon C)$ — a small neighborhood of a compact

[2] Since $\mathcal{B}_{d+1}$ is a proper braid class S_ϵ is contained in its interior.

subset D of the critical manifold $\mathcal{M}$, as in Fig. 9.9.[3]

This proximity of S_ϵ to $\mathcal{M}$ allows one to compare the dynamics of the $\epsilon = 0$ and $\epsilon > 0$ flows. Let $N_0 \subset \mathcal{B}_d \subset \mathcal{M}$ be an isolating neighborhood (isolating block with corners) for the maximal t-dynamics invariant set $S_0 := \mathrm{Inv}(\mathcal{B}_d, \Psi_0^t)$ within the braid class $\mathcal{B}_d$. Combining Lemma 9.19 above, Theorem 2.3C of **[?]**, and the existence theorems for isolating blocks **[?]**, one concludes that if (N_0, N_0^-) is an index pair for the limiting equations $d\boldsymbol{x}/dt = X(\boldsymbol{x}, 0)$ then $N_0(2\epsilon C)$ is an isolating block for Φ_ϵ^t for $0 < \epsilon \leq \epsilon^*(N_0)$ with ϵ^* sufficiently small. A suitable index pair for the flow Φ_ϵ^τ of Equation (9.4.11) is thus given by

$$\big(N_0(2\epsilon C), N_0^-(2\epsilon C)\big). \tag{9.4.12}$$

Clearly, then, the homotopy index of S_0 is equal to the homotopy index of $\mathrm{Inv}(N_0(2\epsilon C))$ for all ϵ sufficiently small. It remains to show that this captures the maximal invariant set S_ϵ.

9.20 Lemma For all sufficiently small ϵ, $\mathrm{Inv}(N_0(2\epsilon C), \Phi_\epsilon^\tau) = S_\epsilon$.

Proof. By the choice of D it holds that $S_\epsilon \subset D(2\epsilon C)$. We start by proving that $S_\epsilon \subset N_0(2\epsilon C)$ for ϵ sufficiently small. Assume by contradiction that $S_{\epsilon_j} \not\subset N_0(2\epsilon_j C)$ for some sequence $\epsilon_j \to 0$. Then, since $N_0(2\epsilon C)$ is an isolating neighborhood for $\epsilon \leq \epsilon^*$, there exist orbits $(\boldsymbol{x}_{\epsilon_j}, \boldsymbol{y}_{\epsilon_j})$ in S_{ϵ_j} such that $(\boldsymbol{x}_{\epsilon_j}, \boldsymbol{y}_{\epsilon_j})(\tau_j) \in D(2\epsilon_j C) - N_0(2\epsilon_j C)$, for some $\tau_j \in \mathbb{R}$. Define $(\widetilde{\boldsymbol{x}}_{\epsilon_j}, \widetilde{\boldsymbol{y}}_{\epsilon_j})(\tau) = (\boldsymbol{x}_{\epsilon_j}, \boldsymbol{y}_{\epsilon_j})(\tau - \tau_j)$, and set $(\mathbf{a}_{\epsilon_j}, \mathbf{b}_{\epsilon_j})(t) = (\widetilde{\boldsymbol{x}}_{\epsilon_j}, \widetilde{\boldsymbol{y}}_{\epsilon_j})(\tau)$. The sequence $(\mathbf{a}_{\epsilon_j}, \mathbf{b}_{\epsilon_j})$ satisfies the equations

$$\frac{d}{dt}\mathbf{a}_{\epsilon_j} = X(\mathbf{a}_{\epsilon_j}, \mathbf{b}_{\epsilon_j}), \quad \frac{d}{dt}\mathbf{b}_{\epsilon_j} = -\frac{1}{\epsilon}\mathbf{b}_{\epsilon_j} + Y(\mathbf{a}_{\epsilon_j}). \tag{9.4.13}$$

By assumption $\|\mathbf{b}_{\epsilon_j}(t)\| \leq C\epsilon_j$, and $\|\mathbf{a}_{\epsilon_j}\|, \|d\mathbf{a}_{\epsilon_j}/dt\| \leq C$, for all $t \in \mathbb{R}$ and all ϵ_j. An Arzela-Ascoli argument then yields the existence of an orbit $(\mathbf{a}_*(t), 0) \subset \mathcal{B}_d$, with $(\mathbf{a}_*(0), 0) \in \mathrm{cl}(\mathcal{B}_d - N_0)$, satisfying the equation $\frac{d\mathbf{a}_*}{dt} = X(\mathbf{a}_*, 0)$. By definition, $(\mathbf{a}_*, 0) \in \mathrm{Inv}(\mathcal{B}_d) = \mathrm{Inv}(N_0) \subset \mathrm{int}(N_0)$, a contradiction, which proves that $S_\epsilon \subset N_0(2\epsilon C)$ for ϵ sufficiently small.

The boundary of $N_0(2\epsilon C)$ splits as $b_1 \cup b_2$, with

$$b_1 = \{(\boldsymbol{x}, \boldsymbol{y}) \mid \|\boldsymbol{y}\| = 2\epsilon C\}, \quad \text{and} \quad b_2 = \{(\boldsymbol{x}, \boldsymbol{y}) \mid \boldsymbol{x} \in \partial N_0\}.$$

Since the compact set N_0 is contained in $\mathcal{B}_d$, the boundary component b_2 is contained in $\mathcal{B}_{d+1}$ provided that ϵ is sufficiently small. If the set $\Sigma_{\mathbb{E}}$ is non-empty then the boundary component b_1 never lies entirely in $\mathcal{B}_{d+1}$ regardless of ϵ. As $\epsilon \to 0$ the set $N_0(2\epsilon C) - \big(\mathcal{B}_{d+1} \cap N_0(2\epsilon C)\big)$ is contained is arbitrary small neighborhood of $\Sigma_{\mathbb{E}}$. Independent of the parabolic flow in question, and thus of ϵ, there exists a

[3] If one applies singular perturbation theory it is possible to construct an invariant manifold $\mathcal{M}_\epsilon \subset D(\epsilon C)$. The manifold $\mathcal{M}_\epsilon$ lies strictly within $\mathcal{B}_{d+1}$ and intersects $\mathcal{M}$ at rest points of the Φ_0^t.

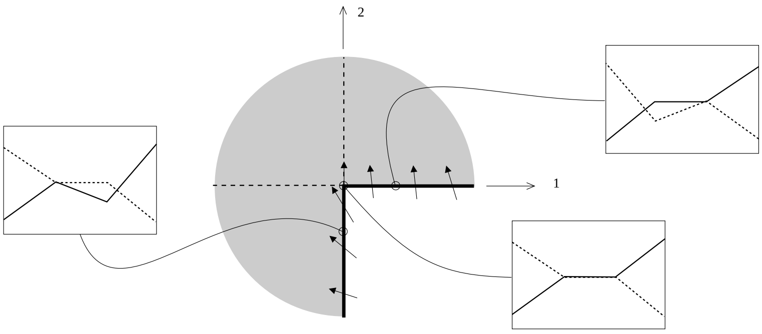


Figure 9.10: The local picture of a generic singular tangency between strands α (solid) and α' (dashed). The shaded region represents $\mathcal{B}_{d+1}$.

neighborhood $K \subset \Sigma^n_{d+1}$ rel v of $\Sigma_{\mathbb{E}}$ on which the co-orientation of the boundary is pointed inside the braid class $\mathcal{B}_{d+1}$. In other words for every parabolic system the points in K enter $\mathcal{B}_{d+1}$ under the flow, see Fig. 9.10. By using coordinates $u^\alpha_i - u^{\alpha'}_i$ and $u^\alpha_{i+1} - u^{\alpha'}_{i+1}$ adapted to the singular strands, it it easily seen (Fig. 9.10) that the braids are simplified by moving into the set $\mathcal{B}_{d+1}$.

We now show that $\mathrm{Inv}(N_0(2\epsilon C)) \subset \mathcal{B}_{d+1} \cap N_0(2\epsilon C)$. If not, then there exist points $(\boldsymbol{x}_{\epsilon_j}, \boldsymbol{y}_{\epsilon_j}) \in \left[N_0(2\epsilon_j C) - \left(\mathcal{B}_{d+1} \cap N_0(2\epsilon_j C)\right)\right] \cap \mathrm{Inv}(N_0(2\epsilon_j C))$ for some sequence $\epsilon_j \to 0$. Consider the α-limit sets $\alpha_{\epsilon_j}((\boldsymbol{x}_{\epsilon_j}, \boldsymbol{y}_{\epsilon_j}))$. Since $(\boldsymbol{x}_{\epsilon_j}, \boldsymbol{y}_{\epsilon_j}) \in \mathrm{Inv}(N_0(2\epsilon_j C))$, and since $\Phi^\tau_{\epsilon_j}((\boldsymbol{x}_{\epsilon_j}, \boldsymbol{y}_{\epsilon_j}))$ cannot enter $\mathcal{B}_{d+1} \cap N_0(2\epsilon_j C)$ in backward time due to the co-orientation of K, it follows that $\alpha_{\epsilon_j}((\boldsymbol{x}_{\epsilon_j}, \boldsymbol{y}_{\epsilon_j}))$ is contained in $N_0(2\epsilon_j C) - \left(\mathcal{B}_{d+1} \cap N_0(2\epsilon_j C)\right)$.

By a similar Arzela-Ascoli argument as before, this yields a set $\alpha_0 \subset \Sigma_{\mathbb{E}}$ which is invariant for the flow Ψ^t_0. However due to the form of the vector field the associated flow Ψ^t_0 cannot contain an invariant set in $\Sigma_{\mathbb{E}}$, which proves that $\mathrm{Inv}(N_0(2\epsilon C)) \subset \mathcal{B}_{d+1} \cap N_0(2\epsilon C)$ for ϵ sufficiently small.

Finally, knowing that $S_\epsilon \subset \mathrm{Inv}(N_0(2\epsilon C))$, and that for sufficiently small ϵ it holds $\mathrm{Inv}(N_0(2\epsilon C) = \mathrm{Inv}(\mathcal{B}_{d+1} \cap N_0(2\epsilon C)) = S_\epsilon$, it follows that $S_\epsilon = \mathrm{Inv}(N_0(2\epsilon C))$, which proves the lemma.

Theorem 9.18 now follows. Since, by Theorem **??**, the homotopy index is independent of the parabolic flow used to compute it, one may choose the parabolic flow Φ^τ_ϵ for $\epsilon > 0$ sufficiently small. The homotopy index of Φ^τ_ϵ on the maximal invariant set S_ϵ yields the wedge of all the connected components: $\mathbf{h}(\mathbb{E}\mathbf{u} \text{ rel } \mathbb{E}v)$. We have computed that this index is equal to the index of Ψ^t on the original braid class: $\mathbf{h}(\mathbf{u} \text{ rel } v)$.

■ **9.21 Remark** The proof of Theorem 9.18 implies that any component of the period-$(d+1)$ braid class $\mathcal{B}_{d+1}$ which does not intersect $\mathcal{M}$ must necessarily have trivial index. ■

■ **9.22 Remark** The above procedure also yields a stabilization result for bounded proper classes which are not bounded as topological classes. In this case one simply augments the skeleton v by two constant strands as follows. Define the *augmented braid* $\boldsymbol{v}^* := \boldsymbol{v} \cup \boldsymbol{v}^- \cup \boldsymbol{v}^+$, where

$$v_i^- := \min_{\alpha,i} v_i^\alpha - 1, \quad v_i^+ := \max_{\alpha,i} v_i^\alpha + 1. \tag{9.4.14}$$

Suppose $[\mathbf{u} \;\text{rel}\; \boldsymbol{v}] \subset \mathcal{D}^n_{d_0} \;\text{rel}\; \boldsymbol{v}$ is bounded for some period d_0. It now holds that $h(\mathbf{u} \;\text{rel}\; \boldsymbol{v}) = h(\mathbf{u} \;\text{rel}\; \boldsymbol{v}^*)$, and $\{\mathbf{u} \;\text{rel}\; \{\boldsymbol{v}^*\}\}$ is a bounded class. It therefore follows from Theorem 9.17 that

$$\bigvee_{j=0}^{m_{d_0}} h(\mathbf{u}(j) \;\text{rel}\; \boldsymbol{v}) = \mathbf{h}(\mathbf{u} \;\text{rel}\; \boldsymbol{v}^*), \tag{9.4.15}$$

where $\mathbf{h}$ can be evaluated via any discrete representative of $\{\mathbf{u} \;\text{rel}\; \{\boldsymbol{v}^*\}\}$ of any admissible period. ∎

9.4.c Eventually free classes

At the end of this subsection, we complete the proof of Theorem 9.17. The preliminary step is to show that discretized braid classes are eventually free under $\mathbb{E}$.

Given a braid $\mathbf{u} \in \mathcal{D}^n_d$, consider the extension $\mathbb{E}\mathbf{u}$ of period $d+1$. Assume at first the simple case in which $d = 1$, so that $\mathbb{E}\mathbf{u}$ is a period-2 braid. Draw the braid diagram $\beta(\mathbb{E}\mathbf{u})$ as defined in §**??** in the domain $[0,2] \times \mathbb{R}$. Choose any 1-parameter family of curves $\gamma_s : t \mapsto (f_s(t), t) \in (0,2) \times \mathbb{R}$ such that $\gamma_0 : t \mapsto (1,t)$ and so that γ_s is transverse[4] to the braid diagram $\beta(\mathbb{E}\mathbf{u})$ for all s. Define the braid $\gamma_s \cdot \mathbb{E}\mathbf{u}$ as follows:

$$(\gamma_s \cdot \mathbb{E}\mathbf{u})_i^\alpha := \begin{cases} (\mathbb{E}\mathbf{u})_i^\alpha & : i = 0,2 \\ \gamma_s \cap (\mathbb{E}\mathbf{u})^\alpha & : i = 1 \end{cases} . \tag{9.4.16}$$

The point $\gamma_s \cap (\mathbb{E}\mathbf{u})^\alpha$ is well-defined since γ_s is always transverse to the braid strands and γ_0 intersects each strand but once.

9.23 Lemma For any such family of curves γ_s, $[\gamma_s \cdot \mathbb{E}\mathbf{u}] = [\mathbb{E}\mathbf{u}]$.

Proof. It suffices to show that this path of braids does not intersect the singular braids Σ. Since $\mathbf{u}$ is assumed to be a nonsingular braid, every crossing of two strands in the braid diagram of $\mathbb{E}\mathbf{u}$ is a transversal crossing between $i = 0$ and $i = 1$. Thus, if for some s, $\gamma_s(t) \cap (\mathbb{E}\mathbf{u})^\alpha = \gamma_s(t) \cap (\mathbb{E}\mathbf{u})^{\alpha'}$ for distinct strands α and α', then

$$\left(\mathbb{E}\mathbf{u}_0^\alpha - \mathbb{E}\mathbf{u}_0^{\alpha'}\right)\left(\mathbb{E}\mathbf{u}_1^\alpha - \mathbb{E}\mathbf{u}_1^{\alpha'}\right) < 0. \tag{9.4.17}$$

[4] At the anchor points, the transversality should be topological as opposed to smooth.

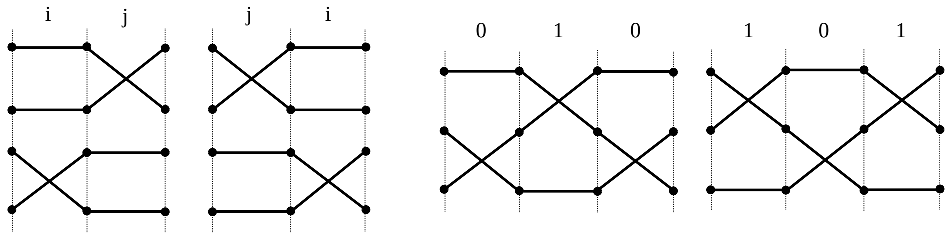


Figure 9.11: Relations in the braid group via discrete isotopy.

The braid $\gamma_s \cdot \mathbb{E}\mathbf{u}$ has a crossing of the α and α' strands at $i = 1$. Checking the transversality of this crossing yields

$$\begin{aligned} &\left((\gamma_s \cdot \mathbb{E}\mathbf{u})_0^\alpha - (\gamma_s \cdot \mathbb{E}\mathbf{u})_0^{\alpha'}\right)\left((\gamma_s \cdot \mathbb{E}\mathbf{u})_2^\alpha - (\gamma_s \cdot \mathbb{E}\mathbf{u})_2^{\alpha'}\right) \\ &= \left((\mathbb{E}\mathbf{u})_0^\alpha - (\mathbb{E}\mathbf{u})_0^{\alpha'}\right)\left((\mathbb{E}\mathbf{u})_2^\alpha - (\mathbb{E}\mathbf{u})_2^{\alpha'}\right) \\ &= \left((\mathbb{E}\mathbf{u})_0^\alpha - (\mathbb{E}\mathbf{u})_0^{\alpha'}\right)\left((\mathbb{E}\mathbf{u})_1^\alpha - (\mathbb{E}\mathbf{u})_1^{\alpha'}\right) < 0. \end{aligned} \tag{9.4.18}$$

Thus the crossing is transverse and the braid is never singular.

Note that the proof of Lemma 9.23 does not require the braid $\mathbb{E}\mathbf{u}$ to be a closed braid diagram since the isotopy fixes the endpoints: the proof is equally valid for any localized region of a braid in which one spatial segment has crossings and the next segment has flat strands.

9.24 Corollary The "shifted" extension operator which inserts a trivial period-1 braid at the i^{th} discretization point in a braid has the same action on components of $\mathcal{D}_d$ as does $\mathbb{E}$.

9.25 Proposition The period-d discretized braid class $[\mathbf{u}]$ is free when $d > |\mathbf{u}|_{\text{word}}$.

Proof. We must show that any braid $\mathbf{u}' \in \mathcal{D}_d^n$ which has the same topological type as $\mathbf{u}$ is discretely isotopic to $\mathbf{u}$. Place both $\mathbf{u}$ and $\mathbf{u}'$ in general position so as to record the sequences of crossings using the generators of the n-strand positive braid semigroup, $\{\sigma_i\}$, as in §**??**. Recall the braid group has relations $\sigma_i\sigma_j = \sigma_j\sigma_i$ for $|i-j| > 1$ and $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$; closure requires making conjugacy classes equivalent.

The conjugacy relation can be realized by a discrete isotopy as follows: since $d > |\mathbf{u}|_{\text{word}}$, $\mathbf{u}$ must possess some discretization interval on which there are no crossings. Lemma 9.23 then implies that this interval without crossings commutes with all neighboring discretization intervals via discrete isotopies. Performing d consecutive exchanges shifts the entire braid over by one discretization interval. This generates the conjugacy relation.

To realize the remaining braid relations in a discrete isotopy, assume first that $\mathbf{u}$ and $\mathbf{u}'$ are of the form that there is at most one crossing per discretization interval.

It is then easy to see from Fig. 9.11 that the braid relations can be executed via discrete isotopy.

In the case where $\mathbf{u}$ (and/or $\mathbf{u}'$) exhibits multiple crossings on some discretization intervals, it must be the case that a corresponding number of other discretization intervals do not possess any crossings (since $d > |\mathbf{u}|_{\text{word}}$). Again, by inductively utilizing Lemma 9.23, we may redistribute the intervals-without-crossing and "comb" out the multiple crossings via discrete isotopies so as to have at most one crossing per discretization interval.

Proof of Theorem 9.17: Assume that $\{\mathbf{u} \ \text{rel} \ \{v\}\} = \{\mathbf{u}' \ \text{rel} \ \{v'\}\}$. This implies that there is a path of topological braid diagrams taking the pair $(\mathbf{u}, v)$ to $(\mathbf{u}', v')$. This path may be chosen so as to follow a sequence of standard relations for closed braids. From the proof of Proposition 9.25, these relations may be performed by a discretized isotopy to connect the pair $(\mathbb{E}^j\mathbf{u}, \mathbb{E}^j v)$ to $(\mathbb{E}^k\mathbf{u}', \mathbb{E}^k v')$ for j and k sufficiently large, and of the right relative size to make the periods of both pairs equal. For this choice, then, $[\mathbb{E}^j\mathbf{u} \ \text{rel} \ [\mathbb{E}^j v]] = [\mathbb{E}^k\mathbf{u}' \ \text{rel} \ [\mathbb{E}^k v']]$, and their homotopy indices agree. An application of Theorem 9.18 completes the proof.

We suspect that all braids in the image of $\mathbb{E}$ are free: a result which, if true, would simplify index computations yet further.

9.5 Duality

For purposes of computation of the index, we will often pass to the homological level. In this setting, there is a natural duality made possible by the fact that the index pair used to compute the index of a braid class can be chosen to be a manifold pair.

9.26 Definition The *duality operator* on discretized braids is the map $\mathbb{D} : \bar{\mathcal{D}}^n_{2p} \to \bar{\mathcal{D}}^n_{2p}$ given by

$$(\mathbb{D}\mathbf{u})^\alpha_i := (-1)^i u^\alpha_i. \tag{9.5.19}$$

Clearly $\mathbb{D}$ induces a map on relative braid diagrams by defining $\mathbb{D}(\mathbf{u} \ \text{rel} \ v)$ to be $\mathbb{D}\mathbf{u} \ \text{rel} \ \mathbb{D}v$. The topological action of $\mathbb{D}$ is to insert a half-twist at each spatial segment of the braid. This has the effect of linking unlinked strands, and, since $\mathbb{D}$ is an involution, linked strands are unlinked by $\mathbb{D}$: see Fig. 9.12.

For the duality statements to follow, we assume that all braids considered have even periods and that all of the braid classes and their duals are proper, so that the homotopy index is well-defined.

9.27 Lemma The duality map $\mathbb{D}$ respects braid classes: if $[\mathbf{u}] = [\mathbf{u}']$ then $[\mathbb{D}(\mathbf{u})] = [\mathbb{D}(\mathbf{u}')]$. Bounded braid classes are taken to bounded braid classes by $\mathbb{D}$.

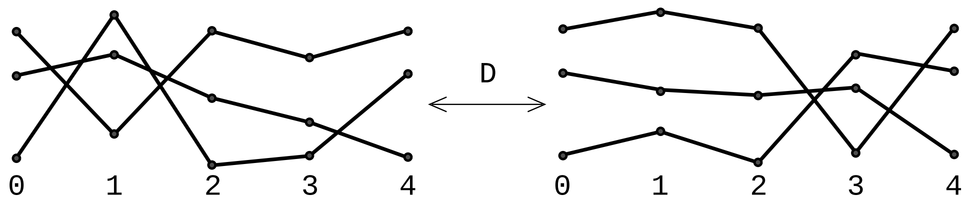


Figure 9.12: The topological action of $\mathbb{D}$.

Proof. It suffices to show that the map $\mathbb{D}$ is a homeomorphism on the pair $(\bar{\mathcal{D}}^n_{2p}, \Sigma)$. This is true on $\bar{\mathcal{D}}^n_{2p}$ since $\mathbb{D}$ is a smooth involution ($\mathbb{D}^{-1} = \mathbb{D}$). If $\mathbf{u} \in \Sigma$ with $u_i^\alpha = u_i^{\alpha'}$ and

$$(u_{i-1}^\alpha - u_{i-1}^{\alpha'})(u_{i+1}^\alpha - u_{i+1}^{\alpha'}) \geq 0, \tag{9.5.20}$$

then applying the operator $\mathbb{D}$ yields points $\mathbb{D}u_i^\alpha = \mathbb{D}u_i^{\alpha'}$ with each term in the above inequality multiplied by -1 (if i is even) or by $+1$ (if i is odd): in either case, the quantity is still non-negative and thus $\mathbb{D}\mathbf{u} \in \Sigma$. Boundedness is clearly preserved. ∎

9.28 Theorem (a) The effect of $\mathbb{D}$ on the index pair is to reverse the direction of the parabolic flow.

(b) For $[\mathbf{u} \text{ rel } \mathbf{v}] \subset \mathcal{D}^n_{2p}$ rel $\mathbf{v}$ of period $2p$ with n free strands,

$$CH_*(h(\mathbb{D}(\mathbf{u} \text{ rel } \mathbf{v})); \mathbb{R}) \cong CH_{2np-*}(h(\mathbf{u} \text{ rel } \mathbf{v}); \mathbb{R}). \tag{9.5.21}$$

(c) For $[\mathbf{u} \text{ rel } \mathbf{v}] \subset \mathcal{D}^n_{2p}$ rel $\mathbf{v}$ of period $2p$ with n free strands,

$$CH_*(\mathbf{h}(\mathbb{D}(\mathbf{u} \text{ rel } \mathbf{v})); \mathbb{R}) \cong CH_{2np-*}(\mathbf{h}(\mathbf{u} \text{ rel } \mathbf{v}); \mathbb{R}). \tag{9.5.22}$$

Proof: For (a), let (N, N^-) denote an index pair associated to a proper relative braid class $[\mathbf{u} \text{ rel } \mathbf{v}]$. Dualizing sends N to a homeomorphic space $\mathbb{D}(N)$. The following local argument shows that the exit set of the dual braid class is in fact the complement (in the boundary) of the exit set of the domain braid: specifically,

$$(\mathbb{D}(N))^- = \mathrm{cl}\left\{\partial(\mathbb{D}(N)) - \mathbb{D}(N^-)\right\}.$$

Let $\mathbf{w} \in [\mathbf{u} \text{ rel } \mathbf{v}] \cap \Sigma$. At any singular anchor point of $\mathbf{w}$, i.e., where $w_i^\alpha = w_i^{\alpha'}$ and the transversality condition is not satisfied, then it follows from Axiom (A2) that

$$\mathrm{SIGN}\left\{\frac{d}{dt}(w_i^\alpha - w_i^{\alpha'})\right\} = \mathrm{SIGN}\left\{w_{i-1}^\alpha - w_{i-1}^{\alpha'}\right\}. \tag{9.5.23}$$

(Depending on the form of (A2) employed, one might use $w_{i+1}^\alpha - w_{i+1}^{\alpha'}$ on the right hand side without loss.) Since the subscripts on the left side have the opposite parity of the subscripts on the right side, taking the dual braid (which multiplies the anchor points by $(-1)^i$ and $(-1)^{i-1}$ respectively) alters the sign of the terms. Thus, the operator $\mathbb{D}$ reverses the direction of the parabolic flow.

From this, we may compute the Conley index of the dual braid by reversing the time-orientation of the flow. Since one can choose the index pair used to compute the index to be an oriented manifold pair (specifically, an isolating block: see, cf. [**?**]), one may then apply a Poincaré-Lefschetz duality argument as in [**?**] and use the fact that the dimension is $2np$ to obtain the duality formula for homology. This yields (b).

The final claim (c) follows from (b) by showing that $\mathbb{D}$ is bijective on *topological* braid classes within $\bar{\mathcal{D}}^n_{2p}$. Assume that $[\mathbf{u} \ \mathrm{rel} \ v]$ and $[\mathbf{u}' \ \mathrm{rel} \ v]$ are distinct braid classes in $\mathcal{D}^n_{2p}$ of the same topological type. Since $\mathbb{D}$ is a homeomorphism on $\mathcal{D}^n_{2p}$, the dual classes $[\mathbb{D}\mathbf{u} \ \mathrm{rel} \ \mathbb{D}v]$ and $[\mathbb{D}\mathbf{u}' \ \mathrm{rel} \ \mathbb{D}v]$ are distinct. Claim (c) follows upon showing that these duals are still topologically the same braid class.

Proposition 9.25 implies that $[(E^r(D)^{2k}\mathbf{u}) \ \mathrm{rel} \ (E^r(D)^{2k}v)] = [(E^r(D)^{2k}\mathbf{u}') \ \mathrm{rel} \ (E^r(D)^{2k}v)]$ for k sufficiently large since $\{\mathbf{u} \ \mathrm{rel} \ v\} = \{\mathbf{u}' \ \mathrm{rel} \ v\}$. By Lemma 9.27,

$$\mathbb{D}\left[(E^r(D)^{2k}\mathbf{u}) \ \mathrm{rel} \ (E^r(D)^{2k}v)\right] = \mathbb{D}\left[(E^r(D)^{2k}\mathbf{u}') \ \mathrm{rel} \ (E^r(D)^{2k}v)\right],$$

which, by Lemma **??** means that these braids are topologically the same. The topological action of dualizing the $2k$-stabilizations of $\mathbf{u} \ \mathrm{rel} \ v$ and $\mathbf{u}' \ \mathrm{rel} \ v$ is to add k full twists. Since the full twist is in the center of the braid group (this element commutes with all other elements of the braid group [**?**]), one can factor the dual braids within the topological braid group and mod out by k full twists, yielding that $\{\mathbb{D}\mathbf{u} \ \mathrm{rel} \ \mathbb{D}v\} = \{\mathbb{D}\mathbf{u}' \ \mathrm{rel} \ \mathbb{D}v\}$. ∎

We use this homological duality to complete a crucial computation in the proof of the forcing theorems (e.g., Theorem 10.1.2) at the end of this paper. The following small corollary uses duality to give the first step towards answering the question of just what the homotopy index measures topologically about a braid class. Recall the definition of an augmented braid from Remark 9.22.

9.29 Corollary Consider the dual of any augmented proper relative braid. Adding a full twist to this dual braid shifts the homology of the index up by two dimensions.

Proof. Assume that $\mathbb{D}[\mathbf{u} \ \mathrm{rel} \ v^*]$ is the dual of an augmented braid in period $2p$ (the augmentation is required to keep the braid class bounded upon adding a full twist). The prior augmentation implies that the outer two strands of $\mathbb{D}v$ "maximally link" the remainder of the relative braid. The effect of adding a full twist to this braid can be realized by instead stabilizing $[\mathbf{u} \ \mathrm{rel} \ v^*]$ twice and then dualizing. The homological duality implies that for each connected component of the topological class,

$$\begin{aligned} CH_*(h(\mathbb{D}E^r(D)^2(\mathbf{u}\ \text{rel}\ v^*))) &\cong CH_{2np+2-*}(h(E^r(D)^2(\mathbf{u}\ \text{rel}\ v^*))) \\ &\cong CH_{2np+2-*}(h(\mathbf{u}\ \text{rel}\ v^*)) \\ &\cong CH_{*-2}(h(\mathbb{D}(\mathbf{u}\ \text{rel}\ v^*))), \end{aligned} \tag{9.5.24}$$

which gives the desired result for the index $\mathbf{h}$ via Theorem 9.28. ∎

▪ **9.30 Remark** *The homotopy version of* (9.5.24) *can be achieved by following a similar procedure as in §**??**. One obtains a double-suspension of the homotopy index, as opposed to a shift in homology.* ▪

▪ **9.31 Remark** *Given a braid class* $[\mathbf{u}]$ *of odd period* $p = 2d + 1$*, the image under* $\mathbb{D}$ *is* not *necessarily a discretized braid at all: without some symmetry condition, the braid will not "close up" at the ends. To circumvent this, define the dual of* $\mathbf{u}$ *to be the braid* $\mathbb{D}(\mathbf{u}^2)$ *— the dual of the period* $2p$ *extension of* $\mathbf{u}$*. The analogue of Theorem 9.28 above is that*

$$CH_*(\mathbf{h}(\text{SYM}(\mathbb{D}(\mathbf{u}\ \text{rel}\ v)));\mathbb{R}) \cong CH_{np-*}(\mathbf{h}(\mathbf{u}\ \text{rel}\ v);\mathbb{R}), \tag{9.5.25}$$

where SYM *denotes the subset of the braid class which consists of symmetric braids:* $u_i^\alpha = u_{2p-i}^\alpha$ *for all* i*.* ▪

9.6 Morse theory

It is clear that the Morse-theoretic content of the homotopy index on braids holds implications for the dynamics of parabolic flows and thus zeros of parabolic recurrence relations. With this in mind, we restrict ourselves to bounded proper braid classes.

Recall that the *characteristic polynomial* of an index pair (N, N^-) is the polynomial

$$CP_t(N) := \sum_{k \geq 0} \beta_k t^k; \qquad \beta_k(N) := \dim CH_k(N;) = \dim H_k(N, N^-;). \tag{9.6.26}$$

The *Morse relations* in the setting of the Conley index (see [**?**]) state that, if N has a Morse decomposition into distinct isolating subsets $\{N_a\}_{a=1}^C$, then

$$\sum_{a=1}^C CP_t(N_a) = CP_t(N) + (1+t)Q_t, \tag{9.6.27}$$

for some polynomial Q_t with *nonnegative* integer coefficients.

9.6.a The exact, nondegenerate case

For parabolic recurrence relations which satisfy (A3) (gradient type) it holds that if $h(\mathbf{u} \text{ rel } v) \neq 0$, then R has at least one fixed point in $[\mathbf{u} \text{ rel } v]$. Indeed, one has:

> **9.32 Lemma** For an exact nondegenerate parabolic flow on a bounded proper relative braid class, the sum of the Betti numbers β_k of h, as defined in (9.6.26), is a lower bound on the number of fixed points of the flow on that braid class.

The details of this standard Morse theory argument are provided for the sake of completeness. Choose Ψ^t a nondegenerate gradient parabolic flow on $[\mathbf{u} \text{ rel } v]$ (in particular, Ψ^t fixes v for all time). Enumerate the [finite number of] fixed points $\{\mathbf{u}_a\}_{a=1}^C$ of Ψ^t on this [bounded] braid class. By nondegeneracy, the fixed point set may be taken to be a Morse decomposition of $\mathrm{Inv}(N)$. The characteristic polynomial of each fixed point is merely $t^{\mu^*(\mathbf{u}_a)}$, where $\mu^*(\mathbf{u}_a)$ is the Morse co-index of $\mathbf{u}_a$. Substituting $t = 1$ into Equation (9.6.27) yields the lower bound

$$\#\mathrm{Fix}([\mathbf{u} \text{ rel } v], \Psi^t) \geq \sum_k \beta_k(h). \tag{9.6.28}$$

On the level of the topological braid invariant $\mathbf{h}$, one needs to sum over all the path components as follows. As in Theorem 9.17, choose period-d representatives $\mathbf{u}(j)$ (j from 0 to m) for each path component of the topological class $\{\mathbf{u} \text{ rel } \{v\}\}$. If we consider fixed points in the union $\cup_{j=0}^m [\mathbf{u}(j) \text{ rel } v]$, we obtain the following Morse inequalities from (9.6.28) and Theorem 9.17:

$$\#\mathrm{Fix}(\cup_{j=0}^m [\mathbf{u}(j) \text{ rel } v], \Psi^t) \geq \sum_k \beta_k(\mathbf{h}), \tag{9.6.29}$$

where $\beta_k(\mathbf{h})$ is the k^{th} Betti number of $\mathbf{h}(\mathbf{u} \text{ rel } v^*)$. Thus, again, the sum of the Betti numbers is a lower bound, with the proviso that some components may not contain any critical points.

If the topological class $\{\mathbf{u} \text{ rel } \{v\}\}$ is bounded the inequality (9.6.29) holds with the invariant $\mathbf{h}(\mathbf{u} \text{ rel } v)$.

9.6.b The exact, degenerate case

Here a coarse lower bound still exists.

> **9.33 Lemma** For an arbitrary exact parabolic flow on a bounded relative braid class, the number of fixed points is bounded below by the number of distinct nonzero monomials in the characteristic polynomial $CP_t(h)$.

Proof. Assuming that #Fix is finite, all critical points are isolated and form a Morse decomposition of $\mathrm{Inv}(N)$. The specific nature of parabolic recurrence relations reveals that the dimension of the null space of the linearized matrix at an isolated critical point is at most 2, see e.g. [**?**]. Using this fact Dancer proves [**?**], via the degenerate version of the Morse lemma due to Gromoll and Meyer, that $CH_k(\mathbf{u}_a) \neq 0$ for *at most* one index $k = k_0$. Equation (9.6.27) implies that,

$$\sum_{a=1}^{C_d} CP_t(\mathbf{u}_a) \geq CP_t(h) \tag{9.6.30}$$

on the level of polynomials. As the result of Dancer implies that for each a, $CP_t(\mathbf{u}_a) = At^k$, for some $A \geq 0$, it follows that the number of critical points needs to be *at least* the number of non-trivial monomials in $CP_t(h)$. ∎

As before, if we instead use the topological invariant $\mathbf{h}$ for $\{\mathbf{u} \ \text{rel} \ \{v^*\}\}$ we obtain that the number of monomials in $CP_t(\mathbf{h})$ is a lower bound for the total sum of fixed points over the topologically equivalent path-components.

More elaborate estimates in some cases can be obtained via the extension of the Conley index due to Floer [**?**].

9.6.c The non-exact case

If we consider parabolic recurrence relations that are not necessarily exact, the homotopy index may still provide information about solutions of $R = 0$. This is more delicate because of the possibility of periodic solutions for the flow $u_i' = R_i(u_{i-1}, u_i, u_{i+1})$. For example, if $CP_t(h) \bmod (1+t) = 0$, the index does not provide information about additional solutions for $R = 0$, as a simple counterexample shows. However, if $CP_t(h) \bmod (1+t) \neq 0$, then there exists at least one solution of $R = 0$ with the specified relative braid class. Specifically,

9.34 Lemma An arbitrary parabolic flow on a bounded relative braid class is forced to have a fixed point if $\chi(h) := CP_{-1}(h)$ is nonzero. If the flow is non-degenerate, then the number of fixed points is bounded below by the quantity

$$\left(CP_t(h) \bmod_{\mathbb{Z}^+[t]} (1+t)\right)\Big|_{t=1} \tag{9.6.31}$$

Proof. Set $N = \mathrm{cl}([\mathbf{u} \ \text{rel} \ \mathbf{v}])$. As the vector field R has no zeros at ∂N, the Brouwer degree, $\deg(R, N, 0)$, may be computed via a small perturbation $\widetilde{R}$ and is given by[5]

$$\deg(R, N, 0) := \sum_{\mathbf{u} \in N, \widetilde{R}(\mathbf{u}) = 0} \text{sign} \det(-d\widetilde{R}(\mathbf{u})).$$

[5]We choose to define the degree via $-d\widetilde{R}$ in order to simplify the formulae.

For a generic perturbation $\widetilde{R}$ the associated parabolic flow $\widetilde{\Psi}^t$ is a Morse-Smale flow **[?]**. The (finite) collection of rest points $\{\mathbf{u}_a\}$ and periodic orbits $\{\gamma_b\}$ of $\widetilde{\Psi}^t$ then yields a Morse decomposition of $\mathrm{Inv}(N)$, and the Morse inequalities are

$$\sum_a CP_t(\mathbf{u}_a) + \sum_b CP_t(\gamma_b) = CP_t(h) + (1+t)Q_t.$$

The indices of the fixed points are given by $CP_t(\mathbf{u}_a) = t^{\mu^*(\mathbf{u}_a)}$, where μ^* is the number of eigenvalues of $d\widetilde{R}(\mathbf{u}_a)$ with positive real part, and the indices of periodic orbits are given by $CP_t(\gamma_b) = (1+t)t^{\mu^*(\gamma_b)}$. Upon substitution of $t=-1$ we obtain

$$\begin{aligned} \deg(R,N,0) &= \deg(\widetilde{R},N,0) = \sum_a (-1)^{\mu^*(\mathbf{u}_a)} \\ &= \sum_a CP_{-1}(\mathbf{u}_a) = CP_{-1}(h) = \chi(h). \end{aligned}$$

Thus, if the Euler characteristic of h is non-trivial, then R has at least one zero in N.

In the generic case the Morse relations give even more information. One has $CP_t(h) = p_1(t) + (1+t)p_2(t)$, with $p_1, p_2 \in \mathbb{Z}_+[t]$, and $CP_t(h) \bmod_{\mathbb{Z}^+[t]} (1+t) = p_1(t)$. It then follows that $\sum_a CP_t(\mathbf{u}_a) \geq CP_t(h) \bmod_{\mathbb{Z}^+[t]} (1+t)$, proving the stated lower bound. ∎

9.7 Morse decompositions, Morse relations and connecting orbits

10 — Conservative Differential Equations

In Classical Mechanics the existence of closed integral curves of the Euler-Lagrange equations is a fundamental problem, eg. periodic motions in Celestial Mechanics. The Lagrangians that occurs in Classical mechanics are first-order Langrangians. In this chapter we study closed characteristics on second- and higher-order Lagrangian. The Periodic motions of higher-order Lagrangians are restricted to non-compact, connected, energy manifolds. We focus here on second-order Lagrangians and we utilize the Brouwer degree to show that the number of closed characteristics on a prescribed energy manifold is bounded below by its second Betti number, which is easily computable from the Lagrangian.[1][16]

10.1 Second-Order Lagrangian Systems

The Lagrangian formulation of Classical Mechanics in its simplest form is given by the Principle of Least Action:

$$\delta \int_I L(u,u')dt = 0,$$

where $L = L(u,u') = \frac{1}{2}m|u'|^2 - V(u)$ and $u\colon I \to \mathbb{R}^n$ describes the position of a particle. The (smooth) function V represents the potential energy and $\frac{1}{2}m|u'|^2$ the kinetic energy.

If we consider variations δu that are compactly supported on I_E then integra-

[1] This chapter of a summary of the results in Kalies and Vandervorst.

tion by parts yields

$$\begin{aligned}\delta\int_I L(u,u')dt &= \int_I \frac{\partial L}{\partial u}\delta u dt + \int_I \frac{\partial L}{\partial u'}\delta u' dt\\ &= \int_I \frac{\partial L}{\partial u}\delta u dt - \frac{d}{dt}\int_I \frac{\partial L}{\partial u'}\delta u dt = 0.\end{aligned}$$

This implies the Euler-Lagrange equations of motion

$$-\frac{d}{dt}\frac{\partial L}{\partial u'} + \frac{\partial L}{\partial u} = 0.$$

The Euler-Lagrange equation are conservative since the following expression is constant along orbits:

$$\frac{\partial L}{\partial u'}u' - L(u,u') = \text{constant}.$$

10.1 Exercise Show that the above expression is constants along solutions of the Euler-Lagrange equations. ■

Higher-order Lagrangian models are given by the following Principle of Least Action:

$$\delta\int_I L(u,u',u'',\cdots)dt = 0,$$

where $L = L(u,u',u'',\cdots)$ is called a higher-order Lagrangian. Of special interest are second-order Lagrangians $L = L(u,u',u'')$. In Physics second-oder Lagrangians are used in models of non-linear optics, higher-order phase transitions of solids, non-linear elasticity, etc. We refer the reader to [17][18][26] and the references therein for more information.

Second-order Lagrangian systems are defined variationally by extremizing action functionals of the form $J[u] = \int_I L(u,u',u'')\,dt$. The Euler-Lagrange equations of such systems are given by

$$\frac{d^2}{dt^2}\frac{\partial L}{\partial u''} - \frac{d}{dt}\frac{\partial L}{\partial u'} + \frac{\partial L}{\partial u} = 0, \tag{10.1.1}$$

and are in essence fourth-order differential equations.

10.2 Exercise Derive the above Euler-Lagrange equations of motion from the Principle of Least Action for second-order Lagrangians. ■

Under the natural hypothesis that L is convex in u'', a second-order Lagrangian system is equivalent to a two degrees of freedom Hamiltonian system in $\mathbb{R}^4$ endowed with its standard symplectic form ω. The Hamiltonian is given by

$$H(u,u',u'',u''') = \left(\frac{\partial L}{\partial u'} - \frac{d}{dt}\frac{\partial L}{\partial u''}\right)u' + \frac{\partial L}{\partial u''}u'' - L(u,u',u''). \tag{10.1.2}$$

The Hamiltonian is constant along orbits of the Euler-Lagrange equations (10.1.1).

> **10.3 Exercise** Show that the above Hamiltonian is constants along solutions of the Euler-Lagrange equations given by (10.1.1). ■

Introducing the symplectic coordinates $x = (u, v, p_u, p_v)$, the Hamiltonian becomes $H(x) = p_u v + L^*(u, v, p_v)$, where L^* is the Legendre transform[2] of L with respect to u''. Hamilton's equations of motion are equivent to (10.1.1) and yield a dynamical system ϕ^t on $\mathbb{R}^4$. The Hamiltonian H foliates $\mathbb{R}^4$ with three-dimensional energy manifolds $M_E = \{x \in \mathbb{R}^4 \mid H(x) = E\}$. These manifolds are invariant under the flow ϕ^t, and the dynamical behavior of a system can be studied on an individual energy manifold. An energy manifold is *regular* if E is a regular value of H. In the context of second order Lagrangians this is equivalent to the condition that $\partial_{u''} L(u, 0, 0) \neq 0$ whenever $L(u, 0, 0) + E = 0$. For more details on second order Lagrangians cf. [33]

Recently the analysis of periodic orbits, or closed characteristics, on given energy manifolds has become an important issue in the study of general Hamiltonian systems.[13][29, 36] Weinstein [36] conjectured in the 1970's, motivated by a novel result by Rabinowitz [29], that any compact hypersurface $M \in (\mathbb{R}^{2n}, \omega)$, with the additional requirement that

$$\alpha(\xi) \neq 0, \quad 0 \neq \xi \in \mathscr{E}_M,$$

for 1-form α with $d\alpha = \omega$, has at least one closed characteristic. Such manifolds are said to be of contact type in $(\mathbb{R}^4, \omega)$. The conjecture was proved by Viterbo.[34]

Energy manifolds M_E coming from second-order Lagrangians do not fit within this theory for two reasons. The energy manifolds for second-order Lagrangians are always non-compact and are not necessarily of contact type in $(\mathbb{R}^4, \omega)$. Even with a more general formulation via Reeb vector fields, the latter issue cannot be resolved necessarily cf. [4]

> **10.4 Exercise** Show that M_E is a non-compact set for every $E \in \mathbb{R}$. ■

10.2 Twist sytems

In this final third of the paper, we apply the developed machinery to the problem of forcing closed characteristics in *second* order Lagrangian systems of twist type. The vast literature on fourth order differential equations coming from second order Lagrangians includes many physical models in nonlinear elasticity, nonlinear

[2] The Legendre transform is defined by $L^*(u, v, p_v) = \max_{w \in \mathbb{R}} \{p_v w - L(u, v, w)\}$, which is well-defined due to the convexity of L in w.

optics, physics of solids, Ginzburg-Landau equations, etc. (see §**??**). In this context we mention the work of [**?**, **?**, **?**, **?**].

We recall from §**??** that closed characteristics at an energy level E are concatenations of monotone laps between minima and maxima $(u_i)_{i\in\mathbb{Z}}$, which are periodic sequences with even period $2p$. The extrema are restricted to the set U_E, whose connected components are denoted by I_E: interval components (see §**??** for the precise definition). The problem of finding closed characteristics can, in most cases, be formulated as a finite dimensional variational problem on the extrema (u_i). The following *twist hypothesis*, introduced in [**?**], is key:

(T): $\inf\{J_E[u]=\int_0^\tau\big(L(u,u_x,u_{xx})+E\big)dx\,|\,u\in X_\tau(u_1,u_2),\,\tau\in\mathbb{R}^+\}$ *has a minimizer* $u(x;u_1,u_2)$ *for all* $(u_1,u_2)\in\{I_E\times I_E\mid u_1\neq u_2\}$*, and* u *and* τ *are* C^1*-smooth functions of* (u_1,u_2).

Here $X_\tau=X_\tau(u_1,u_2)=\{u\in C^2([0,\tau])\mid u(0)=u_1,\ u(\tau)=u_2,\ u_x(0)=u_x(\tau)=0 \text{ and } u_x|_{(0,\tau)}>0\}$.

Hypothesis (T) is a weaker version of the hypothesis that assumes that the monotone laps between extrema are unique (cf. [**?**, **?**, **?**]). Hypothesis (T) is valid for large classes of Lagrangians L. For example, if $L(u,v,w)=\frac{1}{2}w^2+K(u,v)$, the following two inequalities ensure the validity of (T):

(a) $\frac{\partial K}{\partial v}v-K(u,v)-E\leq 0$, and

(b) $\frac{\partial^2 K}{\partial v^2}|v|^2-\frac{5}{2}\big\{\frac{\partial K}{\partial v}v-K(u,v)-E\big\}\geq 0$ for all $u\in I_E$ and $v\in\mathbb{R}$.

Many physical models, such as the Swift-Hohenberg equation (**??**), meet these requirements, although these conditions are not always met. In those cases numerical calculations still predict the validity of (T), which leaves the impression that the results obtained for twist systems carry over to many more systems for which Hypothesis (T) is hard to check.[3] For these reasons twist systems play a important role in understanding second order Lagrangian systems. For a direct application of this see [**?**].

The existence of minimizing laps is valid under very mild hypotheses on K (see [**?**]). In that case (b) above is enough to guarantee the validity of (T). An example of a Lagrangian that satisfies (T), but not (a) is given by the Erickson beam-model [**?**, **?**, **?**] $L(u,u_x,u_{xx})=\frac{\alpha}{2}|u_{xx}|^2+\frac{1}{4}(|u_x|^2-1)^2+\frac{\beta}{2}u^2$.

10.2.a Discretization of the variational principle

We commence by repeating the underlying variational principle for obtaining closed characteristics as described in [**?**]. In the present context a *broken geodesic* is a C^2-concatenation of monotone laps (alternating between increasing and decreasing laps) given by Hypothesis (T). A closed characteristic u at energy level E is a (C^2-smooth) function $u:\ [0,\tau]\to\mathbb{R}$, $0<\tau<\infty$, which is stationary for the action $J_E[u]$

[3] Another method to implement the ideas used in this paper is to set up a curve-shortening flow for second order Lagrangian systems in the (u,u') plane.

with respect to variations $\delta u \in C^2_{\rm per}(0,\tau)$, and $\delta\tau \in \mathbb{R}^+$, and as such is a 'smooth broken geodesic'.

The following result, a translation of results implicit in [**?**], is the motivation and basis for the applications of the machinery in the first two-thirds of this paper.

10.5 Theorem Extremal points $\{u_i\}$ for bounded solutions of second order Lagrangian twist systems are solutions of an exact parabolic recurrence relation with the constraints that (i) $(-1)^i u_i < (-1)^i u_{i+1}$; and (ii) the recurrence relation blows up along any sequence satisfying $u_i = u_{i+1}$.

Proof: For simplicity, we restrict to the case of a nonsingular energy level E: for singular energy levels, a slightly more involved argument is required. Denote by I the interior of I_E, and by $\Delta(I) = \Delta := \{(u_1,u_2) \in I \times I \mid u_1 = u_2\}$ the diagonal. Then define the *generating function*

$$S : (I \times I) - \Delta \to \mathbb{R} \quad ; \quad S(u_1,u_2) := \int_0^\tau \big(L(u,u_x,u_{xx}) + E\big)dx; \tag{10.2.3}$$

the action of the minimizing lap from u_1 to u_2. That S is a well-defined function is the content of Hypothesis (T). The *action functional* associated to S for a period $2p$ system is the function

$$W_{2p}(\mathbf{u}) := \sum_{i=0}^{2p-1} S(u_i,u_{i+1}).$$

Several properties of S follow from [**?**]:

(a) [*smoothness*] $S \in C^2(I \times I \backslash \Delta)$.

(b) [*monotonicity*] $\partial_1\partial_2 S(u_1,u_2) > 0$ for all $u_1 \neq u_2 \in I$.

(c) [*diagonal singularity*] $\lim_{u_1 \nearrow u_2} -\partial_1 S(u_1,u_2) = \lim_{u_2 \searrow u_1} \partial_2 S(u_1,u_2) = \lim_{u_1 \searrow u_2} \partial_1 S(u_1,u_2) = \lim_{u_2 \nearrow u_1} -\partial_2 S(u_1,u_2) = +\infty$.

In general the function $\partial_1 S(u_1,u_2)$ is strictly increasing in u_2 for all $u_1 \le u_2 \in I_E$, and similarly $\partial_2 S(u_1,u_2)$ is strictly increasing in u_1. The function S also has the additional property that $S|_\Delta \equiv 0$.

Critical points of W_{2p} satisfy the exact recurrence relation

$$R_i(u_{i-1},u_i,u_{i+1}) := \partial_2 S(u_{i-1},u_i) + \partial_1 S(u_i,u_{i+1}) = 0, \tag{10.2.4}$$

where $R_i(r,s,t)$ is both well-defined and C^1 on the domains

$$\Omega_i \quad = \quad \{(r,s,t) \in I^3 \mid (-1)^{i+1}(s-r) > 0,\ (-1)^{i+1}(s-t) > 0\},$$

by Property (a). The recurrence function R is periodic with $d = 2$, as are the domains Ω.[4] Property (b) implies that Axiom (A1) is satisfied. Indeed, $\partial_1 R_i = \partial_1\partial_2 S(u_{i-1},u_i) > 0$, and $\partial_3 R_i = \partial_1\partial_2 S(u_i,u_{i+1}) > 0$.

[4] We could also work with sequences $\mathbf{u}$ that satisfy $(-1)^{i+1}(u_{i+1} - u_i) > 0$.

Property (c) provides information about the behavior of R at the diagonal boundaries of Ω_i, namely,

$$\begin{aligned} \lim_{s\searrow r} R_i(r,s,t) = \lim_{s\searrow t} R_i(r,s,t) = +\infty \\ \lim_{s\nearrow r} R_i(r,s,t) = \lim_{s\nearrow t} R_i(r,s,t) = -\infty \end{aligned} \tag{10.2.5}$$

The parabolic recurrence relations generated by second order Lagrangians are defined on the constrained polygonal domains Ω_i.

10.6 Definition A parabolic recurrence relation is said to be of up-down type if (10.2.5) is satisfied.

In the next subsection we demonstrate that the up-down recurrence relations can be embedded into the standard theory as developed in §**??**-§9.6.

10.2.b Up-down restriction

The variational set-up for second order Lagrangians introduces a few complications into the scheme of parabolic recurrence relations as discussed in §**??**-§9.6. The problem of boundary conditions will be considered in the following section. Here, we retool the machinery to deal with the fact that maxima and minima are forced to alternate. Such braids we call *up-down* braids.[5]

10.7 Definition The spaces of general/nonsingular/singular up-down braid diagrams are defined respectively as:

$$\begin{aligned} \bar{\mathcal{E}}^n_{2p} &:= \bar{\mathcal{D}}^n_{2p} \cap \left\{ \mathbf{u} : (-1)^i \left(u^\alpha_{i+1} - u^\alpha_i\right) > 0 \quad \forall i,\alpha \right\}, \\ \mathcal{E}^n_{2p} &:= \mathcal{D}^n_{2p} \cap \left\{ \mathbf{u} : (-1)^i \left(u^\alpha_{i+1} - u^\alpha_i\right) > 0 \quad \forall i,\alpha \right\}, \\ \Sigma_{\mathcal{E}} &:= \bar{\mathcal{E}}^n_{2p} - \mathcal{E}^n_{2p}. \end{aligned}$$

Path components of $\mathcal{E}^n_{2p}$ comprise the up-down braid types $[\mathbf{u}]_{\mathcal{E}}$, and path components in $\mathcal{E}^n_{2p}$ rel $\mathbf{v}$ comprise the relative up-down braid types $[\mathbf{u} \text{ rel } \mathbf{v}]_{\mathcal{E}}$.

The set $\bar{\mathcal{E}}^n_{2p}$ has a boundary in $\bar{\mathcal{D}}^n_{2p}$

$$\partial\bar{\mathcal{E}}^n_{2p} = \partial\left(\bar{\mathcal{D}}^n_{2p} \cap \left\{ \mathbf{u} : (-1)^i \left(u^\alpha_{i+1} - u^\alpha_i\right) \geq 0 \quad \forall i,\alpha \right\}\right) \tag{10.2.6}$$

Such braids, called *horizontal singularities*, are not included in the definition of $\bar{\mathcal{E}}^n_{2p}$ since the recurrence relation (10.2.4) does *not* induce a well-defined flow on the boundary $\partial\bar{\mathcal{E}}^n_{2p}$.

[5]The more natural term *alternating* has an entirely different meaning in knot theory.

10.8 Lemma For any parabolic flow of up-down type on $\bar{\mathcal{E}}^n_{2p}$, the flow blows up in a neighborhood of $\partial\bar{\mathcal{E}}^n_{2p}$ in such a manner that the vector field points into $\bar{\mathcal{E}}^n_{2p}$. All of the conclusions of Theorem **??** hold upon considering the ϵ-closure of braid classes $[\mathbf{u}\ \ \mathrm{rel}\ \ v]_{\mathcal{E}}$ in $\bar{\mathcal{E}}^n_{2p}$, denoted

$$\mathrm{cl}_{\bar{\mathcal{E}},\epsilon}[\mathbf{u}\ \ \mathrm{rel}\ \ v]_{\mathcal{E}} := \Big\{\mathbf{u}\ \ \mathrm{rel}\ \ v \in \mathrm{cl}_{\bar{\mathcal{E}}}[\mathbf{u}\ \ \mathrm{rel}\ \ v]_{\mathcal{E}}\ :\ (-1)^i\big(u^\alpha_{i+1} - u^\alpha_i\big) \geq \epsilon \quad \forall i,\alpha\Big\},$$

for all $\epsilon > 0$ sufficiently small.

Proof: The proof that any parabolic flow Ψ^t of up-down type acts here so as to strictly decrease the word metric at singular braids is the same proof as used in Proposition **??**. The only difficulty arises in what happens at the boundary of $\bar{\mathcal{E}}^n_{2p}$: we must show that Ψ^t respects the up-down restriction in forward time.

Define the function

$$\epsilon(\mathbf{u}) = \min_{i,\alpha} |u^\alpha_i - u^\alpha_{i+1}|.$$

Clearly, if $\epsilon(\mathbf{u}) = 0$, then $\mathbf{u} \in \partial\bar{\mathcal{E}}^n_{2p}$. Let $\mathbf{u} \in \bar{\mathcal{E}}^n_{2p}$, and consider the evolution $\Psi^t(\mathbf{u})$, $t > 0$. We compute $\frac{d}{dt}\epsilon(\Psi^t(\mathbf{u}))$ as $\epsilon(\Psi^t(\mathbf{u}))$ becomes small. Using (10.2.4) it follows that

$$\begin{aligned}
\frac{d}{dt}(u^\alpha_i - u^\alpha_{i+1}) &= R_i(u^\alpha_{i-1}, u^\alpha_i, u^\alpha_{i+1}) - R_{i+1}(u^\alpha_i, u^\alpha_{i+1}, u^\alpha_{i+2}) \to \infty, \\
&\quad \text{as} \quad u_i \searrow u_{i+1}, \quad (i \ \text{ odd}), \\
\frac{d}{dt}(u^\alpha_{i+1} - u^\alpha_i) &= R_{i+1}(u^\alpha_i, u^\alpha_{i+1}, u^\alpha_{i+2}) - R_i(u^\alpha_{i-1}, u^\alpha_i, u^\alpha_{i+1}) \to \infty, \\
&\quad \text{as} \quad u_i \nearrow u_{i+1}, \quad (i \ \text{ even}).
\end{aligned}$$

These inequalities show that $\frac{d}{dt}\epsilon(\Psi^t(\mathbf{u})) > 0$ as soon as $\epsilon(\Psi^t(\mathbf{u}))$ becomes too small. Due to the boundedness of $[\mathbf{u}\ \ \mathrm{rel}\ \ v]_{\mathcal{E}}$ and the infinite repulsion at $\partial\bar{\mathcal{E}}^n_{2p}$, we can choose a uniform $\epsilon(\mathbf{u}\ \ \mathrm{rel}\ \ v) > 0$ so that $\frac{d}{dt}\epsilon(\Psi^t(\mathbf{u})) > 0$ for $\epsilon(\Psi^t(\mathbf{u})) \leq \epsilon(\mathbf{u}\ \ \mathrm{rel}\ \ v)$, and thus $\mathrm{cl}_{\bar{\mathcal{E}},\epsilon}[\mathbf{u}\ \ \mathrm{rel}\ \ v]_{\mathcal{E}}$ is an isolating neighborhood for all $0 < \epsilon \leq \epsilon(\mathbf{u}\ \ \mathrm{rel}\ \ v)$.

10.2.c Universality for up-down braids

We now show that the topological information contained in up-down braid classes can be continued to the canonical case described in §**??**. As always, we restrict attention to proper, bounded braid classes, proper being defined as in Definition **??**, and bounded meaning that the set $[\mathbf{u}\ \ \mathrm{rel}\ \ v]_{\mathcal{E}}$ is bounded in $\bar{\mathcal{D}}^n_{2p}$. Note that an up-down braid class $[\mathbf{u}\ \ \mathrm{rel}\ \ v]_{\mathcal{E}}$ can sometimes be bounded while $[\mathbf{u}\ \ \mathrm{rel}\ \ v]$ is not. To bounded proper up-down braids we assign a homotopy index. From Lemma 10.8 it follows that for ϵ sufficiently small the set $N_{\mathcal{E},\epsilon} := \mathrm{cl}_{\bar{\mathcal{E}},\epsilon}[\mathbf{u}\ \ \mathrm{rel}\ \ v]_{\mathcal{E}}$ is an isolating neighborhood in $\bar{\mathcal{E}}^n_{2p}$ whose Conley index,

$$h(\mathbf{u}\ \ \mathrm{rel}\ \ v, \mathcal{E}) := h(N_{\mathcal{E},\epsilon}),$$

is well-defined with respect to any parabolic flow Ψ^t generated by a parabolic recurrence relation of up-down type, and is independent of ϵ. As before, non-triviality of $h(N_{\mathcal{E},\epsilon})$ implies existence of a non-trivial invariant set inside $N_{\mathcal{E},\epsilon}$ (see §10.2.d).

The obvious question is what relationship holds between the homotopy index $h(\mathbf{u} \text{ rel } v, \mathcal{E})$ and that of a braid class without the up-down restriction. To answer this, augment the skeleton v as follows: define $v^* = v \cup v^- \cup v^+$, where

$$v_i^- := \min_{\alpha,i} v_i^\alpha - 1 + (-1)^{i+1}, \quad v_i^+ := \max_{\alpha,i} v_i^\alpha + 1 + (-1)^{i+1}.$$

The topological braid class $\{\mathbf{u} \text{ rel } v^*\}$ is bounded and proper. Indeed, boundedness follows from adding the strands $v^\pm$ which bound $\mathbf{u}$, since $\min_{\alpha,i} v_i^\alpha \le u_i^\alpha \le \max_{\alpha,i} v_i^\alpha$. Properness is satisfied since $\{\mathbf{u} \text{ rel } v\}$ is proper.

10.9 Theorem For any bounded proper up-down braid class $[\mathbf{u} \text{ rel } v]_{\mathcal{E}}$ in $\mathcal{E}_{2p}^n$ rel v,

$$h(\mathbf{u} \text{ rel } v, \mathcal{E}) = h(\mathbf{u} \text{ rel } v^*).$$

Proof. From Lemma **??** in Appendix A we obtain a parabolic recurrence relation R^0 (not necessarily up-down type) for which v^* is a solution. We denote the associated parabolic flow by Ψ_0^t. Define two functions k_1 and k_2 in $C^1(\mathbb{R})$, with $k_1' \ge 0 \ge k_2'$, and $k_1(\tau) = 0$ for $\tau \le -2\delta$, $k_1(-\delta) \ge K$, and $k_2(\tau) = 0$ for $\tau \ge 2\delta$, $k_2(\delta) \ge K$, for some $\delta > 0$ and $K > 0$ to be specified later. Introduce a new recurrence function $R_i^1(r,s,t) = R_i^0(r,s,t) + k_2(s-r) + k_1(t-s)$ for i odd, and $R_i^1(r,s,t) = R_i^0(r,s,t) - k_1(s-r) - k_2(t-s)$ for i even. The associated parabolic flow will be denoted by Ψ_1^t, and $\Psi_1^t(v^*) = v^*$ by construction by choosing δ sufficiently small. Indeed, if we choose $\delta < \epsilon(v)$, the augmented skeleton is a fixed point for Ψ_1^t.

Since the braid class $[\mathbf{u} \text{ rel } v^*]$ is bounded and proper, $N_1 = \text{cl}[\mathbf{u} \text{ rel } v^*]$ is an isolating neighborhood with invariant set $\text{Inv}(N_1)$. If we choose K large enough, and δ sufficiently small, then the invariant set $\text{Inv}(N_1)$ lies entirely in $\text{cl}_{\bar{\mathcal{E}},\epsilon}[\mathbf{u} \text{ rel } v^*]_{\mathcal{E}} = \text{cl}_{\bar{\mathcal{E}},\epsilon}[\mathbf{u} \text{ rel } v]_{\mathcal{E}} = N_{\mathcal{E},\epsilon}$. Indeed, for large K we have that for each i, $R_i^1(r,s,t)$ has a fixed sign on the complement of $N_{\mathcal{E},\epsilon}$. Therefore, $h(\mathbf{u} \text{ rel } v^*) = h(N_1) = h(N_{\mathcal{E},\epsilon})$. Now restrict the flow Ψ_1^t to $N_{\mathcal{E},\epsilon} \subset \bar{\mathcal{E}}_{2p}^n$ rel v. We may now construct a homotopy between Ψ_1^t and Ψ^t, via $(1-\lambda)R + \lambda R^1$ (see Appendix A), where R and the associated flow Ψ^t are defined by (10.2.4). The braid v^* is stationary along the homotopy and therefore

$$h(N_1) = h(N_{\mathcal{E},\epsilon}, \Psi_1^t) = h(N_{\mathcal{E},\epsilon}, \Psi^t),$$

which proves the theorem.

We point out that similar results can be proved for other domains Ω_i with various boundary conditions. The key observation is that the up-down constraint is really just an addition to the braid skeleton.

10.2.d Morse theory

For bounded proper up-down braid classes $[\mathbf{u} \ \mathrm{rel} \ v]_{\mathcal{E}}$ the Morse theory of §9.6 applies. Combining this with Lemma 10.8 and Theorem 10.9, the topological information is given by the invariant $\mathbf{h}$ of the topological braid type $\{\mathbf{u} \ \mathrm{rel} \ \{v^*\}\}$.

10.10 Corollary On bounded proper up-down braid classes, the total number of fixed points of an exact parabolic up-down recurrence relation is bounded below by the number of monomials in the critical polynomial $CP_t(\mathbf{h})$ of the homotopy index.

Proof. Since all critical point are contained in $N_{\mathcal{E},\epsilon}$ the corollary follows from the Lemmas 9.33, 10.8 and Theorem 10.9. ∎

10.3 Multiplicity of closed characteristics

We now have assembled the tools necessary to prove Theorem 10.1.2, the general forcing theorem for closed characteristics in terms of braids, and Theorems **??** and **??**, the application to singular and near-singular energy levels. Given one or more closed characteristics, we keep track of the braiding of the associated strands, including at will any period-two shifts. Fixing these strands as a skeleton, we add hypothetical free strands and compute the homotopy index. If nonzero, this index then forces the existence of the free strand as an existing solution, which, when added to the skeleton, allows one to iterate the argument with the goal of producing an infinite family of forced closed characteristics.

The following lemma (whose proof is straightforward and thus omitted) will be used repeatedly for proving existence of closed characteristics.

10.11 Lemma Assume that R is a parabolic recurrence relation on $\mathcal{D}_d^n$ with $\mathbf{u}$ a solution. Then, for each integer $N > 1$, there exists a lifted parabolic recurrence relation on $\mathcal{D}_{Nd}^n$ for which every lift of $\mathbf{u}$ is a solution. Furthermore, any solution to the lifted dynamics on $\mathcal{D}_{Nd}^n$ projects to some period-d solution. [a]

[a] This does not imply a d-periodic solution, but merely a braid diagram $\mathbf{u}$ of period d.

The primary difficulties in the proof of the forcing theorems are (i) computing the index (we will use all features of the machinery developed thus far, including stabilization and duality); and (ii) asymptotics/boundary conditions related to the three types of closed interval components I_E: a compact interval, the entire real line, and the semi-infinite ray.

All of the forcing theorems are couched in a little braid-theoretic language:

10.12 Definition The *intersection number* of two strands $\mathbf{u}^\alpha$, $\mathbf{u}^{\alpha'}$ of a braid $\mathbf{u}$ is the number of crossings in the braid diagram, denoted

$$\iota(\mathbf{u}^\alpha, \mathbf{u}^{\alpha'}) := \#\text{ of crossings of strands}$$

The *trivial braid* on n strands is any braid (topological or discrete) whose braid diagram has no crossings whatsoever, i.e., $\iota(\mathbf{u}^\alpha, \mathbf{u}^{\alpha'}) = 0$, for all α, α'. The *full-twist braid* on n strands, is the braid of n connected components, each of which has exactly two crossings with every other strand, i.e., $\iota(\mathbf{u}^\alpha, \mathbf{u}^{\alpha'}) = 2$ for all $\alpha \neq \alpha'$.

Among discrete braids of period two, the trivial braid and the full twist are duals in the sense of §9.5.

10.3.a Compact interval components

Let E be a regular energy level for which the set U_E contains a compact interval component I_E.

10.13 Theorem Suppose that a twist system with compact I_E possesses one or more closed characteristics which, as a discrete braid diagram, form a nontrivial braid. Then there exists an infinity of non-simple, geometrically distinct closed characteristics in I_E.

In preparation for the proof of Theorem 10.13 we state a technical lemma, whose [short] proof may be found in [**?**].

10.14 Lemma Let $I_E = [u_-, u_+]$, then there exists a $\delta_0 > 0$ such that

1. $R_1(u_- + \delta, u_-, u_- + \delta) > 0$, $R_1(u_+, u_+ - \delta, u_+) < 0$, and
2. $R_2(u_-, u_- + \delta, u_-) > 0$, $R_2(u_+ - \delta, u_+, u_+ - \delta) < 0$,

for any $0 < \delta \leq \delta_0$.

Proof of Theorem 10.13. Via Theorem 10.5, finding closed characteristics is equivalent to solving the recurrence relation given by (10.2.4). Define the domains

$$\Omega_i^\delta = \begin{cases} \{(u_{i-1}, u_i, u_{i+1}) \in I_E^3 \mid u_- + \delta < u_{i\pm1} < u_i - \delta/2 < u_+ - \delta\}, & i \text{ odd}, \\ \{(u_{i-1}, u_i, u_{i+1}) \in I_E^3 \mid u_- + \delta < u_i + \delta/2 < u_{i\pm1} < u_+ - \delta\}, & i \text{ even}, \end{cases}$$

For any integer $p \geq 1$ denote by Ω_{2p} the set of $2p$-periodic sequences $\{u_i\}$ for which $(u_{i-1}, u_i, u_{i+1}) \in \Omega_i^\delta$. By Lemma 10.14, choosing $0 < \delta < \delta_0$ small enough forces the vector field $R = (R_i)$ to be everywhere transverse to $\partial\Omega_{2p}$, making Ω_{2p} positively invariant for the induced parabolic flow Ψ^t.

By Lemma 10.11, one can lift the assumed solution(s) to a pair of period $2p$ single-stranded solutions to (10.2.4), v^1 and v^2, satisfying $\iota(v^1, v^2) \neq 0$, for some

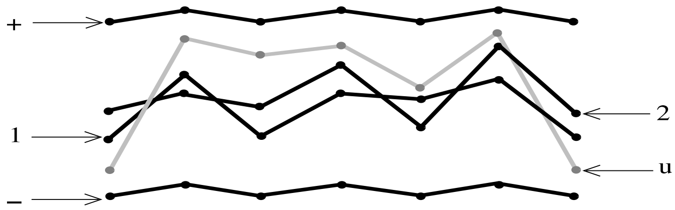


Figure 10.1: A representative braid class for the compact case: $q=1$, $r=4$, $2p=6$.

$p \geq 1$. Define the cones

$$\begin{aligned} C_- &= \{\mathbf{u} \in \Omega_{2p} \mid u_i \leq v_i^\alpha,\ \alpha = 1,2\}, \text{ and} \\ C_+ &= \{\mathbf{u} \in \Omega_{2p} \mid u_i \geq v_i^\alpha,\ \alpha = 1,2\}. \end{aligned}$$

The combination of the facts $\iota(v^1,v^2) = r > 0$, Axiom (A1), and the behavior of R on $\partial\Omega_{2p}$ implies that on the boundaries of the cones C_- and C_+ the vector field R is everywhere transverse and pointing inward. Therefore, C_- and C_+ are also positively invariant with respect to the parabolic flow Ψ^t. Consequently, W_{2p} has global maxima v^- and v^+ on $\mathrm{int}(C_-)$ and $\mathrm{int}(C_+)$ respectively. The maxima v^- and v^+ have the property that $v_i^- < v_i^\alpha < v_i^+$, $\alpha = 1,2$. As a braid diagram, $v = \{v^1, v^2, v^-, v^+\}$ is a stationary skeleton for the induced parabolic flow Ψ^t.

Having found the solutions v^- and v^+ we now choose a compact interval $I \subsetneq I_E$, such that the skeletal strands are all contained in I. In this way we obtain a proper parabolic flow (circumventing boundary singularities) which can be extended to a parabolic flow on $\bar{\mathcal{E}}^1_{2p}$ rel v. Let $[\mathbf{u} \text{ rel } v]_{\mathcal{E}}$ be the relative braid class with a period $2p$ free strand $\mathbf{u} = \{u_i\}$ which links the strands v^1 and v^2 with intersection number $2q$ while satisfying $v_i^- < u_i < v_i^+$: see Fig. 10.1 below.

As an up-down braid class, $[\mathbf{u} \text{ rel } v]_{\mathcal{E}}$ is a bounded proper braid class provided $0 < 2q < r \leq 2p$, and the Morse theory discussed in §9.6 and §10.2.d then requires the evaluation of the invariant $\mathbf{h}$ of the topological class $\{\mathbf{u} \text{ rel } \{v^*\}\}$. In this case, since $h(\mathbf{u} \text{ rel } v, \mathcal{E}) = h(\mathbf{u} \text{ rel } v^*) = h(\mathbf{u} \text{ rel } v)$, augmentation is not needed, and $\mathbf{h}(\mathbf{u} \text{ rel } v^*) = \mathbf{h}(\mathbf{u} \text{ rel } v)$. The nontriviality of the homotopy index $\mathbf{h}$ is given by the following lemma, whose proof we delay until §10.5.

10.15 Lemma The Conley homology of $\mathbf{h}(\mathbf{u} \text{ rel } v)$ is given by:

$$CH_k(\mathbf{h}) = \begin{cases} \mathbb{R} & : \quad k = 2q-1,\ 2q \\ 0 & : \quad \text{else.} \end{cases} \tag{10.3.7}$$

In particular $CP_t(\mathbf{h}) = t^{2q-1}(1+t)$.

From the Morse theory of Corollary 10.10 we derive that for each q satisfying $0 < 2q < r \leq 2p$ there exist at least two distinct period-$2p$ solutions of (10.2.4), which generically are of index $2q$ and $2q-1$. In this manner, the number of solutions depends on r and p. To construct infinitely many, we consider m-fold coverings of the skeleton v, i.e., one periodically extends v to a skeleton contained in $\mathcal{E}^4_{2pm}$, $m \geq 1$. Now q must satisfy $0 < 2q < rm \leq 2pm$. By choosing triples (q,p,m) such that (q,pm) are relative prime, we obtain the same Conley homology as above, and therefore an infinity of pairs of geometrically distinct solutions of (10.2.4), which, via Lemma 10.11 and Theorem 10.5 yield an infinity of closed characteristics.

Note that if we set $q_m = q$ and $p_m = pm$, then the admissible ratios $\frac{q_m}{p_m}$ for finding closed characteristics are determined by the relation

$$0 < \frac{q_m}{p_m} < \frac{r}{2p}. \tag{10.3.8}$$

Thus if v^1 and v^2 are maximally linked, i.e. $r = 2p$, then closed characteristics exist for all ratios in $\mathcal{Q}^\epsilon \cap (0,1)$.

10.3.b Non-compact interval components: $I_E = \mathbb{R}$

On non-compact interval components, closed characteristics need not exist. An easy example of such a system is given by the quadratic Lagrangian $L = \frac{1}{2}|u_{xx}|^2 + \frac{\alpha}{2}|u_x|^2 + \frac{1}{2}|u|^2$, with $\alpha > -2$. Clearly $I_E = \mathbb{R}$ for all $E > 0$, and the Lagrangian system has no closed characteristics for those energy levels. For $\alpha < -2$ the existence of closed characteristics strongly depends on the eigenvalues of the linearization around 0. To treat non-compact interval components, some prior knowledge about asymptotic behavior of the system is needed. We adopt an asymptotic condition shared by most physical Lagrangians: *dissipativity*.

10.16 Definition A second order Lagrangian system is *dissipative* on an interval component $I_E = \mathbb{R}$ if there exist pairs $u_1^* < u_2^*$, with $-u_1^*$ and u_2^* arbitrarily large, such that

$$\begin{aligned} -\partial_1 S(u_1^*, u_2^*) > 0, \quad \partial_2 S(u_1^*, u_2^*) > 0, \quad \text{and} \\ \partial_1 S(u_2^*, u_1^*) > 0, \quad -\partial_2 S(u_2^*, u_1^*) > 0. \end{aligned}$$

Dissipative Lagrangians admit a strong forcing theorem:

10.17 Theorem Suppose that a dissipative twist system with $I_E = \mathbb{R}$ possesses one or more closed characteristic(s) which, as discrete braid diagram in the period-two projection, forms a link which is not a full-twist (Definition 10.12). Then there exists an infinity of non-simple, geometrically distinct closed characteristics in I_E.

Proof. After taking the p-fold covering of the period-two projection for some $p \geq 1$, the hypotheses imply the existence two sequences v^1 and v^2 that form a braid diagram in $\mathcal{E}^2_{2p}$ whose intersection number is not maximal, i.e. $0 \leq \iota(v^1, v^2) = r < 2p$. Following Definition 10.16, choose $I = [u_1^*, u_2^*]$, with $u_1^* < u_2^*$ such that

Figure 10.2: A representative braid class for the case $I_E = \mathbb{R}$: $q = 2$, $r = 1$, $2p = 6$.

$u_1^* < v_i^1, v_i^2 < u_2^*$ for all i, and let Ω_i^δ and Ω_{2p} be as in the proof of Theorem 10.13, with u_1^* and u_2^* playing the role of $u_- + \delta$ and $u_+ - \delta$ respectively for some $\delta > 0$ small. Furthermore define the set

$$C := \{\mathbf{u} \in \Omega_{2p} \mid \iota(\mathbf{u}, v^1) = \iota(\mathbf{u}, v^2) = 2p\}.$$

Since $0 \leq \iota(v^1, v^2) < 2p$, the vector field R given by (10.2.4) is transverse to ∂C. Moreover, the set C is contractible, compact, and R is pointing outward at the boundary ∂C. The set C is therefore negatively invariant for the induced parabolic flow Ψ^t. Consequently, there exists a global minimum v^3 in the interior of C. Define the skeleton v to be $v := \{v^1, v^2, v^3\}$.

Consider the up-down relative braid class $[\mathbf{u} \text{ rel } v]_\mathcal{E}$ described as follows: choose $\mathbf{u}$ to be a $2p$-periodic strand with $(-1)^i u_i \geq (-1)^i v_i^3$, such that $\mathbf{u}$ has intersection number $2q$ with each of the strands $v^1 \cup v^2$, $0 \leq r < 2q < 2p$, as in Fig. 10.2. For $p \geq 2$, $[\mathbf{u} \text{ rel } v]_\mathcal{E}$ is a bounded proper up-down braid class. As before, in order to apply the Morse theory of Corollary 10.10, it suffices to compute the homology index of the topological braid class $\{\mathbf{u} \text{ rel } \{v^*\}\}$:

10.18 Lemma The Conley homology of $\mathbf{h}(\mathbf{u} \text{ rel } v^*)$ is given by:

$$CH_k(\mathbf{h}) = \begin{cases} \mathbb{R} & : \quad k = 2q,\ 2q+1 \\ 0 & : \quad \text{else} \end{cases} \tag{10.3.9}$$

In particular $CP_t(\mathbf{h}) = t^{2q}(1+t)$.

By the same covering/projection argument as in the proof of Theorem 10.13, infinitely many solutions are constructed within the admissible ratios

$$\frac{r}{2p} < \frac{q_m}{p_m} < 1. \tag{10.3.10}$$

Theorem 10.17 also implies that the existence of a single *non-simple* closed characteristic yields infinitely many other closed characteristics. In the case of two unlinked closed characteristics all possible ratios in $\mathcal{Q}^\epsilon \cap (0,1)$ can be realized.

10.3.c Half spaces $I_E \simeq \mathbb{R}^\pm$

The case $I_E = [\bar{u}, \infty)$ (or $I_E = (-\infty, \bar{u}]$) shares much with both the compact case and the the case $I_E = \mathbb{R}$. Since these I_E are non-compact we again impose a dissipativity condition.

10.19 Definition A second order Lagrangian system is *dissipative* on an interval component $I_E = [\bar{u}, \infty)$ if there exist arbitrarily large points $u^* > \bar{u}$ such that

$$\begin{aligned} &\partial_1 S(\bar{u}, u^*) > 0, \quad \partial_2 S(\bar{u}, u^*) > 0, \quad \text{and} \\ &\partial_1 S(u^*, \bar{u}) > 0, \quad \partial_2 S(u^*, \bar{u}) > 0. \end{aligned}$$

For dissipative Lagrangians we obtain the same general result as Theorem 10.13.

10.20 Theorem Suppose that a dissipative twist system with $I_E \simeq \mathbb{R}^{\pm}$ possesses one or more closed characteristics which, as a discrete braid diagram, form a nontrivial braid. Then there exists an infinity of non-simple, geometrically distinct closed characteristics in I_E.

Proof. We will give an outline of the proof since the arguments are more-or-less the same as in the proofs of Theorems 10.13 and 10.17. Assume without loss of generality that $I_E = [\bar{u}, \infty)$. By assumption there exist two sequences v^1 and v^2 which form a nontrivial braid in $\mathcal{E}^2_{2p}$, and thus $0 < r = \iota(v^1, v^2) \leq 2p$. Defining the cone C_- as in the proof of Theorem 10.13 yields a global maximum v^- which contributes to the skeleton $\widetilde{v} = \{v^1, v^2, v^-\}$. Consider the braid class $[\mathbf{u} \ \text{rel} \ \widetilde{v}]_{\mathcal{E}}$ defined by adding the strand $\mathbf{u}$ such that $u_i > v_i^-$ and $\mathbf{u}$ links with the strands v^1 and v^2 with intersection number $2q$, $0 < 2q < r$.

Notice, in contrast to our previous examples, that $[\mathbf{u} \ \text{rel} \ \widetilde{v}]_{\mathcal{E}}$ is not bounded. In order to incorporate the dissipative boundary condition that $u_i \to u^*$ is attracting, we add one additional strand v^+. Set $v_i^+ = \bar{u}$ for i even, and $v_i^+ = u^*$, for i odd. As in the proof of Theorem 10.17 choose u^* large enough such that $v_i^1, v_i^2 < u^*$. Let $R^\dagger$ be a parabolic recurrence relation such that $R^\dagger(v^+) = 0$. Using $R^\dagger$ one can construct yet another recurrence relation $R^{\dagger\dagger}$ which coincides with R on $[\mathbf{u} \ \text{rel} \ \widetilde{v}]_{\mathcal{E}}$ and which has v^+ as a fixed point (use cut-off functions). By definition the skeleton $v = \{v^1, v^2, v^-, v^+\}$ is stationary with respect to the recurrence relation $R^{\dagger\dagger} = 0$.

Now let $[\mathbf{u} \ \text{rel} \ v]_{\mathcal{E}}$ be as before, with the additional requirement that $(-1)^{i+1} u_i < (-1)^{i+1} v_i^+$. This defines a bounded proper up-down braid class. The homology index of the topological class $\{\mathbf{u} \ \text{rel} \ \{v^*\}\}$ is given by the following lemma (see §10.5).

10.21 Lemma The Conley homology of $\mathbf{h}(\mathbf{u} \ \text{rel} \ v^*)$ is given by

$$CH_k(\mathbf{h}) = \begin{cases} \mathbb{R} & : \quad k = 2q-1, \ 2q \\ 0 & : \quad \text{else} \end{cases} \tag{10.3.11}$$

In particular $CP_t(\mathbf{h}) = t^{2q-1}(1+t)$.

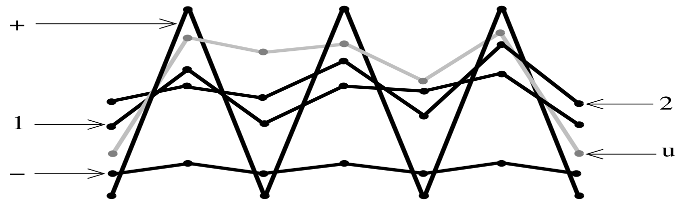


Figure 10.3: A representative braid class for the case $I_E = \mathbb{R}^\pm$: $q = 1$, $r = 4$, $2p = 6$.

For the remainder of the proof we refer to that of Theorem 10.13.

10.4 A general multiplicity result and singular energy levels

10.4.a Proof of Theorem 10.1.2

Lagrangians for which the above mentioned dissipativity conditions are satisfied for all (non-compact) interval components at energy E, are called *dissipative* at E.[6] For such Lagrangians the results for the three different types of interval components are summarized in Theorem 10.1.2 in §**??**. The fact that the presence of a non-simple closed characteristic, when represented as a braid, yields a non-trivial, non-maximally linked braid diagram, allows us to apply all three Theorems 10.13, 10.17, and 10.20, proving Theorem 10.1.2.

10.4.b Singular energy levels

The forcing theorems in §10.3.a - §10.3.c are applicable for all regular energy levels provided the correct configuration of closed characteristics can be found a priori. In this section we will discuss the role of singular energy levels; they may create configurations which force the existence of (infinitely) many periodic orbits. The equilibrium points in these singular energy levels act as seeds for the infinite family of closed characteristics.

For singular energy levels the set U_E is the union of several interval components, for which at least one interval component contains an equilibrium point. If $\partial_u^2 L(u_*,0,0) > 0$ at an equilibrium point u_*, then such a point is called *non-degenerate* and is contained in the interior of an interval component. For applying our results of the previous section the nature of the equilibrium points may play a role.

10.4.c Case I: $I_E = \mathbb{R}$

We examine the case of a singular energy level $E = 0$ such that $I_E = \mathbb{R}$ and I_E contains at least two equilibrium points. One observes that if the equilibria can

[6] One class of Lagrangians that is dissipative on all its regular energy levels is described by

$$\lim_{\lambda\to\infty} \lambda^{-s} L(\lambda u, \lambda^{\frac{2+s}{4}} v, \lambda^{\frac{s}{2}} w) = c_1|w|^2 + c_2|u|^s, \text{for some } s > 2, \text{ and } c_1, c_2 > 0,$$

pointwise in (u, v, w).

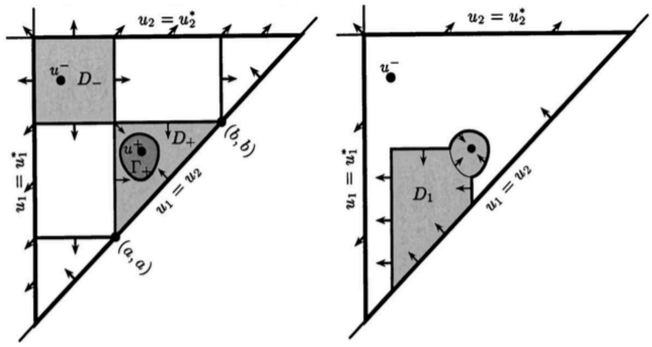


Figure 10.4: The gradient of W_2 for the case with two equilibria and dissipative boundary conditions. On the left, for $E = 0$, the regions $\mathbb{D}_\pm$ with the maxima and minima $\mathbf{u}^\pm$ are depicted, as well as the superlevel set Γ^+. On the right, for $E \in (0, c_0)$, the region D_1, containing an index 1 point, is indicated.

be regarded as periodic orbits then Theorem 10.17 would apply: a regularization argument makes this rigorous. Let $E = 0$ be the energy level in which U_E is the concatenation of three interval components $(-\infty, a] \cup [a, b] \cup [b, \infty)$, i.e., the equilibria are a and b. We remark that the nature of the equilibrium points is irrelevant; there is a global reason for the existence of two unlinked periodic orbit in the energy levels $E \in (0, c_0)$ for some small $c_0 > 0$, see [**?**]. In these regular energy levels we can apply Theorem 10.17, and a limit procedure ensures that the periodic solutions persist to the degenerate energy level $E = 0$, proving Theorem **??**.

Recall from [**?**] that two equilibrium points imply the existence of maximum $\mathbf{u}^+$ and minimum $\mathbf{u}^-$, both simple closed characteristics, see Fig. 10.4. Define the regions $D_+ = \{(u_1, u_2) \,|\, u_2 - u_1 > 0,\ u_1 \geq a,\ u_2 \leq b\}$, and $D_- = \{(u_1, u_2) \,|\, u_1^* \leq u_1 \leq a,\ \ b \leq u_2 \leq u_2^*\}$, where (u_1^*, u_2^*) is the point where the dissipativity condition of Definition 10.16 is satisfied. Then $\mathbf{u}^+ \in D_+$ and $\mathbf{u}^- \in D_-$.

Since W_2 is a C^2-function on $\mathrm{int}(D_+)$ it follows from Sard's theorem that there exists a regular value e^+ such that $0 \leq \max_{\partial D_+} W_2 < e^+ < \max_{D_+} W_2$. Consider the connected component of the super-level set $\{W_2 \geq e^+\}$ which contains $\mathbf{u}^+$. The outer boundary of this component is a smooth circle and ∇W_2 points inwards on this boundary circle. Let Γ^+ be the *interior* of the outer boundary circle in question. By continuity it follows that there exists a positive constant c_0 such that Γ^+ remains an isolating neighborhood for $E \in (0, c_0)$. In the following let $E \in (0, c_0)$ be arbitrary.

Define $D_1 = \{(u_1, u_2) \,|\, u_2 - u_1 \geq \epsilon,\ u_1^- \leq u_1 \leq u_1^+,\ u_2 \leq u_2^+\} \cup \Gamma^+$. It follows from the properties of S (see §10.2) that D_1 is again an isolating neighborhood, see Fig. 10.4. It holds that $CP_t(D_1) = 0$, and $\{D_1 \backslash \Gamma^+, \Gamma^+\}$ forms a Morse decomposition. The Morse relations (9.6.27) yield

$$CP_t(\Gamma^+) + CP_t(D_1 \backslash \Gamma^+) = 1 + CP_t(D_1 \backslash \Gamma^+) = (1+t)Q_t,$$

where Q_t is a nonnegative polynomial. This implies that $D_1 \backslash \Gamma^+$ contains an index 1 solution $\mathbf{u}^1$. We can now define $D_2 = \{(u_1, u_2) \,|\, u_2 - u_1 \geq \epsilon,\ u_1^+ \leq u_1,\ u_2^+ \leq u_2 \leq u_2^-\} \cup \Gamma^+$. In exactly the same way we find an index 1 solution $\mathbf{u}^2 \in D_2$. Notice, that by construction $\iota(\mathbf{u}^1, \mathbf{u}^2) = 0$. Theorem 10.17 now yields an infinity of closed characteristics for all $0 < E < c_0$. As described in §10.3.b these periodic solutions can be characterized by p and q, where (p,q) is any pair of integers such that $q < p$ and p and q are relative prime (or $p = q = 1$). Here $2p$ is the period of the solution $\mathbf{u}_{p,q}$ and $2q = \iota(\mathbf{u}_{p,q}, \mathbf{u}^1) = \iota(\mathbf{u}_{p,q}, \mathbf{u}^2)$.

In the limit $E \to 0$ the solutions $\mathbf{u}^1$ and $\mathbf{u}^2$ may collapse onto the two equilibrium points (if they are centers). Nevertheless, the infinite family of solutions still exists in the limit $E = 0$, because the extrema of the associated closed characteristics may only coalesce in pairs at the equilibrium points. This follows from the uniqueness of the initial value problem of the Hamiltonian system. Hence in the limit $E \to 0$ the type (p,q) of the periodic solution is conserved when we count extrema *with* multiplicity and intersections *without* multiplicity.

Note that when the equilibria are saddle-foci then $\mathbf{u}^1$ and $\mathbf{u}^2$ stay away from ± 1 in the limit $E \to 0$. Extrema may still coalesce at the equilibrium points as $E \to 0$, but intersections are counted with respect to $\mathbf{u}^1$ and $\mathbf{u}^2$. Finally, in the regular energy levels $E \in (0, c_0)$, Theorem 10.17 provides at least two solutions of each type (except $p = q = 1$); in the limit $E = 0$ one cannot exclude the possibility that two solutions of the same type coincide.

■ **10.22 Remark** Theorem **??**, proved in this subsection, is immediately applicable to the Swift-Hohenberg model (**??**) as described in §**??**. Notice that if the parameter α satisifies $\alpha > 1$, then Theorem **??** yields the existence of infinitely many closed chararacteristics at energy $E = -\frac{(\alpha-1)^2}{4}$, and nearby levels. However from the physical point of view it is also of interest to consider the case $\alpha \leq 1$. In that case there exists only one singular energy level and one equilibrium point. This case can be treated with our theory, but the nature of the equilibrium point comes into play. If an equilibrium point is a saddle or saddle-focus, it is possible that no additional periodic orbits exist (see [**?**]). However, if the equilibrium point is a center an initial non-simple closed characteristic can be found by analyzing an improper braid class, which by, Theorem 10.1.2, then yields infinitely many closed characteristics. The techniques involved are very similar to those used in the present and subsequent sections. We do not present the details here as this falls outside of the scope of this paper. ■

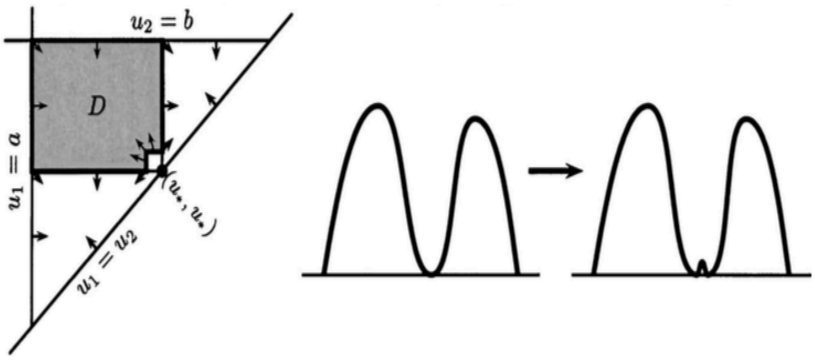


Figure 10.5: [left] The gradient of W_2 for the case of one saddle-focus equilibrium and compact boundary conditions. Clearly a saddle point is found in D. [right] The perturbation of one equilibrium to three equilibria.

10.4.d Cases II and III: $I_E = [a,b]$ or $I_E = \mathbb{R}^{\pm}$

The remaining cases are dealt with in Theorem **??**. We will restrict the proof here to the case that I_E contains an equilibrium point that is a saddle-focus — the center case can be treated as in [**?**].[7] It also follows for the previous that there is no real difference between I_E being compact or a half-line. For simplicity we consider the case that I_E is compact.

Let us first make some preliminary observations. When u_* is a saddle-focus, then in $E = 0$ there exists a solution $\mathbf{u}^1$ such that $u_1^1 < u_* < u_2^1$. This follows from the fact that there is a point (u_1^*, u_2^*), $u_1^* < u_2^*$, close to (u_*, u_*) at which the vector ∇W_2 points to the north-west (see Fig. 10.5 and [**?**]). This solution $\mathbf{u}^1$ is a saddle point, its rotation number being unknown. The impression is that (u_*, u_*) is a minimum (with $\tau = 0$), and if u_* were a periodic solution, then one would have a linked pair (u_*, u_*) and $\mathbf{u}^1$ to which one could apply Theorem 10.13. Since u_* is a saddle-focus it does not perturb to a periodic solution for $E > 0$. Hence we need to use a different regularization which conveys the information that u_* acts as a minimum. The form of the perturbation that we have in mind is depicted in Fig. 10.5, where we have drawn the "potential" $L(u,0,0)$.

This idea can be formalized as follows. Choose a function $T \in C_0^\infty[0,\infty)$ such that $0 \le T(s) \le 1$, $T(s) = 1$ for $x \le \frac{1}{2}$, $T(s)$ strictly decreases on $(\frac{1}{2}, 1)$ and $T(s) = 0$ for $x \ge 1$. Add a perturbation

$$\Phi_\epsilon(u) = \int_{u_*}^{u} -2C_0\,(s - u_*)\,T\Big(\frac{|s - u_*|}{\epsilon}\Big)\,ds$$

[7] Indeed, for energy levels $E + c$, c sufficiently small, a small simple closed charecteristic exists due to the center nature of the equilibrium point at E; spectrum $\{\pm ai, \pm bi\}$, $a < b$. This small simple closed characteristic will have a non-trivial rotation number close to $\frac{a}{b}$. The fact that the rotation number is non-zero allows one to use the arguments in [**?**] to construct a non-simple closed characteristic. As a matter of fact a linked braid diagram is created this way.

to the Lagrangian, i.e. $\tilde{L} = L + \Phi_\epsilon(u)$, where $C_0 = \partial_u^2 L(u_*,0,0)$. The new Euler-Lagrange equation near u_* becomes

$$\partial^2_{u_{xx}} L u_{xxxx} + \left[2\partial^2_{u_{xx}u} L - \partial^2_{u_x} L\right] u_{xx} + \partial^2_u L \left[1 - 2T\left(\tfrac{|u-u_*|}{\epsilon}\right)\right](u - u_*) = O(U^2),$$

where all partial derivatives of L are evaluated at $(u_*,0,0)$, and where U is the vector $(u - u_*, u_x, u_{xx}, u_{xxx})$ in phase space. Hence for all small ϵ there are now two additional equilibria near u_*, denoted by $\widehat{u} \in (u_* - \epsilon, u_* - \epsilon/2)$ and $\widetilde{u} \in (u_* + \epsilon/2, u_* + \epsilon)$. Since $(u_* - \widehat{u}) - (\widetilde{u} - u_*) = O(\epsilon^2)$, the difference between $\widetilde{E}(\widehat{u})$ and $\widetilde{E}(\widetilde{u})$ is $O(\epsilon^2)$. To level this difference we add another small perturbation to $\widetilde{L}$ of the form $\Psi(u) = \int_{u_*}^u C_\epsilon T(\frac{|s-u_*|}{2\epsilon})ds$, i.e. $\widehat{L}(u) = \widetilde{L}(u) + \Psi(u)$, where C_ϵ is chosen so that $\widehat{E}(\widehat{u}) = \widehat{E}(\widetilde{u})$ (of course $\widehat{u}$ and $\widetilde{u}$ shift slightly), and $C_\epsilon = O(\epsilon^2)$. Using the same analysis as before we conclude that a neighborhood of u_* in the energy level $E(\widehat{u})$ looks just like Fig. 10.4. In $B = \{(u_1,u_2) \,|\, u_1^* < u_1 < \widehat{u},\ \widetilde{u} < u_2 < u_2^*\}$ we find a minimum. Choose an regular energy level E_ϵ slightly larger than $\widehat{E}(\widehat{u}) = \widehat{E}(\widetilde{u})$ (with $E_\epsilon = O(\epsilon)$), such that the minimum in B persists. Taking this minimum and the original $\mathbf{u}^1$ — which persists since we have only used small perturbations, preserving D (see Fig. 10.5) as an isolating neighborhood — we apply Theorem 10.13.

Finally, we take the limit $\epsilon \to 0$. The solutions now converge to solutions of the original equation in the degenerate energy level. It follows that in the energy level $E = 0$ a solution of type (p,q) exists, where the number of extrema has to be counted with multiplicity since extrema can coalesce in pairs at u_*.

10.5 Computation of the homotopy index

Theorems 10.13, 10.17, and 10.20 hang on the homology computations of the homotopy invariant for certain canonical braid classes (Lemmas 10.15, 10.18, and 10.21). Our strategy (as in, e.g., [**?**]) is to choose a sufficiently simple system (an integrable Hamiltonian system) which exhibits the braids in question and to compute the homotopy index via knowing the structure of an unstable manifold. By the topological invariance of the homotopy index, any computable case suffices to give the index for any period d.

Consider the first-order Lagrangian system given by the Lagrangian $L_\lambda(u,u_x) = \frac{1}{2}|u_x|^2 + \lambda F(u)$, where we choose $F(u)$ to be an even four-well potential, with $F''(u) \geq -1$, and $F''(0) = -1$. The Lagrangian system (L_λ, dx) defines an integrable Hamiltonian system on $\mathbb{R}^2$, with phase portrait given in Fig. 10.6.

Linearization about bounded solutions $u(x)$ of the above Lagrangian system yields the quadratic form

$$Q[\phi] = \int_0^1 |\phi_x|^2 dt + \lambda \int_0^1 F''(u(x))\phi^2 dx \geq \int_0^1 (\pi^2 - \lambda)\phi^2 dx, \quad \phi \in H_0^1(0,1),$$

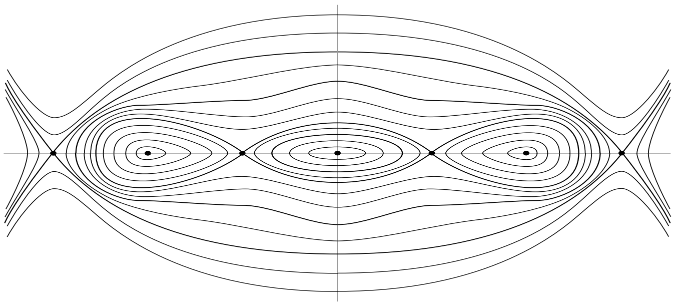


Figure 10.6: The integrable model in the (u,u_x) plane; there are centers at $0,\pm 2$ and saddles at $\pm 1,\pm 3$.

which is strictly positive for all $0<\lambda<\pi^2$. For such choices of λ the time-1 map defined via the induced Hamiltonian flow ψ^x, i.e., $(u,p_u)=(u,u_x)\mapsto\psi^1(u,p_u)$, is an area preserving monotone twist map. The generating function of the twist map is given by the minimization problem

$$S_\lambda(u_1,u_2)=\inf_{q\in X(u_1,u_2)}\int_0^1 L_\lambda(u,u_x)dx,$$

where $X(u_1,u_2)=\{u\in H^1(0,1)\mid u(0)=u_1,\ u(1)=u_2\}$.[8] The function S_λ is a smooth function on $\mathbb{R}^2$, with $\partial_1\partial_2 S_\lambda>0$. The recurrence function $R_\lambda(u_{i-1},u_i,u_{i+1})=\partial_2 S_\lambda(u_{i-1},u_i)+\partial_1 S_\lambda(u_i,u_{i+1})$ satisfies Axioms (A1)-(A3), and thus defines an exact (autonomous) parabolic recurrence relation on $\mathbf{X}=\mathbb{R}^{\mathbb{Z}}$. We choose the potential F such that the bounded solutions within the heteroclinic loop between $u=-1$ and $u=+1$ have the property that the period T_λ is an increasing function of the amplitude A, and $T_\lambda(A)\to\frac{2\pi}{\sqrt{\lambda}}$, as $A\to 0$.

This single integrable system is enough to compute the homotopy index of the three families of braid classes in Lemmas 10.15, 10.18, and 10.21 in §10.3.

We begin by identifying the following periodic solutions. Set $v^{1,\pm}=\{v_i^{1,\pm}\}$, $v_i^{1,\pm}=\pm 3$, and $v^{2,\pm}=\{v_i^{2,\pm}\}$, $v_i^{2,\pm}=\pm 1$. Let $\widehat{u}(t)$ be a solution of (L_λ,dx) with $\widehat{u}_x(0)=0$ (minimum), $|\widehat{u}(x)|<1$, and $T_1(A(\widehat{u}))=2\tau_0>2\pi$, $\tau_0\in\mathbb{N}$. For arbitrary $\lambda\le 1$ this implies that

$$T_\lambda(A(\widehat{u}))=\frac{T_1(A(\widehat{u}))}{\sqrt{\lambda}}=\frac{2\tau_0}{\sqrt{\lambda}},$$

where we choose λ so that $\frac{1}{\sqrt{\lambda}}\in\mathbb{N}$. For $r\ge 1$ set $d:=\frac{\tau_0 r}{\sqrt{\lambda}}$ and define $v^3:=\{v_i^3\}$, with $v_i^3=\widehat{u}(i)$, and $v^4=\{v_i^4\}$, with $v_i^4=\widehat{u}(i+\tau_0/\sqrt{\lambda})$, $i=0,...,d$. Clearly, $\iota(v^3,v^4)=r$, for all $\frac{1}{\sqrt{\lambda}}\in\mathbb{N}$.

Next choose $\widetilde{u}(x)$, a solution of (L_λ,dx) with $\widetilde{u}_x(0)=0$ (minimum), which oscillates around both equilibria -2 and $+2$, and in between the equilibria -3 and $+3$, and $T_1(A(\widetilde{u}))=2\tau_1>2\pi$, $\tau_1\in\mathbb{N}$. As before

$$T_\lambda(A(\widetilde{u}))=\frac{T_1(A(\widetilde{u}))}{\sqrt{\lambda}}=\frac{2\tau_1}{\sqrt{\lambda}}.$$

[8] The strict positivity of the quadratic form Q via the choice of λ yields a smooth family of hyperbolic minimizers.

Let $2p \geq r$ and choose $\tau_0, \tau_1 \geq 4$ such that

$$\frac{\tau_0}{\tau_1} = \frac{2p}{r} \geq 1 \qquad (\tau_0 \geq \tau_1).$$

Set $v^5 = \{v_i^5\}$, $v_i^5 = \widetilde{u}(i)$, and $v^6 = \{v_i^6\}$, with $v_i^6 = \widetilde{u}(i + \tau_1/\sqrt{\lambda})$, $i = 0, ..., d$. For $x \in [0, d]$ the solutions $\widehat{u}$ and $\widetilde{u}$ have exactly $2p$ intersections. Therefore, if we choose λ sufficiently small, i.e. $\frac{1}{\sqrt{\lambda}} \in \mathbb{N}$ is large, then it also holds that $\iota(v^{3,4}, v^{5,6}) = 2p$.

Finally we choose the unique periodic solution $u(x)$, with $|u(x)| < 1$, $u_x(0) = 0$ (minimum), and $T_1(A(u)) = 2\tau_2 > 2\pi$, $\tau_2 \in \mathbb{N}$. Let $0 < 2q < r \leq 2p$, and choose τ_2, and consequently the amplitude A, so that

$$\frac{\tau_0}{\tau_2} = \frac{2q}{r} < 1 \qquad (\tau_0 < \tau_2, \quad A(\widehat{u})) < A(u)).$$

The solution u is part of a hyperbolic circle of solutions $u_s(x)$, $s \in \mathbb{R}/\mathbb{Z}$. Define $(\mathbf{u}(s))_{s \in \mathbb{R}/\mathbb{Z}}$, with $\mathbf{u}(s) = \{u_i(s)\}$, where $u_i(s) = u_s(i + 2\tau_2 s/\sqrt{\lambda})$. As before, since the intersection number of $\widehat{u}$ and u_s is equal to $2q$, it holds that $\iota(\mathbf{u}(s), v^{3,4}) = 2q$, for λ sufficiently small. Moreover, $\iota(\mathbf{u}(s), v^{5,6}) = 2p$. From this point on λ is fixed. We now consider three different skeleta v.

I: $v = \{v^{2,-}, v^{2,+}, v^3, v^4\}$. The relative braid class $[\mathbf{u} \text{ rel } v]_{\mathrm{I}}$ is defined as follows: $v_i^{2,-} \leq u_i \leq v_i^{2,+}$, and $\mathbf{u}$ links with the strands v^3 and v^4 with intersection number $2q$, $0 \leq 2q < r$. The topological class $\{\mathbf{u} \text{ rel } \{v\}\}$ is precisely that of Lemma 10.15 [Fig. 10.1] and as such is bounded and proper.

II: $v = \{v^{2,-}, v^{1,+}, v^3, v^4, v^5\}$. The relative braid class $[\mathbf{u} \text{ rel } v]_{\mathrm{II}}$ is defined as follows: $v_i^{2,-} \leq u_i \leq v_i^{1,+}$, $\mathbf{u}$ links with the strands v^3 and v^4 with intersection number $2q$, $0 \leq 2q < r$, and $\mathbf{u}$ links with v^5 with intersection number $2p$. The topological class $\{\mathbf{u} \text{ rel } \{v\}\}$ is precisely that of Lemma 10.21 [Fig. 10.3] and as such is bounded and proper.

III: $v = \{v^{2,-}, v^3, v^4, v^5, v^6\}$. The relative braid class $[\mathbf{u} \text{ rel } v]_{\mathrm{III}}$ is defined as follows: $v_i^{2,-} \leq u_i$, $\mathbf{u}$ links with the strands v^3 and v^4 with intersection number $2q$, and $\mathbf{u}$ links with v^5 and v^6 with intersection number $2p$. The topological class $\{\mathbf{u} \text{ rel } \{v\}\}$ is *not* bounded [Fig. 10.7[right]]. The augmentation of this braid class is bounded.

Cases I and II: Since the topological classes are bounded and proper, the invariant $\mathbf{h}$ is independent of period of the chosen representative, and can be easily computed from the integrable model. The closure of the collection of topologically equivalent braid classes is an isolating neighborhood for the parabolic flow Ψ^t induced by the recurrence relation $R_\lambda = 0$ (defined via (L_λ, dx)). The invariant set is given by the normally hyperbolic circle $\{\mathbf{u}(s)\}_{s \in \mathbb{R}/\mathbb{Z}}$. For this reason the index $\mathbf{h}$ can be computed via the connected component that contains the critical circle; we denote this neighborhood by N. The Conley index of N can be determined via computing $W^u(\{\mathbf{u}(s)\})$, the unstable manifold associated to this circle. This computation is precisely that appearing in the calculations of [**?**, pp. 372]: $W^u(\{\mathbf{u}(s)\})$

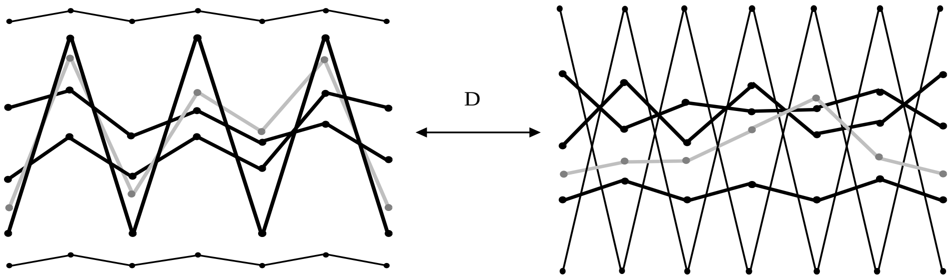


Figure 10.7: The augmentation of the braid from Lemma 10.18 [left] is the dual of the type III braid [right].

is orientable and of dimension $2q$, and thus

$$\mathbf{h}(\mathbf{u} \ \text{rel} \ v) = h(N) \simeq \left(S^1 \times S^{2q-1}\right) / \left(S^1 \times \{\text{pt}\}\right) \simeq S^{2q-1} \vee S^{2q}. \qquad (10.5.12)$$

The Conley homology is given by $CH_k(\mathbf{h}) = \mathbb{R}$ for $k = 2q-1, 2q$, and $CH_k(\mathbf{h}) = 0$ elsewhere. This completes the proofs of the Lemmas 10.15 and 10.21.
Case III: It holds that

$$\{\mathbf{u} \ \text{rel} \ v\} \cap \left(\mathcal{D}^1_{2p} \ \text{rel} \ v\right) \neq \emptyset.$$

The discrete class for period $2p$ is bounded, but for periods $d > 2p$ this is not the case. However, by augmenting the braid, we obtain from (9.4.15) that

$$\mathbf{h}(\mathbf{u} \ \text{rel} \ v) = \mathbf{h}(\mathbf{u} \ \text{rel} \ v^*),$$

where $v^* = v \cup \{v^{1,-}, v^{1,+}\}$. Since the topological class $\{\mathbf{u} \ \text{rel} \ \{v^*\}\}$ is bounded and proper, we may use the previous calculations to conclude that

$$\mathbf{h}(\mathbf{u} \ \text{rel} \ v^*) \simeq (S^1 \times S^{2q-1})/(S^1 \times \{\text{pt}\}) \simeq S^{2q-1} \vee S^{2q}.$$

Our motivation for this computation is to complete the proof of Lemma 10.18. Let $[\mathbf{u}' \ \text{rel} \ v']$ denote the period $2p$ braid class described by Fig. 10.2, with intersection numbers denoted by $2q'$ and $2r'$, and let $[\mathbf{u} \ \text{rel} \ v]$ denote a type-III braid of period $2p$. Then, it is straightforward to see (as illustrated in Fig. 10.7) that, for $q' = p - q$ and $r' = 2p - r$,

$$\left[\mathbf{u}' \ \text{rel} \ [v'^*]\right] = \mathbb{D}\Big(\left[\mathbf{u} \ \text{rel} \ [v]\right]\Big). \qquad (10.5.13)$$

Lemma 10.18 gives the index for the augmented class $\{\mathbf{u}' \ \text{rel} \ \{v'^*\}\}$, which is bounded and proper as a topological class. The above considerations allow us to compute the homology of $\mathbf{h}(\mathbf{u}' \ \text{rel} \ \ v'^*)$ via Theorem 9.28:

$$
\begin{aligned}
CH_*\left(\mathbf{h}(\mathbf{u}' \ \text{rel} \ \ v'^*)\right) &\cong CH_*\left(\mathbf{h}(\mathbb{D}\mathbf{u} \ \text{rel} \ \mathbb{D}v)\right) \\
&\cong CH_{2p-*}\left(\mathbf{h}(\mathbf{u} \ \text{rel} \ v)\right) \\
&\cong CH_{2p-*}\left(\mathbf{h}(\mathbf{u} \ \text{rel} \ v^*)\right) \\
&\cong \begin{cases} \mathbb{R} & : \ 2p-*=2q-1, 2q \\ 0 & : \ \text{else} \end{cases} . \\
&\cong \begin{cases} \mathbb{R} & : \ *=2q', 2q'+1 \\ 0 & : \ \text{else} \end{cases}
\end{aligned} \tag{10.5.14}
$$

The intersection numbers $2q'$ and $2r'$ are exactly those of Lemma 10.18, completing the proof.

10.6 The Geometry of Second-Order Lagrangians

In order to reduce the amount of technical detail we restrict ourselves to Lagrangians that satisfy the following hypotheses:

(H1) $L(u,v,w) = \frac{1}{2}w^2 + K(u,v)$.

(H2) for every $u \in \mathbb{R}$ there exists a constant $c > 0$ such that $K(u,v) \geq -c - c|v|^\gamma, \quad 0 \leq \gamma < 4$.

Note that (H2) is a lower bound on K; an upper bound is not necessary. Hypotheses (H1)-(H2) are very mild restrictions on the second-order Lagrangians that we consider in this chapter. These hypotheses can be further weakened as is discussed in Section 10.9. We now formulate the main result of this paper.

10.23 Theorem Let M_E be a regular energy manifold of a second-order Lagrangian system with Lagrangian L satisfying hypotheses *(H1)* and *(H2)*. Then the number of closed characteristics is bounded below by the second Betti number $\beta_2 := \dim H_2(M)$.

For Lagrangian systems with $\partial_w^2 L \geq \alpha > 0$, the homotopy type of M_E can be determined directly from the sign of its "potential", $L(u,0,0) + E$, see Section 10.9 and cf. [4] This number then yields the second Betti number of M_E.

Theorem 10.23 is a generalization of the situation for first-order Lagrangian systems $L(u,u') = \frac{1}{2}(u')^2 + K(u)$ where $H = \frac{1}{2}(u')^2 - K(u)$. An energy manifold M_E is one-dimensional, and each compact component of (regular) M_E consists of a single periodic solution. Thus the number of closed characteristics is exactly $\beta_1 := \dim H_1(M)$. For second order Lagrangians β_2 is only a lower bound. One can easily gives examples of systems with infinitely many different closed characteristics, cf. [33] Note that for Lagrangians of the special form $L = L(u,u'')$ Hypothesis (H2) becomes void and the similarity between first and second order systems becomes even stronger.

To establish the existence of closed characteristics on energy manifolds of second-order Lagrangian systems, we use their variational structure. A closed characteristic is equivalent to a periodic solution u which are found as critical points of the second-order action, i.e

$$\delta_{u,\tau} \int_0^\tau \big[L(u,u',u'') + E\big]\, dt = 0, \tag{10.6.15}$$

where $\tau > 0$ is the period of u. Note that variations are taken in τ as well as u. By Adding the term E in the Lagrangian action in (10.6.15) solutions are guaranteed to lie on M_E, see Lemma 10.33.

> **10.24 Exercise** Prove that extrema of second-order Lagrangian action in (10.6.15) are confined to the energy manifold M_E. ▪

10.6.a Intersection theory

We consider functions which have a simple profile consisting of two monotone laps, u_+ which increases from some minimal value u_1 to a maximal value u_2 and u_- which decreases from u_2 back to u_1. At u_1 and u_2 it holds that $u' = 0$. If u_+ and u_- are solutions, then their concatenation $u_+ \# u_-$ is called a 'broken geodesic'. The extrema u_1 and u_2 are called concatenation points. Note that a broken geodesic need not be a solution to (10.1.1) at the concatenation points. The third derivatives need not match, cf. [33]

We obtain a periodic solution from the method of broken geodesics in two steps. First we must determine when monotone laps exist between given values of u, and this is accomplished in Section 10.7 via minimization. Then it must be shown that there exists a broken geodesic which is a solution to (10.1.1), which follows from the geometric and topological properties of M_E as we now explain.

From the Hamiltonian (10.1.2), solutions must satisfy

$$\frac{\partial L}{\partial u''} u'' - L(u,0,u'') = E$$

at points where $u' = 0$. We denote this level set in the (u,u'')-plane by N_E. Note that N_E is the section of M_E defined by $M_E \cap \{u' = 0\}$.[9] Due to the convexity of L with respect to u'', the manifold N_E consists of two graphs in the (u,u'')-plane. In particular, the projection of N_E onto the u-axis can be characterized by $\pi N_E = \{u : L(u,0,0) + E \geq 0\}$, and the sets $N_E \cap \{(u,u'') \mid u \geq 0\}$ and $N_E \cap \{(u,u'') \mid u \leq 0\}$ are graphs over πN_E. A particular connected component of πN_E will be denoted by I_E, and will be referred to as an *interval component*.

[9] As a matter of fact $M_E \cap \{u' = 0\}$ is the cylinder $N_E \times \mathbb{R}$, where the $\mathbb{R}$-variable is accounted for by p_u-coordinate.

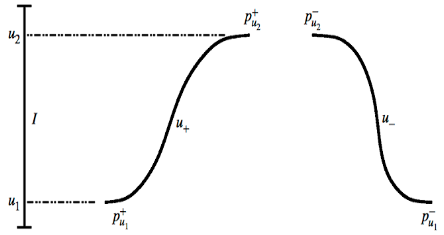


Figure 10.8: The increasing and decreasing laps u_+ and u_- respectively.

We will consider broken geodesics whose values lie in a single interval component I_E. Given such a component I_E of πN_E, define $B = \{(u_1,u_2) \in I \times I : u_1 < u_2\}$. For given laps u_+ and u_- let $p^+_{u_1}$, $p^+_{u_2}$, and $p^-_{u_1}$, $p^-_{u_2}$ be the p_u-values at the concatenation points respectively. As shown in [33], if the condition

$$p^+_{u_1} - p^-_{u_1} = 0 \qquad \text{and} \qquad p^+_{u_2} - p^-_{u_2} = 0 \tag{10.6.16}$$

is satisfied at the concatenation points, then $u_+\#u_-$ is a periodic solution of (10.1.1), and thus a closed characteristic on M_E.

Let $(u_1,u_2) \in B$ and $p^+_{u_1}, p^+_{u_2} \in \mathbb{R}$. Consider the trajectory $x(t) = \phi^t(u_1, 0, p^+_{u_1}, p_v(u_1))$ of the Hamiltonian flow. Here $p_v = u''$ is a function of u_1 since the initial point has $v = u' = 0$, and hence is in N_E. Thus there are two choices for $p_v(u_1)$, and we will choose $p_v(u_1) > 0$.

Define $f_+(u_1, p^+_{u_1})$ and $g_+(u_1, p^+_{u_1})$ to be the values of u and p_u at the first maximum of $\pi_u x(t)$ respectively, see Fig. 10.8. As $p^+_{u_1} \to \infty$, then $f_+(u_1, p^+_{u_1}) \to u_1$. The maps f_+ and g_+ are well-defined for fixed u_1 with decreasing $p^+_{u_1}$ as long as $f_+(u_1, p^+_{u_1}) \le \max I$. In addition the f_+ and g_+ are smooth in $(u_1, p^+_{u_1})$ on the domain of definition P_+. We can define analogous maps $f_-(u_2, p^-_{u_2})$ and $g_-(u_2, p^-_{u_2})$ as the values of u and p_u at the first minimum of a decreasing lap, see Fig. 10.8, with domain of definition P_-. Let

$$\begin{aligned} \mathcal{L}_+ &= \{(u, f_+(u,p_u), p_u, g_+(u,p_u))\}, \quad \text{and} \\ \mathcal{L}_- &= \{(f_-(u,p_u), u, g_+(u,p_u), p_u)\}, \end{aligned}$$

be subsets of the tangent bundle T^*B. Then $\mathcal{L}_\pm$ are two-dimensional submanifolds of T^*B given as graphs over the (u_1, p_{u_1})-plane and the (u_2, p_{u_2})-plane respectively[10], see Fig. 10.9.

The submanifolds $\mathcal{L}_\pm$ are in fact Lagrangian submanifolds of T^*B. Condition (10.6.16) implies that intersection points of the manifolds $\mathcal{L}_+ \cap \mathcal{L}_-$ correspond to broken geodesics which are periodic solutions, i.e. $u_1 = f_-(u_2, p_{u_2})$, $f_+(u_1, p_{u_1}) = u_2$, $p_{u_1} = g_-(u_2, p_{u_2})$, and $g_+(u_1, p_{u_1}) = p_{u_2}$.

[10] One can also define maps f_- and g_- by following the flow backward in time to the next maximum. This way the maps f_- and g_- again depend on (u_1, p_{u_1}), and which leads to the same results.

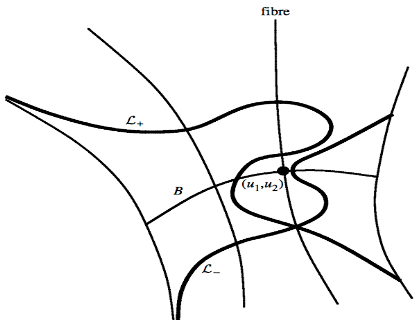


Figure 10.9: The contangent bundle T^*B and the intersecting Lagrangian submanifolds $\mathcal{L}_+$ and $\mathcal{L}_-$.

In the special case that there exist unique laps u_+ and u_- for all $(u_1, u_2) \in B$, then the system is a twist system. In this case, $\mathcal{L}_\pm$ are an exact Lagrangian submanifolds of T^*B, i.e. $\mathcal{L}_\pm$ are the graphs of exact 1-forms on B provided by generating functions for the laps [33]. In this case a direct variational principle exists in terms of just the extrema u_1 and u_2.

10.25 Lemma Let L satisfy Hypotheses (H1) and (H2), then for each pair $(u_1, u_2) \in B$ there exist increasing and decreasing laps u_+ and u_- respectively. In particular $\pi\mathcal{L}_\pm = B$.

This lemma is proved by Theorem 10.38 in Section 10.7, and shows that for any Lagrangian satisfying (H1) and (H2) the projections $\pi\mathcal{L}_\pm$ cover the base B, which plays a crucial role in our intersection theory. The next lemma establishes that the intersection set $\pi(\mathcal{L}_+ \cap \mathcal{L}_-)$ is always strictly contained in B for Lagrangians satisfying (H1) and (H2).

10.26 Lemma Let L satisfy Hypotheses (H1) and (H2), then

$$\pi(\mathcal{L}_+ \cap \mathcal{L}_-) \cap \partial B = \varnothing.$$

Moreover, if $\mathrm{cl}(B)$ is compact then there exists a compact set $B^\dagger \subset \mathrm{int}\, B$ such that $\pi(\mathcal{L}_+ \cap \mathcal{L}_-) \subset B^\dagger$.

Proof. Denote by $\pi: \ T^*B \to B$ the canonical projection onto the base, let $\pi^{-1}(u_1, u_2) = \zeta$ be a fiber in T^*B, and let π^* be the projection onto the (p_{u_1}, p_{u_2}) coordinates of a point in T^*B. From Lemma 10.25 it holds that for each point $(u_1, u_2) \in B$ that $\zeta \cap \mathcal{L}_\pm \neq \varnothing$. Take a point $(u_1, u_2) \in \partial B$, and consider the points $(p^+_{u_1}, p^+_{u_2}) \in \zeta \cap \mathcal{L}_+$ and $(p^-_{u_1}, p^-_{u_2}) \in \zeta \cap \mathcal{L}_-$. It then follows from Lemma 7 in [33] that each pair $(p^+_{u_1}, p^+_{u_2})$ and $(p^-_{u_1}, p^-_{u_2})$ either $p^+_{u_1} - p^-_{u_1}$ or $p^+_{u_2} - p^-_{u_2}$ has a definite sign (strictly negative). Thus, for any boundary point $(u_1,u_2) \in \partial B$ it holds that

$$\pi^*(\zeta \cap \mathcal{L}_+) \neq \pi^*(\zeta \cap \mathcal{L}_-), \tag{10.6.17}$$

which implies that $\pi(\mathcal{L}_+ \cap \mathcal{L}_-) \cap \partial B = \emptyset$.

Now assume that $\mathrm{cl}(B)$ is compact, hence I is a compact interval component. Define $B_\delta = \{(u_1,u_2) \in B \mid u_1 \geq \min I + \delta, \;\; u_2 \leq \max I - \delta\}$. From Lemma 8 in [33], there exists $\delta_0 > 0$ such that for all $\delta \leq \delta_0$ the boundaries $\{u_1 = \min I + \delta\}$ and $\{u_2 = \max I - \delta\}$ satisfy (10.6.17). This proves that $\pi(\mathcal{L}_+ \cap \mathcal{L}_-) \cap \partial B_\delta = \emptyset$. Define the diagonal $\triangle = \{(u_1,u_2) \in \mathrm{cl}(B) \mid u_1 = u_2\}$. Suppose now that there exists a sequence of points (u_1^n,u_2^n) accumulating at $\mathrm{cl}(B_\delta) \cap \triangle$, then it follows from Lemma 5 in [33] that

$$\|(p^+_{u_1^n} - p^-_{u_1^n}, p^+_{u_2^n} - p^-_{u_2^n})\| \to \infty \quad \text{as} \quad n \to \infty$$

for any pair $(p^+_{u_1^n} - p^-_{u_1^n}, p^+_{u_2^n} - p^-_{u_2^n})$ in $(\zeta_n \cap \mathcal{L}_+) \times (\zeta_n \cap \mathcal{L}_-)$, where $\zeta_n = \pi^{-1}(u_1^n,u_2^n)$. The latter combined with the behavior of $\mathcal{L}_+ \cap \mathcal{L}_-$ on ∂B_δ now implies that there exists a compact set $B^\dagger \subset \mathrm{int}B_\delta \subset \mathrm{int}B$ such that $\pi(\mathcal{L}_+ \cap \mathcal{L}_-) \subset B^\dagger$. ■

10.6.b Continuation

To study $\mathcal{L}_+ \cap \mathcal{L}_-$ we use the intersection number $\iota(\mathcal{L}_+,\mathcal{L}_-)$. Our approach is to define $\iota(\mathcal{L}_+,\mathcal{L}_-)$ via the Brouwer degree by constructing proper equations on T^*B whose zero sets are $\mathcal{L}_\pm$. This can be done in many ways and the intersection number $\iota(\mathcal{L}_+,\mathcal{L}_-)$ does not depend on the particular choice of the defining equations.

Define

$$\begin{aligned} F_+(u_1,p_{u_1},u_2,p_{u_2}) &= [u_2 - f_+(u_1,p_{u_1}), p_{u_2} - g_+(u_1,p_{u_1})]; \\ F_-(u_1,p_{u_1},u_2,p_{u_2}) &= [u_1 - f_-(u_2,p_{u_2}), p_{u_1} - g_-(u_2,p_{u_2})], \end{aligned}$$

where the domain of definition of F_+ is $(u_1,p_{u_1}) \in P_+$, $(u_2,p_{u_2}) \in I \times \mathbb{R}$, and the domain of definition of F_- is $(u_2,p_{u_2}) \in P_-$, $(u_1,p_{u_1}) \in I \times \mathbb{R}$. Then $\mathcal{L}_\pm$ are the level sets $F_\pm^{-1}(0)$ in T^*B. Define $F(u_1,p_{u_1},u_2,p_{u_2}) = [F_+,F_-]$ on $P_+ \times P_-$. Then the zero set of F is $F^{-1}(0) = \mathcal{L}_+ \cap \mathcal{L}_-$, which is bounded and contained in $\mathrm{int}(P_+ \times P_-)$. The latter follows from Lemma 10.26. Indeed, for an intersection it holds that $(u_1,u_2) \in B^\dagger$, so $u_1 \in \mathrm{int}I$. If $(u_1,p_{u_1}) \in \partial P_+$, then $u_2 \in \partial I$, a contradiction. Thus $(u_1,p_{u_1}) \in \mathrm{int}P_+$. Similarly it follows that $(u_2,p_{u_2}) \in \mathrm{int}P_-$. Since $\pi(\mathcal{L}_+ \cap \mathcal{L}_-) \subset B^\dagger$ the boundedness of $F^{-1}(0)$ follows from continuity. These facts combined justify the definition

$$\iota(\mathcal{L}_+,\mathcal{L}_-) = \deg(F, P_+ \times P_-, 0),$$

c.f [10]. Since $\dim \mathcal{L}_\pm = 2$, we have $\iota(\mathcal{L}_+,\mathcal{L}_-) = \iota(\mathcal{L}_-,\mathcal{L}_+)$. We are now ready to prove the main result.

Proof of Theorem 10.23: Let M_E be a regular energy manifold corresponding to $H(x) = E$. We compare the Lagrangian system determined by $L_0 = L = \frac{1}{2}w^2 + K(u,v)$ with the system determined by $L_1 = \frac{1}{2}w^2 + K(u,0)$. The latter system is of Swift-Hohenberg type which is shown to be a twist system in [33]. The two systems are related by continuation. Specifically, define $L_\lambda = (1-\lambda)L_0 + \lambda L_1$. Then the energy manifolds M_λ are regular for all $\lambda \in [0,1]$. Hence each M_λ is homotopy equivalent to $M = M_0$. Moreover, from the definition of L_λ it is clear that the sections $N_\lambda = N$ and the base manifolds $B_\lambda = B$ for all $\lambda \in [0,1]$.

Since M_0 and M_1 are homotopy equivalent, the Betti numbers $\dim H_k(M_0)$ and $\dim H_k(M_1)$, $k \geq 0$, are equal. In Section 7 of [4] it was shown that $\dim H_2(M_1)$ is equal to the number of compact components of the section N_E, which can be computed directly from the graph of the potential $K(u,0)$, i.e. the number of compact intervals on which $K(u,0) + E \geq 0$.

Since M_1 is a twist system, the results in [33] imply that for each compact component of N_E there exists a closed characteristic of the system for L_1 for which $\iota(\mathcal{L}^1_+, \mathcal{L}^1_-) = \pm 1$. The sign of $\iota(\mathcal{L}^1_+, \mathcal{L}^1_-)$ depends on the orientations of $\mathcal{L}_\pm$ induced by their definition as level sets of $F_\pm$. Since L_λ satisfies hypotheses (H1) and (H2) and $\partial^2_w L_\lambda = 1 > 0$ for all $\lambda \in [0,1]$, the results of Lemma 10.25 apply for all $\lambda \in [0,1]$. Moreover Lemma 10.26 implies that $\pi(\mathcal{L}^\lambda_+ \cap \mathcal{L}^\lambda_-) \subset B^\dagger$ for some compact set $B^\dagger \subset B$ uniformly for all $\lambda \in [0,1]$. The continuation property of the degree can be used to show that the intersection number can be continued for all $\lambda \in [0,1]$. This fact requires a little argument. For each $\lambda_0 \in [0,1]$ there exists a $\epsilon(\lambda_0) > 0$ such that for all $\lambda \in (\lambda_0 - \epsilon(\lambda_0), \lambda_0 + \epsilon(\lambda_0)) \cap [0,1]$ it holds that $\mathcal{L}^\lambda_+ \cap L^\lambda_- \subset D_{\lambda_0} \subset P^\lambda_+ \times P^\lambda_-$. Therefore $\deg(F, P^\lambda_+ \times P^\lambda_-, 0) = \deg(F, D_{\lambda_0}, 0)$. Consequently $\iota(\mathcal{L}^0_+, \mathcal{L}^0_-) = \iota(\mathcal{L}^{\epsilon(0)}_+, \mathcal{L}^{\epsilon(0)}_-)$, and $\iota(\mathcal{L}^1_+, \mathcal{L}^1_-) = \iota(\mathcal{L}^{1-\epsilon(1)}_+, \mathcal{L}^{1-\epsilon(1)}_-)$. Since the interval $[\epsilon(0), 1-\epsilon(1)]$ is compact the desired result follows via a finite covering, and consequently $\iota(\mathcal{L}^\lambda_+, \mathcal{L}^\lambda_-) = \iota(\mathcal{L}^1_+, \mathcal{L}^1_-) \neq 0$ for all $\lambda \in [0,1]$. Hence for each compact component of N_E, the energy manifold M_E contains a closed characteristic. Therefore the number of closed characteristics is at least $\dim H_2(M)$. ∎

10.7 Existence of Minimizing Laps

Fix $E \in \mathbb{R}$ and a compact component interval component I_E. Given $\mathbf{u} = (u_1, u_2) \in B$ and $\mathbf{b} \in \mathcal{K} = \{(b_1, b_2) \in \mathbb{R}^2 : b_1 b_2 \geq 0 \text{ and } \max\{|b_1|, |b_2|\} < 1/2\}$, define $X_\tau(\mathbf{u}, \mathbf{b}) = \{u \in H^2([0,\tau]) : u(0) = u_1,\, u(\tau) = u_2,\, u'(0) = b_1,\, u'(\tau) = b_2, \text{ and } u'(t) \neq 0 \text{ for } t \in (0,\tau)\}$ and

$$J_E[u] = \int_0^\tau \left[\frac{1}{2}|u''(t)|^2 + K(u(t), u'(t)) + E\right] dt$$

which is well-defined on $X(\mathbf{u},\mathbf{b}) = \cup_{\tau\in\mathbb{R}^+} X_\tau(\mathbf{u},\mathbf{b})$. To prove that Lemma 10.25, we consider the following minimization problem,

$$\mathcal{J}_E(\mathbf{u},\mathbf{b}) = \inf_{\substack{u\in X_\tau \\ \tau\in\mathbb{R}^+}} J_E[u],$$

and establish the existence of minimizers.

Minimization J_E requires a growth condition on the Lagrangian. Hypothesis (H2) implies the following property.

10.27 Lemma If hypothesis (H2) holds, then for every $\epsilon > 0$ there exists $C_\epsilon \geq 0$ such that $K(u,v) + E + \epsilon^{-1}v^4 \geq -C_\epsilon|v|$ for all $u \in I$ and $v \in \mathbb{R}$.

Proof. Since u is bounded, we have $K(u,v) \geq -C - C|v|^\gamma$. Thus $K(u,v) + E + \epsilon^{-1}v^4 \geq -C + E - C|v|^\gamma + \epsilon^{-1}v^4 \geq -C^*_\epsilon$ for all $u \in I$ and $v \in \mathbb{R}$. Since $K(u,0) + E \geq 0$ and $\partial_v K(u,0)$ is bounded for $u \in I$, there exists $C_\epsilon > 0$ such that $K(u,v) + E + \epsilon^{-1}v^4 \geq -C_\epsilon|v|$ for all $u \in I$ and $v \in \mathbb{R}$. ∎

10.28 Lemma If $u \in X(\mathbf{u},\mathbf{b})$, then

$$\int_0^\tau |u''|^2\,dt \geq \frac{4(1-|\mathbf{b}|_\infty)}{9|u_2-u_1|^2}\int_0^\tau |u'|^4\,dt - \frac{4|\mathbf{b}|^2_\infty}{9|u_2-u_1|}.$$

Proof. Since u is monotone, we can reparametrize by $u'(t) = v(u)$ and let $z(u) = v|v|^{1/2}(u)$. Transforming to (u,z)-variables yields

$$J_E[u(t)] = J_E[z(u)] = \int_{u_1}^{u_2} \left[\frac{2}{9}|z'(u)|^2 + \frac{K(u,z^{2/3}(u)) + E}{z^{2/3}(u)}\right] du$$

with $z \in \chi + H^1_0([u_1,u_2])$ where χ is a smooth function satisfying $z(u_1) = b_1^{3/2}$ and $z(u_2) = b_2^{3/2}$. Hence z is absolutely continuous with $z(u) - z(u_1) = \int_{u_1}^{u} z'(\mu)\,d\mu$ for all $u \in [u_1,u_2]$, which implies $|z(u) - b_1^{3/2}|^2 \leq |u_2-u_1|\int_{u_1}^{u_2}|z'|^2\,du$. Note that under this transformation

$$\int_0^\tau |u''(t)|^2\,dt = \frac{4}{9}\int_{u_1}^{u_2}|z'(u)|^2\,du \qquad \text{and} \qquad \int_0^\tau |u'(t)|^4\,dt = \int_{u_1}^{u_2}|z(u)|^2\,du$$

Therefore,

$$\begin{aligned}
\int_0^\tau |u''|^2\,dt &= \frac{4}{9}\int_{u_1}^{u_2} |z'|^2\,du \\
&\geq \frac{4}{9|u_2-u_1|^2}\int_{u_1}^{u_2} |z-b_1^{3/2}|^2\,du \\
&= \frac{4}{9|u_2-u_1|^2}\left[\int_{u_1}^{u_2} z^2\,du - 2b_1^{3/2}\int_{u_1}^{u_2} z\,du + b_1^3|u_2-u_1|\right] \\
&\geq \frac{4}{9|u_2-u_1|^2}\left[\int_{u_1}^{u_2} z^2\,du - 2b_1^{3/2}|u_2-u_1|^{1/2}\left(\int_{u_1}^{u_2} z^2\,du\right)^{1/2} + b_1^3|u_2-u_1|\right] \\
&\geq \frac{4(1-b_1)}{9|u_2-u_1|^2}\int_{u_1}^{u_2} z^2\,du - \frac{4b_1^2}{9|u_2-u_1|} \\
&\geq \frac{4(1-|\mathbf{b}|_\infty)}{9|u_2-u_1|^2}\int_0^\tau |u'|^4\,dt - \frac{4|\mathbf{b}|_\infty^2}{9|u_2-u_1|}
\end{aligned}$$

■

Now we use this inequality to prove that J_E is bounded below on $X(\mathbf{u},\mathbf{b})$, so that the minimization problem is well-posed, i.e. $\mathcal{J}_E > -\infty$.

10.29 Lemma There exists a constant $C(|u_2-u_1|,|\mathbf{b}|_\infty) > 0$ such that $J_E[u] \geq -C$ for all $u \in X(\mathbf{u},\mathbf{b})$.

Proof. Applying Lemma 10.27 and 10.28 we obtain

$$\begin{aligned}
J_E[u] &= \int_0^\tau \left[\frac{1}{2}|u''|^2 + K(u,u') + E\right]dt \\
&\geq \frac{2(1-|\mathbf{b}|_\infty)}{9|u_2-u_1|^2}\int_0^\tau |u'|^4\,dt - \frac{2|\mathbf{b}|_\infty^2}{9|u_2-u_1|} + \int_0^\tau [K(u,u')+E]\,dt \\
&\geq \int_0^\tau \left[K(u,u') + E + \frac{1}{9|u_2-u_1|^2}|u'|^4\right]dt - \frac{2|\mathbf{b}|_\infty^2}{9|u_2-u_1|} \\
&\geq -\int_0^\tau Cu'\,dt - \frac{2|\mathbf{b}|_\infty^2}{9|u_2-u_1|} \\
&\geq -C|u_2-u_1| - \frac{2|\mathbf{b}|_\infty^2}{9|u_2-u_1|},
\end{aligned}$$

which implies that $J_E[u]$ is bounded below on $X(\mathbf{u},\mathbf{b})$. ■

Define the sublevel set $J_E^a(\mathbf{u},\mathbf{b}) = \{u \in X(\mathbf{u},\mathbf{b}) : J_E[u] \le a\}$.

10.30 Lemma There exists positive constants C_1, C_2, and T_1 depending on $a, |u_2 - u_1|$ and $|\mathbf{b}|_\infty$ such that for any $u \in J_E^a(\mathbf{u},\mathbf{b})$ we have $\tau \ge T_1$, $\|u''\|_{L^2([0,\tau])} \le C_1$, and $\|u'\|_{L^4([0,\tau])} \le C_2$.

Proof. We have

$$\begin{aligned} a \ge J_E[u] &= \frac{1}{2}\int_0^\tau |u''|^2\,dt + \int_0^\tau [K(u,u') + E]\,dt \\ &\ge \left[\frac{2(1-|\mathbf{b}|_\infty)}{9|u_2-u_1|^2} - \frac{1}{9|u_2-u_1|^2}\right]\int_0^\tau |u'|^4\,dt \\ &- \frac{2|\mathbf{b}|_\infty^2}{9|u_2-u_1|} - C|u_2-u_1|. \end{aligned}$$

Therefore, $\int_0^\tau |u'|^4\,dt \le C(a, |\mathbf{b}|_\infty, |u_2-u_1|)$, which also implies $\int_0^\tau |u''|^2\,dt \le C(a, |\mathbf{b}|_\infty, |u_2-u_1|)$. As for a lower bound on τ we argue as follows. Integrating u' over $[0,\tau]$, we find that $|u_2-u_1| \le \tau^{1/2}\|u'\|_{L^2} \le \tau^{3/4}\|u'\|_{L^4} \le C\tau^{3/4}$. ∎

10.31 Lemma There exists $C(\tau, a, \mathbf{u}, |\mathbf{b}|_\infty)$ such that $\|u\|_{H^2([0,\tau])} \le C$ for all $u \in J_E^a(\mathbf{u},\mathbf{b})$.

Proof. By Cauchy-Schwarz, $\int_0^\tau |u'|^2\,dt \le C(a, |\mathbf{b}|_\infty, |u_2-u_1|)\tau^{1/2}$ which implies that $\|u\|_{L^\infty([0,\tau])} \le C(a, |\mathbf{b}|_\infty, |u_2-u_1|)\tau^{3/4} + |u_1|$ and $\int_0^\tau u^2\,dt \le (C(a, |\mathbf{b}|_\infty, |u_2 - u_1|)\tau^{3/4} + |u_1|)^2\tau$. Therefore, $\|u\|_{H^2([0,\tau])} \le C(a, |\mathbf{b}|_\infty, |u_2-u_1|, |u_1|, \tau)$. ∎

To find a minimizer we need to establish that J_E is coercive and weakly lower semicontinuous along a minimizing sequence. Lemma 10.31 implies coercivity provided that τ is uniformly bounded, which is proved in Subsection 10.8 for the regular case. We now show that J_E is sequentially weakly lower semicontinuous along sequences for which τ is bounded.

10.32 Lemma Suppose $\mathbf{u}_n \in X(\mathbf{u},\mathbf{b})$ with both $\|u_n\|_{H^2([0,\tau_n])}$ and τ_n uniformly bounded. Then $\liminf_{n_k\to\infty} J_E[u_{n_k}] \ge J_E[u]$ for some $u \in H^2([0,\tau])$.

Proof. We can rescale t to separate the dependence on τ from variations in u. Let $\widehat{X}_\tau(\mathbf{u},\mathbf{b}) = \{q \in H^2([0,1]) : q(0) = u_1, q(1) = u_2, q'(0) = b_1/\tau, q'(1) = b_2/\tau$, and

$q'(s) \neq 0$ for $s \in (0,1)\}$. Let $\widehat{X}(\mathbf{u},\mathbf{b}) = \cup_{\tau\in\mathbb{R}^+} \widehat{X}_\tau(\mathbf{u},\mathbf{b}) \subset H^2([0,1])$. Then

$$J_E[q] = \int_0^1 \left[\frac{1}{2\tau^3}|q''(s)|^2 + \tau K\left(q(s), \frac{q'(s)}{\tau}\right) + \tau E \right] ds$$

for $q \in \widehat{X}(\mathbf{u},\mathbf{b})$.

The functions $q_n(s) = u_n(\tau s)$ are uniformly bounded in $H^2([0,1])$, hence we can extract a weakly convergent subsequence $q_n \to q$ with $\tau_n \to \tau$. Observe that the functional $\int_0^1 \tau[K(q,q'/\tau) + E]\,ds$ is continuous in $\tau > 0$ and weakly continuous in $q \in H^2([0,1])$. The functional $\frac{1}{2\tau^3}\int_0^1 |q''|^2\,ds$ separates the variables τ and q and is continuous in τ and sequentially weakly lower semicontinuous in q. Hence, $\frac{1}{2\tau^3}\int_0^1 |q''|^2\,ds \leq \liminf_{n\to\infty} \frac{1}{2\tau_n^3}\int_0^1 |q_n''|^2\,ds$. Therefore $J_E[q] \leq \liminf_{n\to\infty} J_E[q_n]$. ∎

10.33 Lemma If $\mathcal{J}_E(\mathbf{u},\mathbf{b}) = J_E[u]$ for some $u \in X(\mathbf{u},\mathbf{b})$, then $u \in C^5([0,\tau])$ satisfies the Euler-Lagrange equation (10.1.1) and $H(u,u',u'',u''') = E$.

Proof. This follows from standard regularity theory, c.f [17]. ∎

Lemmas 10.31 and 10.32 imply that a minimizer exists in $H^2([0,\tau])$ provided that τ is bounded along some minimizing sequence. Lemma 10.33 states that a minimizer belonging to $X(\mathbf{u},\mathbf{b})$ is a solution to the Euler-Lagrange equations. Therefore, we must show that minimizing sequences exist for which τ is bounded and the weak limit belongs to $X(\mathbf{u},\mathbf{b})$. This issue will be addressed in Subsection 10.8 for the regular case. We conclude this subsection with a technical lemma concerning the continuity of the infima $\mathcal{J}_E(\mathbf{u},\mathbf{b})$ with respect to the parameter $\mathbf{b}$.

10.34 Lemma Suppose $\mathbf{b}_n \to \mathbf{b} \in \mathcal{K}$ and $\mathbf{u}_n \in X(\mathbf{u},\mathbf{b}_n)$ with $\tau_n \to \tau$, $J_E[u_n] = \mathcal{J}_E(\mathbf{u},\mathbf{b}_n)$ and $u_n \to u$ in $H^2([0,\tau])$. Then $J_E[u] = \mathcal{J}_E(\mathbf{u},\mathbf{b})$.

Proof. Again we can rescale t to separate the dependence on τ from variations in u. Let $\chi[\tau,\mathbf{b})] : [0,1] \to \mathbb{R}$ be a smooth strictly monotone function satisfying $\chi(0) = u_1$, $\chi(1) = u_2$, $\chi'(0) = b_1/\tau$, and $\chi'(1) = b_2/\tau$, and define

$$J_E[q,\tau;\mathbf{b}] = \int_0^1 L\left(q+\chi, \frac{q'+\chi'}{\tau}, \frac{q''+\chi''}{\tau^2}\right)\tau\,ds$$

for $q \in \{q \in H^2([0,1]) : q'(s) + \chi'(s) \neq 0 \text{ for } s \in (0,1)\}$. Then $\inf_{q,\tau} J_E[q,\tau;\mathbf{b}] = \mathcal{J}_E(\mathbf{u},\mathbf{b})$. The family $\chi[\tau,\mathbf{b}]$ can be chosen to vary continuously in $\mathbf{b}$, the family of functionals $J_E[q,\tau;\mathbf{b}]$ is continuous in $\mathbf{b}$ for each fixed q and τ. Therefore, the infimum $\mathcal{J}_E(\mathbf{u},\mathbf{b})$ is upper semicontinuous with respect to $\mathbf{b}$, cf. [30].

Let $q_n(s) = u_n(\tau_n s) - \chi[\tau_n,\mathbf{b}_n](s)$. Since $J_E[\cdot,\cdot;\mathbf{b}]$ is continuous, we have $J_E[q,\tau;\mathbf{b}] = \lim_{n\to\infty} J_E[q_n,\tau_n;\mathbf{b}_n] = \lim_{n\to\infty} \mathcal{J}_E(\mathbf{u},\mathbf{b}_n) \le \mathcal{J}_E(\mathbf{u},\mathbf{b}) \le J_E[q,\tau;\mathbf{b}]$. Therefore $J_E[u] = J_E[q,\tau;\mathbf{b}] = \mathcal{J}_E(\mathbf{u},\mathbf{b})$. ∎

10.8 The Existence of Minimizers

In this section we prove the existence of minimizers when $\mathbf{u} = (u_1,u_2) \in \mathrm{int}B$ for a single interval component I_E, and hence we will assume that $[u_1,u_2]$ is regular. In this case the following property is due to continuity.

(P3) There exist $\rho > 0$ and $\delta_0 > 0$ such that $K(u,v) + E \ge \rho > 0$ for all $(u,v) \in [u_1,u_2] \times [-\delta_0,\delta_0]$.

10.35 Lemma Under hypotheses (H1) and (H2), there exists a constant $T_2 > 0$, depending on $a, |\mathbf{b}|_\infty, |u_2 - u_1|, 1/\delta_0$, and $1/\rho$, such that for any $u \in J_E^a$ we have $\tau \le T_2$.

Proof. Let $S_{\delta_0} = \{t \in [0,\tau] : |u'(t)| \ge \delta_0\}$, where δ_0 is chosen in (P3). Since $|S_{\delta_0}|\delta_0^4 \le \int_0^\tau |u'|^4\,dt$, we have $|S_{\delta_0}| \le C(a,\delta^2,|u_2-u_1|,1/\delta_0^4)$. Let $\epsilon > 0$. Then,

$$
\begin{aligned}
a \ge J_E[u] &\ge \int_{S_{\delta_0}^c} [K(u,u') + E]\,dt + \int_{S_{\delta_0}} [K(u,u') + E]\,dt \\
&\ge \rho(\tau - |S_{\delta_0}|) - \epsilon^{-1} \int_{S_{\delta_0}} |u'|^4\,dt - C_\epsilon |u_2 - u_1|,
\end{aligned}
$$

which implies that $\tau \le T_2(a,\delta^2,|u_2-u_1|,1/\delta_0^4,1/\rho)$, by Lemma 10.30. ∎

Lemmas 10.29 and 10.35 imply that action is bounded below on $X(\mathbf{u},\mathbf{b})$, and the time τ is bounded on sublevel sets of J_E. Therefore, Lemma 10.32 implies that J_E is coercive and sequentially weakly lower semicontinuous along any sequence in $X(\mathbf{u},\mathbf{b})$ on which J_E is bounded. Let $\mathrm{cl}X_\tau(\mathbf{u},\mathbf{b}) = \{u \in H^2([0,\tau]) : u_n \to u$ for some sequence $u_n \in X(\mathbf{u},\mathbf{b})\}$. Functions $u \in \mathrm{cl}X(\mathbf{u},\mathbf{b}) = \cup_{\tau>0}\mathrm{cl}X_\tau(\mathbf{u},\mathbf{b})$ are monotone, possibly with critical inflection points. We have shown that the minimization problem is well-posed in the sense that a minimizer exists in $\mathrm{cl}X(\mathbf{u},\mathbf{b})$. However we must still show that this minimizer lies in $X(\mathbf{u},\mathbf{b})$ to apply Lemma 10.33.

Without loss of generality, we can assume that the following condition holds.

(P4) The constant $\delta_0 > 0$ in (P3) can be chosen such that $K(u,v) + E$ is nonincreasing in v for all $(u,v) \in [u_1,u_2] \times [-\delta_0,\delta_0]$.

Property (P4) is not a restriction on K. Consider the family of Lagrangians $|u''|^2/2 + K(u,v) - \alpha v$. Then $J_{\alpha,E}[u] = J_E[u] - \alpha|u_2 - u_1|$ for all $\alpha \in \mathbb{R}$. Hence, the minimizers of $J_{\alpha,E}$ are the same for all $\alpha \in \mathbb{R}$. Since $[u_1,u_2]$ is compact, we can choose $\alpha \geq 0$ such that $\partial_v K(u,0) - \alpha$ is strictly negative for all $u \in [u_1,u_2]$. Then, replacing $K(u,v)$ by $K(u,v) - \alpha v$ will satisfy (P4) without changing the minimization problem, and since $\alpha \geq 0$, the growth condition (H2) is still satisifed. Also, property (P3) still holds with possibly smaller values of ρ and δ_0. Property (P4) is used in the following lemma which implies that a minimizer must lie in $X(\mathbf{u},\mathbf{b})$.

10.36 Lemma Suppose $[w_1,w_2] \subset [u_1,u_2]$. Let $u \in \mathrm{cl}\widehat{X}_\tau(\mathbf{w},\mathbf{b}_*)$ for some $\mathbf{b}_* = (b,b)$ with $0 < |b| < \delta_0$. Define $\widehat{\tau} = |w_2 - w_1|/b > 0$ and $w \in X_{\widehat{\tau}}(\mathbf{w},\mathbf{b}_*)$ by $w(t) = bt + w_1$. Then $J_E[w] \leq J_E[u]$ and $w'(t) \neq 0$. If $u'' \not\equiv 0$, then $J_E[w] < J_E[u]$.

Proof. As in the proof of Lemma 10.28, transforming $u(t)$ and $w(t)$ into (u,z)-variables, we have

$$\begin{aligned} J_E[u] &= \frac{2}{9}\int_{w_1}^{w_2} |z'|^2\,du + \int_{w_1}^{w_2}\left[\frac{K(u,z^{2/3})+E}{z^{2/3}}\right] du \\ &\geq \int_{w_1}^{w_2}\left[\frac{K(u,|b|)+E}{|b|}\right] du = J_E[w] \end{aligned} \tag{10.8.18}$$

Here we have used properties (P3) and (P4). ∎

10.37 Corollary If $u \in \mathrm{cl}X(\mathbf{u},\mathbf{b})$ is a minimizer of J_E, then $\widehat{u}' \neq 0$ on $[0,\tau]$, hence $u \in X(\mathbf{u},\mathbf{b})$.

Proof. Suppose u has a critical point at t_0. Since u is monotone, t_0 is contained in some maximal compact interval of critical points I_E. By continuity, for any $b_* = (b,b)$ with $|b|$ sufficiently small, there is an interval $[t_1,t_2]$ containing I_E such that $u'(t_1) = u'(t_2) = b$. Let $w_1 = u(t_1)$ and $w_2 = u(t_2)$. Then using Lemma 10.36, we can construct a function $w \in X(\mathbf{w},\mathbf{b}_*)$ such that $J_E[w] < J_E[u|_{[t_1,t_2]}$. Replacing $u|_{[t_1,t_2]}$ by w yields a function $\widehat{u} \in H^2([0,\widehat{\tau}])$ such that $J_E[\widehat{u}] < J_E[u]$, which contradicts the fact that u is a minimizer. ∎

We have proved the following theorem which implies Lemma 10.25.

10.38 Theorem Suppose L satisfies (H1) and (H2) on an interval component I_E. If $\mathbf{u} \in B$ and $\mathbf{b} \in \mathcal{K}$, then there exists a strictly monotone minimizer $u \in X(\mathbf{u},\mathbf{b}) \cap C^5([0,\tau])$ of J_E which satisfies the Euler-Lagrange equation (10.1.1).

In fact the above results prove Theorem 10.38 for $\mathbf{u} \in \mathrm{int}B$. In order to include all of B one can choose a sequence of minimizers $u_n \in X(\mathbf{u}_n,\mathbf{b})$ with $\mathbf{u}_n \to \mathbf{u} \in \partial B$.

To obtain a limit in $X(\mathbf{u},\mathbf{b})$ we need to argue that τ_n is uniformly bounded. Suppose not, i.e. $\tau_n \to \infty$. Since $\|u_n\|_{H^2([0,\tau_n])} \le C$ we would obtain, after appropriate shifts, a solution asymptotic to either u_1 or u_2, or both, which is a contradiction.

10.9 Extensions and Concluding Remarks

10.9.a More General Lagrangians

Essentially, the hypotheses (H1) and (H2) in Section 10.1 are stated in the manner most convenient for implementing the minimization in Section 10.7 without too many technical details. The analysis in that section is needed to establish the surjectivity of the projection $\pi\mathcal{L}_\pm$ onto the base B, which ensures that the continuation to a twist system is well-defined. The geometric and topological considerations in Section 10.6, other than surjectivity, require merely the convexity of L in u''.

Thus, the hypotheses (H1) and (H2) can be weakened. For example the conditions

(H1′) $0 < \alpha \le \partial_w^2 L(u,v,w) \le \alpha^{-1}$ for all (u,v,w)

(H2′) $L(u,v,w) \ge \frac{\alpha}{2}w^2 - C(|u|) - C(|u|)|v|^\gamma, \quad \gamma < 4,$

where $C(|u|)$ is locally bounded, would also be sufficient. The hypothesis (H1′) implies that the action J_E is well-defined on the Sobolev space $H^2(0,T)$ and (H2′) implies that the the action is bounded below. Moreover, the use of other function spaces would allow super-quadratic growth of L in u''.

10.9.b Sharp Lower Bounds

Consider the Lagrangian $L(u,u',u'') = (u'')^2/2 - (u')^4/4$ and $E > 0$. The corresponding action is not bounded below. Indeed if $u_A(t) = A\sin(\pi(t - T/2)/T)$ then

$$J[u_A] = \int_0^T \left[L(u_A, u_A', u_A'') + E\right] dt = \frac{CA^2}{T^3} - \frac{C'A^4}{T^3} + ET.$$

Thus for A large enough $J[u_A] \to -\infty$ as $T \to 0$. This example shows that the minimization procedure can fail when $\gamma \ge 4$ in hypothesis (H2).

This problem is not just a failure of a particular method. For $E = 0$ in the previous example, it is not difficult to show that there are values u_1 and u_2 for which no monotone laps exists. The growth condition (H2) is a geometric restriction. Since M_E is non-compact, it is inevitable that some such restriction is necessary.

10.9.c The Topology of Energy Manifolds

For Lagrangians that satisfy the convexity hypothesis $\partial_w^2 L \ge \alpha > 0$, the topology of the energy manifolds $M_E = H^{-1}(E)$ can be completely determined from the sign changes in the potential $L(u,0,0) + E$. Consider the homotopy

$$L_\lambda(u,v,w) = (1-\lambda)L(u,v,w) + \lambda\left[\frac{\alpha}{2}w^2 + L(u,0,0)\right]$$

of Lagrangians with corresponding Hamiltonians $H_\lambda(x) = p_u v + L^*_\lambda(u,v,p_v)$. If M_E is regular, then it is immediately clear that M_λ is regular for all $\lambda \in [0,1]$. So $H_\lambda(x)$ defines a cobordism between $M = M_0$ and $M_1 = \{\alpha w^2/2 + p_u v - L(u,0,0) = E\}$. A straightforward calculation shows that the height function $(\lambda, x) \to \lambda$ has no critical points, and hence standard Morse theory implies that M_E is homotopy equivalent to M_λ for all $\lambda \in [0,1]$. In fact they are diffeomorphic.

In [4] the homotopy type of M_1 was computed for the regular case, which implies $L(u,0,0) + E$ has simple zeros. There is a deformation retraction of M_1 onto a bouquet of circles and 2-spheres. Consequently, the homology of M_1 is determined by its Betti numbers with $\beta_0 = 1$ and $\beta_n = 0$ for $n > 2$. The second Betti number β_2 is the number of compact components of N_E, i.e. the number of compact intervals in $\mathbb{R}$ on which $L(u,0,0) + E \geq 0$. The first Betti number is the number of compact intervals on which $L(u,0,0) + E \leq 0$, which depends on the behavior of $L(u,0,0)$ as $|u| \to \infty$. In any case, $\beta_1 \in \{\beta_2 - 1, \beta_2, \beta_2 + 1\}$.

A simple example shows that lower bound $\dim H_2(M)$ is sharp. Let $L(u,u',u'') = (u'')^2/2 + u^2/2$ and $E > 0$. Then $M_E \approx S^1 \times \mathbb{R}^2$ with $\dim H_2(M) = 0$, and solving the (linear) Euler-Lagrange equation explicitly shows that there are no closed characteristics.

10.9.d Singular Manifolds

Singularities in M_E occur at critical points of $L(u,0,0) + E$.
Depending on the eigenvalues of these points as equilibrium points of the flow φ^t, there are three types of singularities: saddle (four real eigenvalues), saddle-focus (four complex eigenvlaues), and center (four imaginary eigenvalues).

Consider an energy manifold M_E with (isolated) singular points in the interior of a compact component I_E of πN. The techniques in this paper imply that for each component of $I \setminus \{\text{singular points}\}$ there is a closed characteristic independent of the type of the singularities. Note that this is already different from the first-order Lagrangian case where singular manifolds cannot contain closed characteristics.

However, in the second-order case depending on the type of the singularities, even more closed characteristics must exist. It is shown in [33] that if the twist property holds on each component of $I \setminus \{\text{singular points}\}$ and the singular points are either of saddle-focus or center type, then the twist property holds on all of I_E, and additional closed characteristics exist with nonzero intersection number.

The arguments of this paper should be applicable in this case by continuation of a singular manifold with saddle-focus or center type singularities to a twist system. The main issue is whether the surjectivity criterion in Lemma 10.25 holds over all of I_E. We leave the details for future work, but we do not forsee any major problems in applying the techniques of [18, 17], which provide exactly the tools required to minimize in the presence of a saddle-focus or center equilibrium, to show, again by minimization as in Section 10.7, that the surjectivity condition holds.

10.9.e Forcing of Additional Closed Characteristics

For twist systems, it is shown in [11] that the existence of certain closed characteristics can force the existence of a multitude of closed characteristics due to their braiding and knotting. The above continuation method does not always immediately apply because the intersection numbers corresponding to these additional closed characteristics can be trivial, but in certain cases the topological information obtained from the braid type will imply nontrivial intersection number. One might also attempt to prove the existence of multiple solutions by more carefully studying the intersections using the fact that they are intersections of Lagrangian manifolds, which we leave for future work.

A — Mappings and topology

A.1 Differentiable mappings

A.1 Theorem (The Inverse Function Theorem). Let $\Omega \subset \mathbb{R}^n$ be a neighborhood of x_0, and let $f \in C^1(\Omega;\mathbb{R}^n)$. Assume that $f'(x_0)$ is an invertible mapping on $\mathbb{R}^n$. Then there exist neighborhoods $U \ni x_0$ and $V \ni f(x_0)$ such that f is a local diffeomorphism on U. Moreover,

$$(f^{-1}(x_0))' = (f'(x_0))^{-1},$$

and x_0 is the unique solution of $f(x_0) = p$ in U.

A.2 Theorem (C^1-version of the Implicit Function Theorem). Let $\Omega \times \Lambda \subset \mathbb{R}^n \times \mathbb{R}^k$ be a neighborhood of (x_0, λ_0), and let $f \in C^1(\Omega \times \Lambda;\mathbb{R}^n)$. Assume that $f(x_0,\lambda_0) = 0$, and $d_x f(x_0,\lambda_0)$ is an invertible matrix. Then there exists a neighborhood $\Lambda' \subset \Lambda$ of λ_0 and a C^1-function $g:\ \Lambda' \to \mathbb{R}^n$, such that

$$g(\lambda_0) = x_0, \quad \text{and} \quad f(g(\lambda),\lambda) = 0, \quad \forall\, \lambda \in \Lambda'.$$

In addition $g'(\lambda) = -[d_x f(g(\lambda),\lambda)]^{-1} d_\lambda f(g(\lambda),\lambda)$.

A.3 Theorem Let $f\colon X \to X$ with $d(T(x),T(y)) \le k d(x,y)$, with $k < 1$. Then, $T(x) = x$ has a unique solution.

A.1.a Approximation

A.4 Theorem Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and let $f \in C^\ell(\overline{\Omega};\mathbb{R}^m)$. Then there exists a sequence of maps $f^k \in C^\infty(\overline{\Omega};\mathbb{R}^m)$ such that

$$\|f - f^k\|_{C^\ell} \to 0,$$

as $k \to \infty$.

Proof. Construct a sequence f^k via mollification. Let $\zeta(x)$ be a C^∞-function on $\mathbb{R}^n$, satisfying

$$\begin{array}{lll} \text{(i)} & \zeta(x) \equiv 0, \quad \text{for} \quad |x| \geq 1, \quad \text{and} \\ \text{(ii)} & \displaystyle\int_{\mathbb{R}^n} \zeta(x)dx = 1. \end{array}$$

Clearly, $\zeta \in C_0^\infty(B_1(0))$, and the radius of the ball can be altered by rescaling ζ as follows: $\zeta^\epsilon(x) = \epsilon^{-n}\zeta(x/\epsilon)$. Then, $\zeta^\epsilon \in C_0^\infty(B_\epsilon(0))$, and $\int_{\mathbb{R}^n} \zeta^\epsilon(x)dx = 1$, by the choice of the rescaling. The cut-off function ζ^ϵ is called a mollifier.

Regard f as a function on $\mathbb{R}^n$ by extending it by zero outside Ω, and define

$$f^k(x) := (\zeta^{1/k} * f)(x) := \int_{\mathbb{R}^n} \zeta^{1/k}(x-y)f(y)dy.$$

Since $\zeta^{1/k}$ is smooth and compactly supported, the convolution is a C^∞-function on $\mathbb{R}^n$ for all $k \geq 1$. In particular $f^k \in C^1(\overline{\Omega})$.

A.5 Exercise Show, all partial derivatives of $\zeta^{1/k} * f$ exist, and therefore that $f^k \in C^\infty(\mathbb{R}^n)$. ■

It remains to show that f^k converges to f in the C^0-topology.

$$\begin{aligned} \|f - f^k\|_{C^0} &= \max_{x \in \overline{\Omega}} \|f(x) - (\zeta^{1/k} * f)(x)\| \\ &= \max_{x \in \overline{\Omega}} \left\| \int_{\mathbb{R}^n} \zeta^{1/k}(x-y)\,[f(x) - f(y)]\,dy \right\| \\ &\leq \max_{x \in \overline{\Omega}} \left(\max_{|x-y| \leq 1/k} \|f(x) - f(y)\| \right). \end{aligned}$$

Since f is uniformly continuous on $\overline{\Omega}$, the right hand side goes to zero as $k \to \infty$, which proves the lemma. ■

■ **A.6 Remark** If f has additional regularity on an open subset $\Omega' \subset \Omega$, say the restriction ti $\overline{\Omega}'$ is in $C^{\ell'}(\overline{\Omega}';\mathbb{R}^m)$, then $f^k \to f$ in $C^{\ell'}(\overline{\Omega}',\mathbb{R}^m)$. ■

A.2 The theorem's of Tietze, Sard and Smale

Sard's Theorem and its inifinite dimensional version due to Smale — the Sard-Smale Theorem — form fundamental steps in establishing topological tools as we have seen in the previous section. We start with giving the 'uncensured' version of Sard's Theorem, and proof is given in the simplist case (the version of Sard's Theorem used in the previous section).

A.7 Theorem Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a map of class C^k. Then the set of critical values $f(C_f)$ has (Lebesgue) measure zero, provided $k \geq \max(1, n - m + 1)$.

Proof. We restrict here to the case that $n = m$. Let $D \subset \mathbb{R}^n$ be a cube with sides of size ℓ. For points $x, x_0 \in D$, using the fact that $f \in C^1(\mathbb{R}^n)$, Taylor's Theorem implies that

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + R(x, x - x_0),$$

where $\|R(x, x - x_0)\| = o(\|x - x_0\|)$, uniformly in $x \in D$, i.e. for any $\epsilon > 0$ there exists a $\delta > 0$ such that $\|R(x, x - x_0)\| \leq \epsilon\|x - x_0\|$, for all $\|x - x_0\| < \delta$. Suppose $x_0 \in C_f \cap D$, then the points $L = \{f(x_0) + f'(x_0)(x - x_0)\}$ represent an affine subspace of dimension less or equal to $n - 1$.

The estimates on the remainder term R imply that if $\|x - x_0\| < \delta$, then $\mathrm{dist}(f(x), L) \leq \epsilon\delta$. Since, f is Lipschitz continuous on D with Lipschitz constant N, it also holds that $\|f(x) - f(x_0)\| \leq N\delta$. The image of the ball $B_\delta(x_0$ under f is therefore contained in a 'cuboid' centered at $f(x_0)$ with sides of size less that $2N\delta$ in L, and of size less that $2\epsilon\delta$ orthogonal to L. The volume of the cuboid is less than $2^n\delta^n N^{n-1}\epsilon$.

Next we divide up the cube D into sub-cubes D_δ with sides of length δ. Thus the number of such cubes is Δ^n, with $\Delta > \ell/\delta$. Each sub-cube that contains a critical point x_0 has the property that

$$\mu(f(D_\delta)) \leq 2^n(\ell/\Delta)^n N^{n-1}\epsilon.$$

Consequently, $\mu(f(C_f \cap D)) \leq \Delta^n \mu(f(D_\delta)) = 2^n \ell^n N^{n-1}\epsilon$, which proves the theorem since ϵ can be chosen arbitrarily small. ■

The version of Sard's Theorem presented above holds for any combination of $n, m \geq 0$. Only when $n \geq m$ the proof is non-trivial, and the proof here is restricted to the case $n = m$. In the case $n < m$ we argue as follows: Let μ denote the Lebesgue measure, then $\mu(f(\mathbb{R}^n)) = 0$, since $f(\mathbb{R}^n)$ is an n-dimensional subset of $\mathbb{R}^m$, with $n < m$ (f is a C^1-map). This implies that Sard's Theorem is trivially satisfied when $n < m$.

A.8 Theorem (Tietze's extension Theorem) Let (X,d) be a metric space and $A \subset X$ a closed subset. Suppose $f : A \subset X \to \mathbb{R}$ is continuous.Then, there exists a continuous extension $\tilde{f} : X \to \mathbb{R}$, such that $\tilde{f}|_A = f$.

A.9 Theorem (Dugundji) Let (X,d) be a metric space and $A \subset X$ a closed subset. Let Y be a normed linear space and $f : A \subset X \to Y$ is continuous. Then, there exists a continuous extension $\tilde{f} : X \to \mathbb{R}$, such that $\tilde{f}|_A = f$ and $\tilde{f}(X)$ is contained in the convex hull of $f(A)$.

B — Nemytskii Mappings

B.1 Basic Nemytskii maps

We start off this chapter with a basic result about composition, or substitution mappings — also called superposition mappings. Such maps are also referred to in the literature as Nemytskii maps, or operators. At a later stage when investigating differentiability properties of maps between Banach spaces this result plays a central role.

B.1 Definition Let $D \subset \mathbb{R}^n$ be a bounded domain, and let $g(x,s)$ that satisfy the following conditions:

(i) for all $s \in \mathbb{R}$ the function $x \mapsto g(x,s)$ is Lebesgue measurable in D,

(ii) the function $s \mapsto g(x,s)$ is continuous on $\mathbb{R}$ for $x \in D$ a.e.

The function $g(x,s)$ is called a Carathéodory function.

We will use Carathéodory functions now to define substitution mappings; Nemytskii maps. Let $u(x)$ be a measurable function on D, then the map $u \mapsto g(\cdot,u(\cdot))$, assigns a measurable function to each u, and is denoted by f. Notation;

$$f(u)(x) := g(x,u(x)).$$

Indeed, let u^n be simple functions converging to u. Then by (i) in the definition of Carathéodory functions, the substitution $g(x,u^n(x))$ is measurable for $x \in D$ a.e. By (ii) $g(x,u^n(x))$ converges to $g(x,u(x))$, establishing the measurability of $g(x,u(x))$. The map f is called a Nemytskii map.

For our purposes we want a Nemytskii map to act between L^p-spaces, or more general Sobolev space. For this additional growth conditions are needed. We will see that these conditions are in fact necessay and sufficient.

B.2 Theorem Let $D \subset \mathbb{R}^n$ be a bounded domain, and let $g(x,s)$ a Carathéodory function on $D \times \mathbb{R}$. If there exist $1 \le p,q \le \infty$, a function $h \in L^q(D)$, and a constant $C > 0$ such that

$$|g(x,s)| \le h(x) + C|s|^{\frac{p}{q}}, \tag{B.1.1}$$

then the Nemytskii mapping $u(x) \mapsto f(u)(x)$, introduced above, is a well-defined mapping from $L^p(D)$ to $L^q(D)$. Moreover, f is bounded and continuous.

Proof. We start with the well-definedness of f. Let $u \in L^p(D)$, then

$$\begin{aligned} \|f(u)\|_q^q &= \int_D |g(x,u(x))|^q dx \\ &\le \int_D \Big| h(x) + C|u(x)|^{\frac{p}{q}} \Big|^q dx \le C' \int_D (|h(x)|^q + |u(x)|^p) dx \\ &\le \|h\|_q^q + C\|u\|_p^p, \end{aligned}$$

which proves that f is a well-defined map from L^p to L^q, and maps bounded sets in L^p to bounded sets in L^q.

As for the continuity we argue as follows. Let $u^n \to u$ in $L^p(D)$. Then, we may assume without loss of generality, that $|u^n(x)| \le k(x)$ (possibly along a subsequence), with $k \in L^p(D)$. For the Nemytskii map this implies that $|f(u^n)(x)| \le h(x) + |k(x)|^{\frac{p}{q}}$, with $h + |k|^{\frac{p}{q}} \in L^q(D)$. In order to apply Lebesgue's Dominated Convergence Theorem now we need to show that $f(u^n)(x)$ converges a.e. to $f(u)(x)$. Assuming the latter we then obtain that $f(u^n) \to f(u)$ in $L^q(D)$.

We establish the pointwise convergence following the DeFigueiredo [**?**]. Define $a(x,s) = g(x,s+u(x)) - g(s,u(x))$, then $a(x,0) = 0$, and by the previous $a(x,s)$ is measurable in x for each $s \in \mathbb{R}$, and $a(x,s)$ is continuous in s for all $x \in D$ a.e. Set $v^n = u^n - u$, then $a(x,v^n) = g(x,u^n) - g(x,u)$. Therefore, if $v^n \to 0$ in measure we need to prove that $a(x,v^n) \to 0$ in measure. For an $\epsilon > 0$, end define the sets

$$D_\epsilon^k = \Big\{ x \in D \,:\, |s| < \frac{1}{k} \implies |a(x,s)| < \epsilon \Big\}.$$

Clearly, these sets are nested with $\cup_k D_\epsilon^k = D$, and $\big|\cup_{k=1}^{k_0} D_\epsilon^k\big| = |D_\epsilon^{k_0}|$. Therefore, we can choose $k_0 \ge 1$ such that $|D| - |D_\epsilon^{k_0}| < \frac{\epsilon}{2}$. Let $\Omega^n = \{x \in D \,:\, |v^n(x)| < \frac{1}{k_0}\}$, then then, by assumption, there exists an $n_0 \ge 1$ such that $|D| - |\Omega^n| < \frac{\epsilon}{2}$ for all $n \ge n_0$. Now let $\Delta^n = \{x \in D \,:\, |a(x,v^n(x))| < \epsilon\}$, then construction $\Omega^n \cap D_\epsilon^{k_0} \subset \Delta^n$. Clearly, $|D| - |\Delta^n| \le |D| - |\Omega^n \cap D_\epsilon^{k_0}| \le \big[|D| - |\Omega^n|\big] + \big[|D| - |D_\epsilon^{k_0}|\big] < \epsilon$, which proves that

$$|D \backslash \Delta^n| = \big|\{x \in D \,:\, |a(x,v^n(x)| \ge \epsilon\}\big| < \epsilon,$$

and therefore $a(x,v^n(x)) \to 0$ in measure. ∎

There are numerous generalizations of this result. For example one can consider functions taking values in $\mathbb{R}^m$, or use Orlicz spaces, etc. Another extension is to allow D to be any domain, i.e. the above result remains valid for arbitrary domains. As mentioned before the above result characterizes Nemytskii mappings. We will mention the following theorem without proof (see [?]).

B.3 Theorem Let $D \subset \mathbb{R}^n$ be a bounded domain, and let $g(x,s)$ a Carathéodory function on $D \times \mathbb{R}$. If the associated Nemytskii mapping f maps for $L^p(D)$ to $L^q(D)$, for $1 \leq p,q < \infty$, then there exists a $C > 0$ and a function $h \in L^q(D)$ such that

$$|g(x,s)| \leq h(x) + C|s|^{\frac{p}{q}},$$

Moreover, f is bounded and continuous.

This theorem remains valid for arbitrary domains D, and also for $q = \infty$.

The next question concernes differentiabilty of Nemytskii maps. The Theorems B.2 and B.3 reveal that differentiability of g is not enough to guarantee the well-defined unless we have growth conditions on g_s as well. Given an Carathéodory function g whose partial derivative g_s is also a Carathéodory function. Theorem B.2 gives that if

$$|g_s(x,s)| \leq k(x) + C|s|^{\frac{a}{b}}, \quad \forall s \in \mathbb{R}, \quad \forall x \in D,$$

with $k \in L^b(D)$, and $1 \leq a,b \leq \infty$, then the mappings $u(x) \mapsto g_s(x,u(x))$ is well-defined from $L^a(D)$ to $L^b(D)$. The growth condition on g_s also yields a growth condition on g itself. We consider a family of functions $g(x,s)$ as follows: In the next theorem we assume that

$$g(x,s) = \int_0^s g_t(x,t)dt + c(x), \quad c \in L^q(D).$$

Using this description we obtain

$$|g(x,s)| \leq C|s|^{\frac{a}{b}+1} + k(x)|s| + d(x) \leq C'|s|^{\frac{a}{b}+1} + C'|k(x)|^{\frac{b}{a}+1} + c(x),$$

Now let $(\frac{b}{a}+1)q = b, \quad (\frac{a}{b}+1)q = a$. Under the assumption $a = p$, and $p > q$, we obtain

$$a = p, \quad b = \frac{pq}{p-q}.$$

We conclude that $u(x) \mapsto g(x,u(x))$ is a Nemytskii map from $L^p(D)$ to $L^q(D)$, and $u(x) \mapsto g_s(x,u(x))$ is well-defined from $L^p(D)$ to $L^{\frac{pq}{p-q}}(D)$. Next we investigate the relation between differentiability of the Nemytskii map f and the well-definedness of $u(x) \mapsto g_s(x,u(x))$.

B.4 Theorem Let $D \subset \mathbb{R}^n$ be a bounded domain, and let $g(x,s)$ and $g_s(x,s)$ be Carathéodory functions on $D \times \mathbb{R}$. If there exist $1 \leq q < p \leq \infty$, a function $k \in L^{\frac{pq}{p-q}}(D)$, and a constant $C > 0$ such that

$$|g_s(x,s)| \leq k(x) + C|s|^{\frac{1}{q}-\frac{1}{p}},$$

then for the functions g defined above the Nemytskii mapping $u(x) \mapsto f(u)(x)$ is continuously differentiable. Moreover,

$$[f'(u)\varphi](x) = g_s(x, u(x))\varphi(x), \quad \forall u, \varphi \in L^p(D).$$

Proof. From the previous we know that $u(x) \mapsto g_s(x, u(x))$ is bounded and continuous from $L^p(D)$ to $L^{\frac{pq}{p-q}}(D)$. Moreover, $g_s(x, u(x))\varphi(x) \in L^q(D)$:

$$\int_D |g_s(x, u(x))\varphi(x)|^q dx \le \left(\int_D |g_s(x, u(x))|^{\frac{pq}{p-q}}\right)^{\frac{p-q}{p}} \left(\int_D |\varphi(x)|^p\right)^{\frac{q}{p}}.$$

This shows that $\varphi \mapsto g_s(x, u(x))\varphi$ defines a bounded linear map from $L^p(D)$ to $L^q(D)$ with norm bounded by $\|g_s(\cdot, u(\cdot))\|_{\frac{pq}{p-q}}$, and the map varies continuously with $u \in L^p(D)$. It remains to show that this is indeed the Fréchet derivative of f. We have

$$\begin{aligned} \theta(x) &= g(x, u(x) + \varphi(x)) - g(x, u(x)) - g_s(x, u(x))\varphi(x) \\ &\int_0^1 [g_s(x, u(x) + t\varphi(x)) - g_s(x, u(x))]\,\varphi(x)dt. \end{aligned}$$

For θ we have

$$\|\theta\|_q \le \left(\int_0^1 \|g_s(\cdot, u(\cdot) + t\varphi(\cdot)) - g_s(\cdot, u(\cdot))\|_{\frac{pq}{p-q}} dt\right)^{\frac{p-q}{p}} \|\varphi\|_p \le \epsilon \|\varphi\|_p,$$

for $\|\varphi\|_p < \delta_\epsilon$, which follows from the continuity of the Nemytskii map defined by g_s. Using these estimates it follows that

$$\|f(u + \varphi) - f(u) - g_s(\cdot, u)\varphi\|_q = o(\|\varphi\|_p),$$

for $\|\varphi\|_p$ sufficiently small. ■

There is a lot more that can be said about differentiability of Nemytskii maps, in particular the case $p = q$.

B.5 Exercise *Investigate the case $p = q$.* ■

Let us now consider the case of integrating Nemytskii mappings. Given a Carathéodory function $g(x, s)$ we define the integral

$$G(x, s) = \int_0^s g(x, t)dt,$$

which is also a Carathéodory function. As before we assume that $|g(x, s)| \le h(x) + C|s|^{\frac{p}{q}}$, $h \in L^q(D)$. Integration yields $|G(x, s)| \le C'|s|^{\frac{p}{q}+1} + |h(x)|^{\frac{q}{p}+1}$, and the composition map $u(x) \mapsto G(x, u(x))$ is bounded and continuous from $L^p(D)$ to $L^{\frac{pq}{p+q}}(D)$. In the special case $q = 1 + \frac{1}{p-1}$ we have a map from $L^p(D)$ to $L^1(D)$. Finally we mention the following result.

B.6 Theorem Let $D \subset \mathbb{R}^n$ be a bounded domain, and let $g(x,s)$ be a Carathéodory functions on $D \times \mathbb{R}$. If there exist $1 \leq p \leq \infty$, a function $k \in L^{\frac{p}{p-1}}(D)$, and a constant $C > 0$ such that $|g(x,s)| \leq h(x) + C|s|^{p-1}$, then for the functional defined by

$$F(u) = \int_D G(x,u(x))dx,$$

is a continuously differentiable function on $L^p(D)$. Moreover, its derivative is continuously differentiable from $L^p(D)$ to $L^{\frac{p}{p-1}}(D)$, and $F' = f$.

Proof. The proof follows along the same lines as the proof of Theorem C.18 and is therefore left to the reader. ∎

C — Sobolev Spaces

C.1 Sobolev Spaces

C.1.a Weak derivatives and Sobolov spaces

Assume that $D \subset \mathbb{R}^n$ is an arbitrary domain, i.e. an open set in $\mathbb{R}^n$. A function $u : D \to \mathbb{R}$ is call locally integrable, or $u \in L^1_{\text{loc}}(D)$, if $u \in L^1(D_0)$, for every measurable strict subset $D_0 \subset D$. For every $u \in L^1_{\text{loc}}(D)$ the integral

$$T_u = \int_D u(x)\phi(x)dx, \quad \phi \in C_0^\infty(D),$$

is well-defined and is call a regular distribution.

C.1 Definition A locally integrable function $u \in L^1_{\text{loc}}(D)$ has a weakly partially differentiable with respect to x_i there exists a function $v_i \in L^1_{\text{loc}}(D)$ such that

$$-\int_D u(x)\frac{\partial \phi}{\partial x_i}(x)dx = \int_D v_i(x)\phi(x)dx,$$

for all $\phi \in C_0^\infty(D)$. The fucntion v_i is called the weak partial derivative with respect to x_i, and is again denoted by $v_i = \frac{\partial u}{\partial x_i}$.

The weak derivative, if it exists as defined above, has to be regarded as a regular distribution $\frac{\partial T_u}{\partial x_i} = T_{v_i}$.

Before we go to the definition of the basic Sobolev spaces let us introduce some notation. With the partial-symbol ∂^α, with multi-index $\alpha = (\alpha_1, \cdots, \alpha_n)$, we denote

$$\partial^\alpha u = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}} u.$$

We can now introduce the standard norms. Let $m \in \mathbb{N}$, then

$$\|u\|_{m,p} = \Big(\sum_{0\le|\alpha|\le m} \|\partial^\alpha u\|_p^p\Big)^{\frac{1}{p}}, \quad 1 \le p < \infty,$$

where $\|\cdot\|_p$ is the standard L^p-norm, and $|\alpha| = \sum_i \alpha_i$. For $p = \infty$ we define

$$\|u\|_{m,\infty} = \max_{0\le|\alpha|\le m} \|\partial^\alpha u\|_\infty,$$

where $\|\cdot\|_\infty$ is the standard L^∞-norm. For $m = 0$ these norm reduce to the L^p-norms.

C.2 Definition The Sobolev space $W^{m,p}(D)$ is defined as the subset of functions $u \in L^p(D)$ for which all weak partial derivatives $\partial^\alpha u$ exist for $0 \le |\alpha| \le m$, and all lie in $L^p(D)$.

Another definition is the space $H^{m,p}(D)$ which is defined as the completion of the set

$$\{u \in C^m(D) \;:\; \|u\|_{m,p} < \infty\}$$

with respect to the norm $\|\cdot\|_{m,p}$, and is a Banach space by definition.

At first sight these two definition seem to define different classes of spaces.

C.3 Exercise *Prove the inclusion $H^{m,p}(D) \subset W^{m,p}(D)$.* ■

One can prove that also the Sobolev spaces $W^{m,p}$ are Banach spaces. These spaces have been used in analysis for quite some time, but it wasn't until the early 60's that Meyers and Serrin proved the following result.

C.4 Theorem For $1 \le p < \infty$ it holds that $H^{m,p}(D) = W^{m,p}(D)$.

This result immediately proves that the Sobolev spaces $W^{m,p}$ are Banach spaces, and $\|\cdot\|_{m,p}$ are equivalent norms on $W^{m,p}$. We sometimes use the notation Hm, p, usually for the Hilbert case $p = 2$, and sometimes $W^{m,p}$. Sobolev spaces are 'nice' Banach space as the following theorem reveals:

C.5 Theorem For $1 \le p < \infty$ the Sobolev spaces $W^{m,p}(D)$ are separable. If in addition $1 < p < \infty$, then these spaces are also reflexive and uniformly convex.

From the proof of the Meyers and Serrin result it follows that in fact the set $C^\infty(D)$ is a dense subset of $W^{m,p}(D)$. Under certain conditions on the domain D is also holds that $C^k(\overline{D})$, $k \ge m$, is dense in $W^{m,p}$.

Finally we define the space $W_0^{m,p}(D) = H_0^{m,p}(D)$ as the closure of $C_0^\infty(D)$ in $W^{m,p}(D)$.

Sobolev spaces for $m<0$ can be defined as an interpretation of dual Sobolev spaces. We mention without proof that for $1\le p<\infty$, the dual space $(W^{m,p}(D))'$ consists functionals T of the form

$$Tu=\sum_{0\le|\alpha|\le m}\int_D v_\alpha(x)\partial^\alpha u(x)dx,$$

with $v_\alpha\in L^{p'}(D)$ for all α. If $u\in W_0^{m,p}(D)$, then the bounded linear functionals can be regarded as distributions of the form

$$T=\sum_{0\le|\alpha|\le m}(-1)^{|\alpha|}\partial^\alpha v_\alpha.$$

In this case we define $W^{-m,p'}(D)=(W_0^{m,p}(D))'$.

In the literature there exists many variations and generalizations of Sobolev spaces, but for the purposes of this course the basic Sobolev spaces are sufficient. One generalization that is worth mentioning is that fractional order Sobolev spaces. We will only give an intuitive definition on this account. In the case $D=\mathbb{R}^n$, and $p=2$ the Sobolev spaces $W^{m,2}(\mathbb{R}^n)$ can also be characterized via the Fourier transform Define the Fourier transform:

$$\widehat{u}(\xi)=\frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n}e^{-ix\cdot\xi}u(x)dx.$$

Then $W^{m,2}(\mathbb{R}^n)=\{u\in L^2(\mathbb{R}^n)\ :\ (1+\|\xi\|^m)\widehat{u}\in L^2(\mathbb{R}^n)\}$, immediately yields a definition of fractional Sobolev spaces $W^{m,2}(\mathbb{R}^n)$ for $m\ge 0$.

C.1.b Sobolev inequalities

As spaces of functions the Sobolev spaces $W^{m,p}$ are very useful for studying partial and ordinary differential equations as we have seen in the previous chapter. The Sobolev inequalities provide insight into the nature of Sobolev functions. We start with the case when $D=\mathbb{R}^n$.

C.6 Lemma — Sobolev inequality. Let $u\in C_0^\infty(\mathbb{R}^n)$, and let $1\le p<n$, then there exists a (universal) constant $C(p,n)>0$ such that[a]

$$\|u\|_{p_n}\le C(n,p)\|\nabla u\|_p,$$

where $\frac{1}{p_n}=\frac{1}{p}-\frac{1}{n}$.

[a] We use the convention $\|\nabla u\|_p=\left(\int_{\mathbb{R}^n}\sum_i\left|\frac{\partial u}{\partial x_i}\right|^p dx\right)^{\frac{1}{p}}$.

Proof. We start with the case $p=1$. We have that $u(x)=\int_{-\infty}^{x_i}\frac{\partial u}{\partial x_i}(\widehat{x})dt$, where $\widehat{x}$ is x with x_i replaced by t. Therefore

$$|u(x)|^n\le\prod_{i=1}^n\int_{\mathbb{R}}\left|\frac{\partial u}{\partial x_i}\right|dx_i=f_1(x)\cdots f_n(x),$$

where $f_i(x)$ is independent of x_i. If we now apply Hölder's inequality to $|u|^{\frac{n}{n-1}}$:

$$\begin{aligned}\int_{\mathbb{R}} |u|^{\frac{n}{n-1}} dx_1 &= \int_{\mathbb{R}} (f_1 \cdots f_n)^{\frac{1}{n-1}} dx_1 = f_1^{\frac{1}{n-1}} \int_{\mathbb{R}} (f_2 \cdots f_n)^{\frac{1}{n-1}} dx_1 \\ &\le f_1^{\frac{1}{n-1}} \prod_{i\ge 2} \Big(\int_{\mathbb{R}} f_i dx_1 \Big)^{\frac{1}{n-1}}\end{aligned}$$

Repeat this step now by integrating over x_2:

$$\begin{aligned}\int_{\mathbb{R}^2} |u|^{\frac{n}{n-1}} dx_1 dx_2 &= \Big(\int_{\mathbb{R}} f_2 dx_1 \Big)^{\frac{1}{n-1}} \int_{\mathbb{R}} \Big[f_1^{\frac{1}{n-1}} \prod_{i\ge 3} \Big(\int_{\mathbb{R}} f_i dx_1 \Big)^{\frac{1}{n-1}} \Big] dx_2 \\ &\le \Big(\int_{\mathbb{R}} f_2 dx_1 \Big)^{\frac{1}{n-1}} \Big(\int_{\mathbb{R}} f_1 dx_2 \Big)^{\frac{1}{n-1}} \prod_{i\ge 3} \Big(\int_{\mathbb{R}^2} f_i dx_1 dx_2 \Big)^{\frac{1}{n-1}}\end{aligned}$$

Repeating these steps by integrating over all x_i we obtain:

$$\int_{\mathbb{R}^n} |u|^{\frac{n}{n-1}} dx = \prod_{i=1}^{n} \Big(\int_{\mathbb{R}^n} \Big| \frac{\partial u}{\partial x_i} \Big| dx \Big)^{\frac{1}{n-1}}.$$

By the arithmetic-geometric main inequality $\prod_i a_i \le \frac{1}{n} \sum_i a_i^n$, $a_i \ge 0$, we then have:

$$\begin{aligned}\Big(\int_{\mathbb{R}^n} |u|^{\frac{n}{n-1}} dx \Big)^{\frac{n-1}{n}} &\le \prod_{i=1}^{n} \Big(\int_{\mathbb{R}^n} \Big| \frac{\partial u}{\partial x_i} \Big| dx \Big)^{\frac{1}{n}} \le \frac{1}{n} \sum_i \Big(\int_{\mathbb{R}^n} \Big| \frac{\partial u}{\partial x_i} \Big| dx \Big) \\ &= \frac{1}{n} \int_{\mathbb{R}^n} \sum_i \Big| \frac{\partial u}{\partial x_i} \Big| dx = \frac{1}{n} \|\nabla u\|_1,\end{aligned}$$

which establishes the Sobolev inequality for $p = 1$. Now substitute $|u|^q$, with $q = \frac{n-1}{n} p_n$, into the above inequality. This yields

$$\begin{aligned}\|u\|_{p_n}^q &= \||u|^q\|_{\frac{n}{n-1}} \le \frac{1}{n} \|\nabla |u|^q\|_1 \le \frac{q}{n} \int_{\mathbb{R}^n} |u|^{q-1} \Big(\sum_i \Big| \frac{\partial u}{\partial x_i} \Big| \Big) dx \\ &\le \frac{q}{n} n^{\frac{p-1}{p}} \||u|^{q-1}\|_{p'} \|\nabla u\|_p = \frac{q}{n} n^{\frac{p-1}{p}} \|u\|_{p_n}^{\frac{p_n}{p'}} \|\nabla u\|_p.\end{aligned}$$

Since $q - \frac{p_n}{p'} = 1$ we obtain the desired inequality. ∎

We should point out that the above proof provides a universal constant $C(n,p)$. This is by no means an optimal constant. It is possible to optimize the constant $C(n,p)$. A direct consequence of the Sobolev inequality are the higher order Sobolev inequalities. We will use these later to prove embeddings for the higher order Sobolev spaces.

C.7 Lemma — Morrey's inequality. Let $u \in C_0^\infty(\mathbb{R}^n)$, and let $p > n$, then there exists a (universal) constant $C(p,n) > 0$ such that

$$\big| u(x) - u(y) \big| \le C(n,p) |x-y|^\theta \|\nabla u\|_p,$$

where $\frac{\theta}{n} = \frac{1}{n} - \frac{1}{p}$.

Proof. Let $Q(R)$ be an open cube containing the origin and whose faces are parallel to coordinate hyperplanes. We have

$$u(x) - u(0) = \int_0^1 \frac{d}{dt} u(tx) dt, \quad x \in Q(R).$$

Since $\frac{d}{dt}u(tx) = x \cdot \nabla u(tx)$ we obtain:

$$\begin{aligned} |u(x) - u(0)| &\leq \int_0^1 |x \cdot \nabla u(tx)| dt \leq \int_0^1 \sum_i |x_i| \Big| \frac{\partial u}{\partial x_i}(tx) \Big| dt \\ &\leq R \sum_i \int_0^1 \Big| \frac{\partial u}{\partial x_i}(tx) \Big| dt \end{aligned}$$

Define the average $\bar{u} = \frac{1}{|Q|} \int_Q u(x) dx$. Then we have that

$$\begin{aligned} |\bar{u} - u(0)| &= \Big| \frac{1}{|Q|} \int_Q u(x) dx - u(0) \Big| \leq \frac{1}{|Q|} \int_Q |u(x) - u(0)| dx \\ &\leq \frac{R}{|Q|} \int_Q \Big(\sum_i \int_0^1 \Big| \frac{\partial u}{\partial x_i}(tx) \Big| dt \Big) dx \\ &= \frac{1}{R^{n-1}} \int_0^1 \Big(\sum_i \int_Q \Big| \frac{\partial u}{\partial x_i}(tx) \Big| dx \Big) dt \\ &= \frac{1}{R^{n-1}} \int_0^1 \Big(\sum_i \int_{tQ} \Big| \frac{\partial u}{\partial x_i}(y) \Big| \frac{1}{t^n} dy \Big) dt \end{aligned}$$

The gradient term can be estimated as follows

$$\int_{tQ} \Big| \frac{\partial u}{\partial x_i}(y) \Big| dy \leq \Big(\int_Q \Big| \frac{\partial u}{\partial x_i}(y) \Big|^p dy \Big)^{\frac{1}{p}} t^{\frac{1}{p'}} |Q|^{\frac{1}{p'}}.$$

Combining these estimates yields

$$|\bar{u} - u(0)| \leq R^{\frac{n}{p'} - n + 1} \leq n^{\frac{p-1}{p}} \|\nabla u\|_p \int_0^1 t^{\frac{n}{p'} - n} dt = \frac{n^{\frac{p-1}{p}}}{1 - \frac{n}{p}} R^{1 - \frac{n}{p}} \|\nabla u\|_p.$$

We can translate the origin to an arbitrary point x, and consequently

$$|\bar{u} - u(x)| \leq= \frac{n^{\frac{p-1}{p}}}{1 - \frac{n}{p}} R^{1 - \frac{n}{p}} \|\nabla u\|_p.$$

If we do the same for a point y we obtain

$$|u(x) - u(y)| \leq= 2 \frac{n^{\frac{p-1}{p}}}{1 - \frac{n}{p}} R^{1 - \frac{n}{p}} \|\nabla u\|_p.$$

We can now choose a cube $Q(R)$ with $R = 2\|x - y\|$ so that both x and y are in Q, then

$$|u(x) - u(y)| \leq 2^{2 - \frac{n}{p}} \frac{n^{\frac{p-1}{p}}}{1 - \frac{n}{p}} \|x - y\|^{1 - \frac{n}{p}} \|\nabla u\|_p,$$

which completes the proof. ■

In the borderline case that $p = n$ we can also prove an analogue of the Sobolev inequality — Trudinger-Moser inequality — which yields embeddings into generalizations of L^p-spaces; so-called Orlicz spaces. These space are defined via strictly convex functions generalizing t^p.

C.1.c Continuous and compact embeddings

The fundamental inequalities proved in the previous section can be used now to establish various continuous and compact embeddings of Sobolev spaces. In order to simplify matters let us consider the spaces $W_0^{m,p}(D)$. Since $C_0^\infty(D)$ is a dense subset in $W_0^{1,p}$ we have the Sobolev embedding for these spaces and

$$\|u\|_q \le C\|u\|_{1,p}, \quad u \in W_0^{1,p}(D), \quad p \le q \le p_n.$$

The latter can be proved using the idea of extension maps. For the spaces $W_0^{m,p}(D)$ we can prove the following lemma.

C.8 Lemma Define the map $e : W_0^{m,p}(D) \to W^{m,p}(\mathbb{R}^n)$ by

$$e(u) = u, \quad x \in D \ \text{a.e.}, \quad e(u) = 0, \quad x \in \mathbb{R}^n \backslash D.$$

Then $\|e(u)\|_{m,p,\mathbb{R}^n} = \|u\|_{m,p}$, which implies that the map e is continuous.

Proof. The proof follows from a standard density argument. Let $u^n \in C_0^\infty(D)$ converging to some $u \in W_0^{m,p}(D)$ in the $W^{m,p}$-topology. The definition of e yields that $e(u^n) \in C_0^\infty(\mathbb{R}^n)$, and $\|e(u^n)\|_{m,p,\mathbb{R}^n} = \|u^n\|_{m,p}$. This provides Cauchy sequences in both spaces and thus the map extends to a continuous map from $W_0^{m,p}(D)$ to $W^{m,p}(\mathbb{R}^n)$. ■

Using the Sobolev inequality from the previous section we can now prove the above Sobolev inequality for $W_0^{1,p}(D)$ into $L^q(D)$. From interpolation L^p-spaces we obtain the following interpolation inequality: Let $\frac{1}{q} = \frac{\theta}{p} + \frac{1-\theta}{p_n}$, then

$$\|u\|_q \le \|u\|_p^\theta \|u\|_{p_n}^{1-\theta}.$$

C.9 Exercise *Prove the above interpolation inequality (Hint: use Hölders inequality).* ■

We start with proving the Sobolev embedding for $W^{1,p}(\mathbb{R}^n)$. By definition

$$\|u\|_p \le \|u\|_{1,p}.$$

From the Sobolev inequality we deduce that

$$\|u\|_{p_n} \le C\|u\|_{1,p}.$$

Combining these two estimates together with the interpolation inequality we obtain:

$$\|u\|_q \leq \|u\|_p^\theta \|u\|_{p_n}^{1-\theta} \leq C^{1-\theta} \|u\|_{1,p}^\theta \|u\|_{1,p}^{1-\theta} = C^{1-\theta} \|u\|_{1,p}.$$

Using the extension operator we then find the embedding inequality for $W_0^{1,p}(D)$ for $q \geq p$. In the case that D is a bounded domain the inequality holds for $1 \leq q \leq p_n$ which follows from the fact that $\|u\|_q \leq |D|^{\frac{p-q}{p}} \|u\|_p^q$ on bounded domains. We can now formulate the following theorem.

C.10 Theorem Let $D \subset \mathbb{R}^n$ be a domain and let $m \leq mp < n$. Then

$$\|u\|_q \leq C\|u\|_{m,p}, \quad \forall u \in W_0^{m,p}(D),$$

with $\frac{1}{p} \geq \frac{1}{q} \geq \frac{1}{p_{n,m}} = \frac{1}{p} - \frac{m}{n}$. In the case that D is a bounded domain the result holds for $1 \leq q \leq p_{n,m}$.

Proof. The case $m = 1$ was proved above. We prove this by induction. Assume the embedding statement is true for $m - 1$. Let $u \in W_0^{m,p}(D)$, $n > mp \geq m$, then u and the partial derivatives $v_i = \frac{\partial u}{\partial x_i}$ belong to $W_0^{m-1,p}(D)$. Consequently

$$\|u\|_{p_{n,m-1}} \leq C\|u\|_{m-1,p}, \quad \|v_i\|_{p_{n,m-1}} \leq C\|v_i\|_{m-1,p},$$

which implies that $u \in W_0^{1,p_{n,m-1}}(D)$, where $\frac{1}{p_{n,m-1}} = \frac{1}{p} - \frac{m-1}{n}$. We now combine this with the embedding for $m = 1$;

$$\|u\|_{p_{n,m}} \leq C\|u\|_{1,p_{n,m-1}} \leq C'\|u\|_{m,p},$$

where $\frac{1}{p_{n,m}} = \frac{1}{p_{n,m-1}} - \frac{1}{n} = \frac{1}{p} - \frac{m}{n}$, which proves the inequality into $L^{p_{n,m}}(D)$. The remainder of the proof follows along the same lines as in the case $m = 1$. ∎

A similar theorem can be proved for the spaces $W^{m,p}(D)$. We state this theorem without proof.

C.11 Theorem Let $D \subset \mathbb{R}^n$ be a domain with a C^1 boundary ∂D and let $m \leq mp < n$. Then

$$\|u\|_q \leq C\|u\|_{m,p}, \quad \forall u \in W^{m,p}(D),$$

with $\frac{1}{p} \geq \frac{1}{q} \geq \frac{1}{p_{n,m}} = \frac{1}{p} - \frac{m}{n}$. In the case that D is a bounded domain the result holds for $1 \leq q \leq p_{n,m}$.

C.12 Exercise *Prove the Sobolev inequality*

$$\|u\|_{p_n} \leq \|\nabla u\|_p, \quad u \in W_0^{1,p}(D).$$

Moreover, show that for D a bounded domain, p_n can be replaced by $1 \leq q \leq p_n$. ∎

Sobolev inequalities as in the above exercise can be used to find equivalent norms for $W_0^{m,p}(D)$ in terms of only the highest order derivatives.

Very important for applications to differential equations is fact that the embedding operators are sometimes compact mappings.

C.13 Theorem Let $D \subset \mathbb{R}^n$ be a bounded domain with a C^1 boundary ∂D and let $m \leq mp < n$. Then the embedding

$$i : W^{m,p}(D) \hookrightarrow L^q(D),$$

with $1 \geq \frac{1}{q} > \frac{1}{p_{n,m}}$, is compact

The case $mp = n$ can be considered as a limiting case of the previous embeddings.

C.14 Theorem Let $D \subset \mathbb{R}^n$ be a domain with a C^1 boundary ∂D and let $mp = n$. Then

$$\|u\|_q \leq C\|u\|_{m,p}, \quad \forall u \in W^{m,p}(D),$$

with $p \leq q < \infty$. If the domain is bounded then the embedding $i : W^{m,p}(D) \hookrightarrow L^q(D)$ is compact.

The final case concerns the embedidngs into Hölder spaces.

C.15 Theorem Let $D \subset \mathbb{R}^n$ be a domain with a C^1 boundary ∂D and let $mp > n$. Then

$$\|u\|_{C^{k-1,\theta}} \leq C\|u\|_{m,p}, \quad \forall u \in W^{m,p}(D),$$

with $k = m - \left[\frac{n}{p}\right]$, and $0 < \theta \leq \left[\frac{n}{p}\right] + 1 - \frac{n}{p}$, or $0 < \theta < 1$, when $\frac{n}{p}$ is an integer. If the domain is bounded then the embedding $i : W^{m,p}(D) \hookrightarrow C^{k-1,\theta}(\overline{D})$ is compact.

We will finish this section with a proof of the compactness of the embedding

$$W_0^{m,p}(D) \hookrightarrow L^{p_{n,m}}(D).$$

In order to do so we start with a special case. Consider the $C = [-1,1]^n \subset \mathbb{R}^n$. On this domain we can consider functions that extend periodically, which means we are looking at functions on the periodic lattice $\mathbb{R}^n/\mathbb{Z}^n$, which we denote by $\mathbb{T}^n$; the n-dimensional torus. For such functions we can use Fourier series. Define

$$\sum_{k \in \mathbb{Z}^n} a_k e^{ik \cdot x}, \quad x \in \mathbb{R}^n.$$

Under the condition $a_{-k} = \overline{a_k}$, the above series represent real function on $\mathbb{T}^n$, provided that the series converge. From Fourier theory we have that the space $L^2(\mathbb{T}^n)$ is characterized as follows; a function $u : \mathbb{T}^n to \mathbb{R}$ lies in $L^(\mathbb{T}^n)$ if and only if its Fourier coefficients satisfy

$$\sum_{k\in\mathbb{Z}^n} |a_k|^2 < \infty,$$

in which case we write $u(x) = \sum_{k\in\mathbb{Z}^n} a_k e^{ik\cdot x}$. In may be clear from the previous considerations that the space $W^{1,2}(\mathbb{T}^n)$ is then characterized by the convergence criterion

$$u \in W^{1,2}(\mathbb{T}^n), \quad \text{iff} \quad \sum_{k\in\mathbb{Z}^n} (1+|k|^2)|a_k|^2 < \infty.$$

C.16 Lemma The embedding

$$i\colon W^{1,2}(\mathbb{T}^n) \hookrightarrow L^2(\mathbb{T}^n),$$

is compact.

Proof. We prove that the embedding i is the limit of finite rank maps. Define

$$i_N(u) = \sum_{|k_i|\le N} a_k e^{ik\cdot x}.$$

It is obvious that i_N is a finite rank map. We now estimate

$$\begin{aligned} \|i_N(u) - i(u)\|_2^2 &= \sum_{|k_i|>N} |a_k|^2 = \sum_{|k_i|>N} \frac{1+|k|^2}{1+|k|^2}|a_k|^2 \\ &\le \frac{1}{1+n^2N^2} \sum_{|k_i|>N} |a_k|^2 = \frac{1}{1+n^2N^2}\|u\|_{1,2}^2, \end{aligned}$$

which proves that i_N converges to i in the operator norm, establishing the compactness of the map i. ■

Let $D \subset \mathbb{R}^n$ be a bounded domain. As in the case of $\mathbb{R}^n$ we can now embed the space $W_0^{1,2}(D)$ into $W^{1,2}(\mathbb{T}^n)$ by considering the extension map. We may assume without loss of generality that $\overline{D}$ is contained in the interior of C. Then by using the extension map as in Lemma C.8 we obtain the continuous map:

$$e\colon W_0^{1,2}(D) \longrightarrow W^{1,2}(\mathbb{T}^n).$$

The yields the following lemma.

C.17 Lemma Let $D \subset \mathbb{R}^n$ be a bounded domain, then the embedding

$$i\colon W_0^{1,2}(D) \hookrightarrow L^2(D),$$

is compact.

Proof. We have the composition

$$W_0^{1,2}(D) \longrightarrow W^{1,2}(\mathbb{T}^n) \hookrightarrow L^2(\mathbb{T}^n) \longrightarrow L^2(D),$$

given by $i = r \circ i_{\mathbb{T}^n} \circ e$, where e is the $i_{\mathbb{T}^n}$ is the embedding map from Lemma C.16, and r the restriction map from $L^2(\mathbb{T}^n)$ to $L^2(D)$. Since $i_{\mathbb{T}^n}$ is compact, the so is i. Indeed, for any bounded sequence $\{u^n\}$, the sequence $\{(i_{\mathbb{T}^n} \circ e)(u^n))\}$ has a convergent subsequece, and is thus $\{(r \circ i_{\mathbb{T}^n} \circ e)(u^n))\}$ has a convergent subsequence. ∎

We will now give a compactness proof for the case $p \geq 2$. Consider the embeddings

$$i_0 : W_0^{1,p}(D) \longrightarrow W_0^{1,2}(D) \hookrightarrow L^2(D) \longrightarrow L^1(D).$$

Due to Lemma C.17 the embedding i_0 is compact. We also have the continuous embedding from Theorem C.10:

$$i_1 : W_0^{1,p}(D) \longrightarrow L^{p_{n,1}}(D).$$

If we now use interpolation we obtain:

$$i : W_0^{1,p}(D) \longrightarrow L^q(D),$$

for $1 \leq q \leq p_{n,1}$ with i satsifying the inequality

$$\|i(u)\|_q \leq \|i_0(u)\|_1^{1-\theta} \|i_1(u)\|_{p_{n,1}}^{\theta} \leq C^{\theta} \|u\|_{1,p}^{1-\theta} \|u\|_{1,p}^{\theta} = C^{\theta} \|u\|_{1,p}.$$

From the first half of the inequality we see immediately that that any bounded sequence $\{u^n\}$ yields a convergent subsequence in $L^q(D)$, provided $\theta < 1$, i.e. $q < p_{n,1}$. This proves that the embedding i is compact for all $1 \leq q < p_{n,1}$. The compactness of this embedding can also prove the compactness of the embedding $W^{1,p}(D) \hookrightarrow L^q(D)$ be using an appropriate extension mapping.

As for the higher order Sobolev spaces $W_0^{m,p}(D)$ we have now proved the following result.

C.18 Theorem Let $D \subset \mathbb{R}^n$ be a bounded domain, then the embedding

$$i : \; W_0^{m,p}(D) \hookrightarrow L^q(D), \quad p \geq 2$$

is compact for all $1 \leq q < p_{n,m}$.

Proof. We have that the embedding $i : W_0^{m,p}(D) \to W_0^{1,p}(D) \hookrightarrow L^1(D)$ is compact. Now we use the interpolation again. ∎

There are many ways we can extend these results. For example, by more advanced interpolation theory, the result of Lemma C.17 also yields compactness in Theorem C.18 for $1 \leq p < 2$. Also interpolation can be used to prove that the fractional space $W_0^{m,p}(D)$, $m > 0$, have the following compact embeddings:

$$W_0^{m,p}(D) \hookrightarrow W_0^{m-k,p}(D),$$

with $0 < k < m$.

D — Posets and Lattices

D.1 Posets, lattices and Boolean algebras

D.1 Definition A partially ordered set or poset $(\mathsf{P},\leq)$ is a set P with a binary relation $\leq$, called a partial order, which satisfies the following axioms:

(i) (reflexivity) $p \leq p$, for all $p \in \mathsf{P}$,

(ii) (anti-symmetry) $p \leq q$, and $q \leq p$, then $p = q$,

(iii) (transitivity) if $p \leq q$, and $q \leq r$, then $p \leq r$.

If two elements p,q are not related, or incomparable we write $p \parallel q$. A relation $<$ which satisfies only the transitivity property and $p \not< p$ for all $p \in \mathsf{P}$ is called a strict partial order and is denoted by $(\mathsf{P},<)$.

Posets and strict poset are equivalent. A partial order $\leq$ defines a strict order via: $p < q$ if and only if $p \leq q$ and $p \neq q$. Similarly, a strict order $<$ defines a partial order via: $p \leq q$ if and only if $p < q$, or $p = q$. The notations $(\mathsf{P},\leq)$ and $(\mathsf{P},<)$ are used interchangeably to emphasize order or strict order. When there is no ambiguity about the partial order, we refer to partially ordered sets simply by denoting the set P. A partial order is called a total order if any two elements can be compared.

A chain is a subset which totally ordered. An anti-chain is an unordered set. A poset P has length $\ell(\mathsf{P}) = n$ if every chain in P has at most $n+1$ elements. A poset has width $w(\mathsf{P}) = n$ if every anit-chain has at most n elements. Note that an infinite poset can have finite length. An anti-chain has length $\ell = 1$, and a chain has width $w = 1$.

A least element in a poset, if it exists, is denoted by $\perp$ and has the property that $\perp \leq p$ for all $p \in \mathsf{P}$. A greatest element in a poset, if it exists, is denoted by $\top$ and has the property that $\top \geq p$ for all $p \in \mathsf{P}$. Least and greatest elements need not exist in a poset. A minimal element $p \in \mathsf{P}$ satisfies: $p' \leq p$, implies $p' = p$. Similarly, a maximal element $p \in \mathsf{P}$ satisfies: $p' \geq p$, implies $p' = p$. Minimal and maximal elements always exists and are the least and greatest element, when unique.

D.2 Definition Let P and P' be posets. A mapping $f : \mathsf{P} \to \mathsf{P}'$ is called order preserving if $f(p) \leq f(q)$ for all $p \leq q$ and order reversing if $f(p) \geq f(q)$ for all $p \leq q$.

The set of order-preserving maps, or order homomorphisms is denoted by $\mathrm{Ord}(\mathsf{P}, \mathsf{P}')$ and the set of order reversing maps, or order anti-homomorphisms is denoted by $\mathrm{Ord}^*(\mathsf{P}, \mathsf{P}')$. The set of all posets as objects and order-preserving, or order-reversing maps as morphisms form the small categories *Poset* and *Poset** respectively (see Appendix **??** for details on categories).

Consider two elements $p, q \in (\mathsf{P}, \leq)$. The infimum of p and q (if it exists) is the greatest lower bound with respect to the partial ordering $\leq$ and is denoted by $\inf(p,q)$. Similarly, if there exists a least upper bound with respect to the partial ordering it is called the supremum and is denoted by $\sup(p,q)$.

We can regard the infimum and supremum as binary operations on a poset P if for any two elements $p, q \in \mathsf{P}$ the infimum and supremum exist:

$$p \vee q = \sup(p,q), \quad p \wedge q = \inf(p,q). \tag{D.1.1}$$

A poset for which infimum and supremum exist for all pairs $p, q \in \mathsf{P}$ is called a lattice, and introduces an algebraic structure to P. The operation $\vee$ is called 'vee' or join and the operation $\wedge$ is called 'wedge' or meet. A lattice can also be defined independently as an algebraic structure.

D.3 Definition A lattice $(\mathsf{L}, \vee, \wedge)$ is a set L with the binary operations $\vee, \wedge : \mathsf{L} \times \mathsf{L} \to \mathsf{L}$ satisfying the following axioms:

(i) (idempotent) $a \wedge a = a \vee a = a$ for all $a \in \mathsf{L}$,
(ii) (commutative) $a \wedge b = b \wedge a$, and $a \vee b = b \vee a$ for all $a, b \in \mathsf{L}$,
(iii) (associative) $a \wedge (b \wedge c) = (a \wedge b) \wedge c$ and $a \vee (b \vee c) = (a \vee b) \vee c$ for all $a, b, c \in \mathsf{L}$,
(iv) (absorption) $a \wedge (a \vee b) = a \vee (a \wedge b) = a$ for all $a, b \in \mathsf{L}$.

A distributive lattice satisfies the additional axiom

(v) (distributive) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ and $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$ for all $a, b, c \in \mathsf{L}$.

A lattice is bounded if there exist neutral elements 0 and 1 with property that

(vi) $0 \wedge a = 0$, $0 \vee a = a$, for all $a \in \mathsf{L}$, and $1 \wedge a = a$, $1 \vee a = 1$, for all $a \in \mathsf{L}$.

A set $\mathsf{K} \subset \mathsf{L}$ is a sublattice if $a \wedge b \in \mathsf{K}$ and $a \vee b \in \mathsf{K}$ for all $a, b \in \mathsf{K}$. A sublattice that contains 0 and 1 is called a $(0,1)$-sublattice.

D.4 Remark For the distributivity axiom (v) only one of the identities is required, i.e. the first one implies the second one and vice versa. ∎

When clear from context, we write L and omit the explicit reference to the lattice operations. We should emphasize that posets and lattices need not be finite sets. Finite lattices are always bounded.

D.5 Remark Observe that any lattice $(\mathsf{L},\vee,\wedge)$ has a natural partial order given by

$$a \le b \text{ if } a\wedge b = a \quad \text{or, equivalently,} \quad a\le b \text{ if } a\vee b = b,$$

which makes $(\mathsf{L},\le)$ a poset. ∎

We emphasize that distributivity and boundedness are not automatic for posets with inf and sup. If a lattice satisfies the axiom (v′) (modular) $a \ge c \implies a\wedge(b\vee c) = (a\wedge b)\vee c$, it is called a modular.

D.6 Proposition A poset $(\mathsf{P},\le)$, with $\vee$ and $\wedge$ as defined in (D.1.1) is an algebraic lattice, i.e. the binary operation $\vee$ and $\wedge$ satisfy (i)-(iv) of Definition D.3.

D.7 Definition Let $(\mathsf{L},\wedge,\vee)$ and $(\mathsf{L}',\wedge',\vee')$ be lattices. A function $h:\mathsf{L}\to\mathsf{L}'$ is a lattice homomorphism if

$$h(a\wedge b) = h(a)\wedge' h(b), \quad h(a\vee b) = h(a)\vee' h(b),$$

a lattice anti-homomorphism if

$$h(a\wedge b) = h(a)\vee' h(b), \quad h(a\vee b) = h(a)\wedge' h(b).$$

The set of lattice homomorphisms is denoted by $\mathrm{Hom}(\mathsf{L},\mathsf{L}')$ and the set of lattice anti-homomorphisms is denoted by $\mathrm{Hom}^*(\mathsf{L},\mathsf{L}')$. Lattice homomorphisms for which $h(0)=0$, and $h(1)=1$ ($h(0)=1$, and $h(1)=0$ for anti-homorphisms) are called $(0,1)$-(anti-)homomorphisms. These sets are denoted by $\mathrm{Hom}_{0,1}(L,L')$ and $\mathrm{Hom}^*_{0,1}(L,L')$ respectively.

With lattices as objects and with lattice (anti-)homomorphisms as morphisms we obtain the small categories *Latt* and *Latt** respectively. If distributivity or modularity holds the categories are denoted by $Latt_D$ and $Latt_M$ respectively.

Observe that $(\mathrm{Hom}(\mathsf{L},\mathsf{L}'),\le)$ and $(\mathrm{Hom}^*(\mathsf{L},\mathsf{L}'),\le)$ are posets with the order $\le$ defined by $h\le k$ if and only if $h(a)\le k(a)$, for all $a\in\mathsf{L}$. An anti-homomorphism $h:\mathsf{L}\to\mathsf{L}'$ can be regarded as a lattice homomorphism by redefining the operations $\wedge = \vee'$ and $\vee = \wedge'$ on L'. Observe that the new order on L' is the dual of the original order.

The algebraic structure of lattices does not accommodate inversion. In some of the structures we encounter some inversion relations are satisfied. In this context of lattices this is descibed by Boolean algebras.

D.8 Definition A Boolean algebra is a bounded distributive lattice $(\mathsf{L},\vee,\wedge,^c)$ with the additional complementation relation $a \mapsto a^c \in \mathsf{L}$, which satisfies the axiom:

(vii) $a \wedge a^c = 0$ and $a \vee a^c = 1$, for all $a \in \mathsf{L}$.

A Heyting algebra is a bounded distributive lattice $(\mathsf{L},\vee,\wedge,^c)$ with the additional pseudo-complementation relation $a \mapsto a^c \in \mathsf{L}$, which satisfies the axiom:

(vii′) $a \wedge a^c = 0$, for all $a \in \mathsf{L}$.

■ **D.9 Remark** To visualize partial orders and lattices in small examples we adopt the following conventions. Let $(\mathsf{P},\leq)$ be a poset. For $p,q \in \mathsf{P}$ we say q covers p if $p < q$ and $p \leq r < q$ implies that $p = r$. This relation is denoted by $p \prec q$. Observe that if P is finite, then $<$ determines $\prec$ and vice versa.

As mentioned in Example **??**, a partial order relation can be represented by a directed graph, which is often useful for visualization. It is a common convention to depict these graphs in a Hasse diagram. This diagram is a planar representation which is constructed as follows. In a diagram of $(\mathsf{P},<)$ we associate a circle to each element of P, and an edge is drawn between the circles for $p,q \in \mathsf{P}$ if $p \prec q$. The circles are placed so that if $p \prec q$, then the circle for p is vertically lower than the circle for q, and every edge intersects no circles other than its endpoints. We note without proof that such a diagram is always possible. ■

E — Homology Theory

E.1 Homology and cohomology

In this section we will give an elementary exposition of the basic homology and cohomology theories that are used in this text. This treatment is self-contained and at elementary level. For more detailed accounts of homology and cohomology theory we refer to various texts in algebraic topology. Before defining the homology introduce the concept of simplicial complex.

E.1.a Simplicial homology

Simplicial homology is the simplest homology theory from a conceptual point view and is used for simplicial complexes in $\mathbb{R}^n$. A collection of $q+1$ points, or vertices $\{v^1, \cdots, v^{q+1}\}$ in $\mathbb{R}^n$ is said to be an independent set of points if the vectors $v^2 - v^1, \cdots, v^{q+1} - v^1$ form a linearly independent system. Note that this holds relative to any chosen vertex v^i.

E.1 Definition A q-simplex ($q \leq n$) is a (convex) subset σ_q of $\mathbb{R}^n$ generated by $q+1$ independent vertices $v^1, \cdots, v^{q+1}$ in the following way:

$$\sigma_q := \Big\{ x \in \mathbb{R}^n \mid x = \sum_i \lambda_i v^i,\ \sum_i \lambda_i = 1,\ \lambda_i \geq 0 \Big\}.$$

The notation for a specific q-simplex is $\sigma_k = [v^1 \cdots v^{k+1}]$. By choosing an ordering of the vertices one obtains oriented k-simplices., which are devided into two orientation classes. For a given orientated q-simplex σ_q permuting the vertices by a odd permutation yields an oriented q-simplex of opposite orientation which we denote by $-\sigma_q$.

The boundary of a simplex is given by its faces. The faces of σ_q are the sets $\{x \in \sigma_q \mid \lambda_i = 0\}$ which are $(q-1)$-simplices. The ith face is give by $[v^1 \cdots \widehat{v}^i \cdots v^{q+1}]$ where the notation $\widehat{v}^i$ indicates removing the vertex v^i. This brings us to considering collections of simplices.

E.2 Definition A finite collection of simplices K is called q-dimensional simplicial complex if;

(i) $\sigma_\ell \in K$, $\ell \leq q$ then all its faces are contained in K;

(ii) two simplices σ_ℓ and $\sigma_{\ell'}$ can only intersect along common faces.

The dimension q of a simplicial complex K is determined by the maximal dimension of the simplices in K.

The morphisms between simplicial complexes K and K' are called simplicial mappings $f : K \to K'$ and are defined by the property that each simplex in K is mapped onto a simplex in K' of the same dimension or lower. This makes simplicial complexes with the simplicial mappings a category.

For oriented simplicial complexes we can build free groups over the complex as follows. Let K be a simplicial complex of dimension q, then define

$$C_k(K;\mathbb{Z}) := \{\boldsymbol{\sigma}_k = \sum a_i \sigma_k^i \mid \sigma_k^i \in K,\ a_i \in \mathbb{Z}\}, \quad \forall k \leq q,$$

which are finite linear combinations of the k-simplices in K. An element $\boldsymbol{\sigma}_k \in C_k(K)$ is called a k-chain in K. A k-chain $\boldsymbol{\sigma}_k$ is carried by a subcomplex $L \subset K$ if $\boldsymbol{\sigma}_k = \sum a_i \sigma_k^i$ with $\sigma_k^i \in L$. We set $C_k = 0$ for all $k < 0$ and $k > q$. By allowing coefficient in $\mathbb{Z}$ we consider oriented simplices and an oriented complex K. A simplicial mapping $f : K \to K'$ induces a homomorphism $f_\# : C_k(K) \to C_k(K')$ as follows:

$$f_\#([v^1 \cdots v^{k+1}]) = \begin{cases} [f(v^1) \cdots f(v^{k+1})] & \text{if } f(v^i) \neq f(v^j)\ \forall i \neq j, \\ 0 & \text{otherwise} \end{cases}$$

and $f_\#(-\sigma_k) = -f_\#(\sigma_k)$.

For an oriented simplex we can define a operation that assign to a simplex its boundary. Let $\sigma_q = [v^1 \cdots v^{q+1}]$ be an oriented q-simplex, then

$$\partial_q [v^1 \cdots v^{q+1}] := \sum_{i=1}^{q+1} (-1)^i [v^1 \cdots \widehat{v}^i \cdots v^{q+1}],$$

which is called the q-boundary operator.

E.3 Example Let $\sigma_2 = [0\ e_1\ e_2] \subset \mathbb{R}^2$, where e_i are the standard basis vectors in $\mathbb{R}^2$. Then $\partial_2 \sigma_2 = [e_1\ e_2] - [0\ e_2] + [0\ e_1]$ which is a 1-complex which represents the boundary of σ_2 with counter clockwise orientation.

Since the boundary operator is defined for all simplices the defintion extends to all elements in q-simplicial complex K, and

$$\partial_k : C_k(K;\mathbb{Z}) \to C_{k-1}(K;\mathbb{Z}), \quad \forall k \le q.$$

The following lemma about the boundary operator is completely clear from an intuitive point of view. If we apply the boundary operator twice, i.e. the boundary of an oriented boundary, then the answer is the empty set.

E.4 Lemma Let $\partial_k = 0$ for all $k \le 0$ and $k > q$. Then $\partial_{k-1} \circ \partial_k = 0$.

E.5 Exercise Prove Lemma E.4. ▪

The groups $C_k(K)$ are called chain groups and the combination $\{C_k(K), \partial_k\}_{k\in\mathbb{Z}}$ is called a chain complex (due to the property of ∂_k given in Lemma E.4). For a chain complex in general we can define its homology. Let

$$\begin{aligned} Z_k(K;\mathbb{Z}) &= \{\boldsymbol{\sigma}_k \in C_k(K) \mid \partial_k \boldsymbol{\sigma}_k = 0\} = \ker(\partial_k), \\ B_k(K;\mathbb{Z}) &= \{\boldsymbol{\sigma}_k \in C_k(K) \mid \exists \widetilde{\boldsymbol{\sigma}}_{k+1} \in C_{k+1}(K) \ni \boldsymbol{\sigma}_k = \partial_k \widetilde{\boldsymbol{\sigma}}_{k+1}\} = \mathrm{im}(\partial_{k+1}). \end{aligned}$$

The elements in Z_k are called k-cycles and the elements in B_k are called k-boundaries. From Lemma E.4 it follows that $B_k(K) \subset Z_k(K)$ as linear subgroups.

E.6 Definition The kth simplicial homology group $H_k(K;\mathbb{Z})$ of a q-simplicial complex K is defined as the quotient $H_k(K;\mathbb{Z}) := Z_k(K;\mathbb{Z})/B_k(K;\mathbb{Z})$.

The homology classes are equivalence classes under the equivalence relation: $\boldsymbol{\sigma}_k \sim \boldsymbol{\sigma}'_k$, $\boldsymbol{\sigma}_k, \boldsymbol{\sigma}'_k \in Z_k(K)$ if $\boldsymbol{\sigma}_k - \boldsymbol{\sigma}'_k \in B_k(K)$. Therefore an homology class $[\boldsymbol{\sigma}_k]$ can be represented by cycles of the form $\boldsymbol{\sigma}_k + \partial_{k+1}\widetilde{\boldsymbol{\sigma}}_{k+1}$, $\widetilde{\boldsymbol{\sigma}}_{k+1} \in C_{k+1}(K)$. Computing the homology of a q-simplicial complex becomes a linear algebra problem with coefficients in $\mathbb{Z}$. The addition in the abelian groups $H_k(K)$ is given by: $[\boldsymbol{\sigma}_k] + [\boldsymbol{\sigma}'_k] = [\boldsymbol{\sigma}_k + \boldsymbol{\sigma}'_k]$ by the above considerations. The above construction can also be carried out over different groups $\mathbb{F}$, e.g. $\mathbb{F} = \mathbb{R}$, $\mathbb{F} = \mathcal{Q}^\epsilon$, or $\mathbb{F} = \mathbb{Z}_p$. Simplicial homology $H_k(K;\mathbb{F})$ may be different for different groups. As a matter of fact $H_k(K,\mathbb{Z})$ may contain more detailed information than $H_k(K;\mathcal{Q}^\epsilon)$.

E.7 Exercise Let K be a 2-dimensional complex consisting of $\sigma_2 = [0\ e_1\ e_2] \subset \mathbb{R}^2$ and all its lower dimensional faces. Compute the homology groups $H_k(K;\mathbb{Z})$, $k = 0,1,2$. ▪

E.8 Exercise Let K be a q-simplicial complex. Show that $\dim H_0(K;\mathbb{Z})$ is equal to the number of connected component of K. ▪

E.9 Exercise Figure below is an example of a simplicial Möbius strip M. Compute the homology $H_*(M;\mathbb{Z})$ and $H_*(M;\mathcal{Q}^\epsilon)$. Do these homology calculations give the same answer? ■

E.10 Exercise Let $K = \{\sigma_0\}$, a simplicial complex consisting of a single 0-simplex. Show that $H_0(K;\mathbb{Z}) \cong \mathbb{Z}$. ■

Simplicial mapping induce homomorphisms in homology:

E.11 Lemma Let $f : K \to K'$ be a simplicial mapping then $f_\# \partial_k = \partial_k f_\#$ for all k and $f_\#$ induces a homomorphism $f_* : H_k(K;\mathbb{Z}) \to H_k(K';\mathbb{Z})$.

Proof. For a single simplex we need to verify the relation $f_\#(\partial_k[v^1 \cdots v^k] = \partial_k f_\#([v^1 \cdots v^k])$. If $f_\#([v^1 \cdots v^k])$ is a simplex of dimension k, then the relation follows from the defintion. ■

E.12 Lemma For f_* the following properties hold:

(i) if $\mathrm{id} : K \to K$ is the identity mapping, then $\mathrm{id}_* : H_k(K) \to H_k(K)$ is the identity homomorphism;

(ii) Let K, K', K'' be simplicial complexes. If $f : K \to K'$ and $g : K' \to K''$ are simplicial mappings, then $(g \circ f)_* = g_* \circ f_*$,

where $f_* : H_k(K) \to H_k(K')$ and $g_* : H_k(K') \to H_k(K'')$.

E.13 Exercise Prove Lemma E.12. ■

A second important ingredient of homology theory is the relative homology of a pair of simplicial complexes (K, L) with $L \subset K$, where L is a subcomplex of K. The groups $C_k(L;\mathbb{Z})$ are subgroups of $C_k(K;\mathbb{Z})$. Define

$$C_k(K, L;\mathbb{Z}) := C_k(K;\mathbb{Z})/C_k(L;\mathbb{Z}),$$

and equivalence classes are given by $\boldsymbol{\sigma}_k + C_k(L;\mathbb{Z})$, $\boldsymbol{\sigma}_k \in C_k(K\backslash L;\mathbb{Z})$ (a chain consisting of simplices that are not in L). The boundary operator $\bar{\partial}_k : C_k(K, L) \to C_{k-1}(K, L)$ is defined by $\bar{\partial}_k(\boldsymbol{\sigma}_k + C_k(L)) = \partial_k \boldsymbol{\sigma}_k + C_{k-1}(L)$. From the properties of ∂_k it follows that $\widehat{\partial}_{k-1} \circ \widehat{\partial}_k \boldsymbol{\sigma}_k = \partial_{k-1} \circ \partial_k \boldsymbol{\sigma}_k + C_{k-2}(L) = C_{k-2}(L)$, which is zero in $C_{k-2}(K, L)$ and therefore $\widehat{\partial}_{k-1} \circ \widehat{\partial}_k = 0$.

As before set $Z_k(K, L;\mathbb{Z}) = \ker(\widehat{\partial}_k)$ and $B_k(K, L;\mathbb{Z}) = \mathrm{im}(\widehat{\partial}_{k+1})$ and are called relative k-cycles and relative k-boundaries respectively. It follows that elements in $Z_k(K, L)$ are of the form $\boldsymbol{\sigma}_k + C_k(L)$ with $\partial_k \boldsymbol{\sigma}_k$ carried by L and elements in $B_k(K, L)$ are of the form $\boldsymbol{\sigma}_k = \partial_{k+1} \widetilde{\boldsymbol{\sigma}}_{k+1} + C_k(L)$ with $\boldsymbol{\sigma}_k - \partial_{k+1} \widetilde{\boldsymbol{\sigma}}_{k+1}$ carried by L. Relative homology classes $[\boldsymbol{\sigma}_k]$ are represented by relative k-cycles of the form $\boldsymbol{\sigma}_k + \partial_{k+1} \widetilde{\boldsymbol{\sigma}}_{k+1} + C_k(L)$.

E.14 Definition The kth relative simplicial homology group $H_k(K,L;\mathbb{Z})$ of a simplicial pair (K,L) is defined as the quotient $H_k(K,L;\mathbb{Z}) := Z_k(K,L;\mathbb{Z})/B_k(K,L;\mathbb{Z})$.

Consider the short exact sequence of maps:

$$0 \longrightarrow L \xrightarrow{i} K \xrightarrow{j} (K,L) \longrightarrow 0,$$

where i is the embedding and $j(K) = (K,\varnothing)$. It holds that $\ker(i) = 0$ and $\ker(j) = \mathrm{im}(i) = L$ and thus the sequence is exact.

E.15 Lemma The above short exact sequence yields a long exact sequence in homology:

$$\cdots \xrightarrow{\partial_*} H_k(L) \xrightarrow{i_*} H_k(K) \xrightarrow{j_*} H_k(K,L) \xrightarrow{\partial_*} H_{k-1}(L) \xrightarrow{i_*} \cdots,$$

where i_*, j_* and ∂_* are the induced homomorphisms on homology.

Proof. The homomorphism i_* is induced by the simplicial mapping i as described above. Define $j_\# : C_k(K) \to C_k(K,L)$ by $j_\#(\boldsymbol{\sigma}_k) = \boldsymbol{\sigma}_k + C_k(L)$, where $\boldsymbol{\sigma}_k \in C_k(K)$. Now $j_\#(\partial_k\boldsymbol{\sigma}_k) = \partial_k\boldsymbol{\sigma}_k + C_{k-1}(L)$ and $\widehat{\partial}_k(j_\#\boldsymbol{\sigma}_k) = \widehat{\partial}_k(\boldsymbol{\sigma}_k + C_{k-1}(L)) = \partial_k\boldsymbol{\sigma}_k$. Therefore, $j_\#\partial_k = \widehat{\partial}_k j_\#$. Let $[\boldsymbol{\sigma}_k] \in H_k(K)$, then $\widehat{\partial}_k j_\#([\sigma_k]) = j_\#\partial_k([\boldsymbol{\sigma}_k]) = 0$ and thus $j_\#([\sigma_k]) \in Z_k(K,L)$. The latter defines the homology class $[j_\#([\sigma_k])] \in H_k(K,L)$ and thereby the mapping j_*.

Under construction. ∎

E.1.b Definition of De Rham cohomology

In the previous chapters we introduced and integrated m-forms over manifolds M. We recall that k-form $\omega \in \Gamma^k(M)$ is closed if $d\omega = 0$, and a k-form $\omega \in \Gamma^k(M)$ is exact if there exists a $(k-1)$-form $\sigma \in \Gamma^{k-1}(M)$ such that $\omega = d\sigma$. Since $d^2 = 0$, exact forms are closed. We define

$$\begin{aligned} Z^k(M) &= \{\omega \in \Gamma^k(M) \,:\, d\omega = 0\} = \mathrm{Ker}(d), \\ B^k(M) &= \{\omega \in \Gamma^k(M) \,:\, \exists\, \sigma \in \Gamma^{k-1}(M) \;\ni\; \omega = d\sigma\} = \mathrm{Im}(d), \end{aligned}$$

and in particular

$$B^k(M) \subset Z^k(M).$$

The sets Z^k and B^k are real vector spaces, with B^k a vector subspace of Z^k. This leads to the following definition.

E.16 Definition *Let M be a smooth m-dimensional manifold then the* de Rham cohomology groups *are defined as*

$$H^k_{dR}(M) := Z^k(M)/B^k(M), \qquad k = 0, \cdots m, \tag{E.1.1}$$

where $B^0(M) := 0$.

It is immediate from this definition that $Z^0(M)$ are smooth functions on M that are constant on each connected component of M. Therefore, when M is connected, then $H^0_{\mathrm{dR}}(M) \cong \mathbb{R}$. Since $\Gamma^k(M) = \{0\}$, for $k > m = \dim\ M$, we have that $H^k_{dR}(M) = 0$ for all $k > m$. For $k < 0$, we set $H^k_{dR}(M) = 0$.

■ **E.17 Remark** The de Rham groups defined above are in fact real vector spaces, and thus groups under addition in particular. The reason we refer to de Rham cohomology groups instead of de Rham vector spaces is because (co)homology theories produce abelian groups. ■

An equivalence class $[\omega] \in H^k_{\mathrm{dR}}(M)$ is called a *cohomology class*, and two form $\omega, \omega' \in Z^k(M)$ are *cohomologous* if $[\omega] = [\omega']$. This means in particular that ω and ω' differ by an exact form, i.e.

$$\omega' = \omega + d\sigma.$$

Now let us consider a smooth mapping $f : N \to M$, then we have that the pullback f^* acts as follows: $f^* : \Gamma^k(M) \to \Gamma^k(N)$. From Theorem **??** it follows that $d \circ f^* = f^* \circ d$ and therefore f^* descends to homomorphism in cohomology. This can be seen as follows:

$$df^*\omega = f^*d\omega = 0, \quad \text{and} \quad f^*d\sigma = d(f^*\sigma),$$

and therefore the closed forms $Z^k(M)$ get mapped to $Z^k(N)$, and the exact form $B^k(M)$ get mapped to $B^k(N)$. Now define

$$f^*[\omega] = [f^*\omega],$$

which is well-defined by

$$f^*\omega' = f^*\omega + f^*d\sigma = f^*\omega + d(f^*\sigma)$$

which proves that $[f^*\omega'] = [f^*\omega]$, whenever $[\omega'] = [\omega]$. Summarizing, f^* maps cohomology classes in $H^k_{\mathrm{dR}}(M)$ to classes in $H^k_{\mathrm{dR}}(N)$:

$$f^* : H^k_{\mathrm{dR}}(M) \to H^k_{\mathrm{dR}}(N),$$

E.18 Theorem Let $f: N \to M$, and $g: M \to K$, then

$$g^* \circ f^* = (f \circ g)^* : \; H^k_{\mathrm{dR}}(K) \to H^k_{\mathrm{dR}}(N),$$

Moreover, id^* is the identity map on cohomology.

Proof. Since $g^* \circ f^* = (f \circ g)^*$ the proof follows immediately. ∎

As a direct consequence of this theorem we obtain the invariance of de Rham cohomology under diffeomorphisms.

E.19 Theorem If $f: N \to M$ is a diffeomorphism, then $H^k_{\mathrm{dR}}(M) \cong H^k_{\mathrm{dR}}(N)$.

Proof. We have that $\mathrm{id} = f \circ f^{-1} = f^{-1} \circ f$, and by the previous theorem

$$\mathrm{id}^* = f^* \circ (f^{-1})^* = (f^{-1})^* \circ f^*,$$

and thus f^* is an isomorphism. ∎

E.1.c Homotopy invariance of cohomology

We will prove now that the de Rham cohomology of a smooth manifold M is even invariant under homeomorphisms. As a matter of fact we prove that the de Rham cohomology is invariant under homotopies of manifolds.

E.20 Definition *Two smooth mappings $f, g: N \to M$ are said to be* homotopic *if there exists a continuous map $H: N \times [0,1] \to M$ such that*

$$\begin{aligned} H(p,0) &= f(p) \\ H(p,1) &= g(p), \end{aligned}$$

for all $p \in N$. Such a mapping is called a homotopy *from/between f to/and g. If in addition H is smooth then f and g are said to be* smoothly homotopic, *and H is called a* smooth homotopy.

Using the notion of smooth homotopies we will prove the following crucial property of cohomology:

E.21 Theorem Let $f, g: N \to M$ be two smoothly homotopic maps. Then for $k \geq 0$ it holds for $f^*, g^*: H^k_{\mathrm{dR}}(M) \to H^k_{\mathrm{dR}}(N)$, that

$$f^* = g^*.$$

▪ **E.22 Remark** It can be proved in fact that the above results holds for two homotopic (smooth) maps f and g. This is achieved by constructing a smooth homotopy from a homotopy between maps. ▪

Proof of Theorem E.21: A map $\mathbf{h} : \Gamma^k(M) \to \Gamma^{k-1}(N)$ is called a *homotopy map* between f^* and g^* if

$$d\mathbf{h}(\omega) + \mathbf{h}(d\omega) = g^*\omega - f^*\omega, \quad \omega \in \Gamma^k(M). \tag{E.1.2}$$

Now consider the embedding $i_t : N \to N \times I$, and the trivial homotopy between i_0 and i_1 (just the identity map). Let $\omega \in \Gamma^k(N \times I)$, and define the mapping

$$\mathbf{h}(\omega) = \int_0^1 i_{\frac{\partial}{\partial t}} \omega dt,$$

which is a map from $\Gamma^k(N \times I) \to \Gamma^{k-1}(N)$. Choose coordinates so that either

$$\omega = \omega_I(x,t)dx^I, \quad \text{or} \quad \omega = \omega_{I'}(x,t)dt \wedge d^x I'.$$

In the first case we have that $i_{\frac{\partial}{\partial t}} \omega = 0$ and therefore $d\mathbf{h}(\omega) = 0$. On the other hand

$$\begin{aligned}
\mathbf{h}(d\omega) &= \mathbf{h}\Big(\frac{\partial \omega_I}{\partial t} dt \wedge dx^I + \frac{\partial \omega_I}{\partial x_i} dx^i \wedge dx^I\Big) \\
&= \Big(\int_0^1 \frac{\partial \omega_I}{\partial t} dt\Big) dx^I \\
&= \big(\omega_I(x,1) - \omega_I(x,0)\big) dx^I = i_1^*\omega - i_0^*\omega,
\end{aligned}$$

which prove (E.1.2) for i_0^* and i_1^*, i.e.

$$d\mathbf{h}(\omega) + \mathbf{h}(d\omega) = i_1^*\omega - i_0^*\omega.$$

In the second case we have

$$\begin{aligned}
\mathbf{h}(d\omega) &= \mathbf{h}\Big(\frac{\partial \omega_I}{\partial x_i} dx^i \wedge dt \wedge dx^{I'}\Big) \\
&= \int_0^1 \frac{\partial \omega_I}{\partial dx_i} i_{\frac{\partial}{\partial x_i}} \big(dx^i \wedge dt \wedge dx^{I'}\big) dt \\
&= -\Big(\int_0^1 \frac{\partial \omega_I}{\partial x_i} dt\Big) dx^i \wedge dx^{I'}.
\end{aligned}$$

On the other hand

$$\begin{aligned}
d\mathbf{h}(\omega) &= d\Big(\Big(\int_0^1 \omega_{I'}(x,t)dt\Big) dx^{I'}\Big) \\
&= \frac{\partial}{\partial x_i}\Big(\int_0^1 \omega_{I'}(x,t)dt\Big) dx^i \wedge dx^{I'} \\
&= \Big(\int_0^1 \frac{\partial \omega_I}{\partial x_i} dt\Big) dx^i \wedge dx^{I'} \\
&= -\mathbf{h}(d\omega).
\end{aligned}$$

This gives the relation that

$$d\mathbf{h}(\omega) + \mathbf{h}(d\omega) = 0,$$

and since $i_1^*\omega = i_0^*\omega = 0$ in this case, this then completes the argument in both cases, and $\mathbf{h}$ as defined above is a homotopy map between i_0^* and i_1^*.

By assumption we have a smooth homotopy $H : N \times [0,1] \to M$ bewteen f and g, with $f = H \circ i_0$, and $g = H \circ i_1$. Consider the composition $\widetilde{\mathbf{h}} = \mathbf{h} \circ H^*$. Using the relations above we obtain

$$\begin{aligned}
\widetilde{\mathbf{h}}(d\omega) + d\widetilde{\mathbf{h}}(\omega) &= \mathbf{h}(H^* d\omega) + d\mathbf{h}(H^*\omega) \\
&= \mathbf{h}(d(H^*\omega)) + d\mathbf{h}(H^*\omega) \\
&= i_1^* H^*\omega - i_0^* H^*\omega \\
&= (H \circ i_1)^*\omega - (H \circ i_0)^*\omega \\
&= g^*\omega - f^*\omega.
\end{aligned}$$

If we assume that ω is closed then

$$g^*\omega - f^*\omega = d\mathbf{h}(H^*\omega),$$

and thus

$$0 = [d\mathbf{h}(H^*\omega)] = [g^*\omega - f^*\omega] = g^*[\omega] - f^*[\omega],$$

which proves the theorem. ■

■ **E.23 Remark** Using the same ideas as for the Whitney embedding theorem one can prove, using approximation by smooth maps, that Theorem E.21 holds true for continuous homotopies between smooth maps. ■

E.24 Definition *Two manifolds N and M are said to be* homotopy equivalent, *if there exist smooth maps $f : N \to M$, and $g : M \to N$ such that*

$$g \circ f \cong \mathrm{id}_N, \qquad f \circ g \cong \mathrm{id}_M \quad \text{(homotopic maps)}.$$

We write $N \sim M$., The maps f and g are homotopy equivalences *are each other* homotopy inverses . *If the homotopies involved are smooth we say that N and M* smoothly homotopy equivalent.

E.25 Example Let $N = S^1$, the standard circle in $\mathbb{R}^2$, and $M = \mathbb{R}^2 \backslash \{(0,0)\}$. We have that $N \sim M$ by considering the maps

$$f = i : S^1 \hookrightarrow \mathbb{R}^2 \backslash \{(0,0)\}, \qquad g = \mathrm{id}/|\cdot|.$$

Clearly, $(g \circ f)(p) = p$, and $(f \circ g)(p) = p/|p|$, and the latter is homotopic to the identity via $H(p,t) = tp + (1-t)p/|p|$.

E.26 Theorem Let N and M be smoothly homotopically equivalent manifold, $N \sim M$, then

$$H^k_{\mathrm{dR}}(N) \cong H^k_{\mathrm{dR}}(M),$$

and the homotopy equivalences f g between N and M, and M and N respectively are isomorphisms.

As before this theorem remains valid for continuous homotopy equivalences of manifolds.

Solutions

Answers Exercises

1.1 Let $A \subset \mathbb{R}^n$ be a closed set. In order to show that $f(A)$ is closed we show that the complement $(f(A))^c$ is an open set. Let $p \in (f(A))^c$ and let $\overline{B_\epsilon(p)}$ be a compact neighborhood. By assumption $f^{-1}(\overline{B_\epsilon(p)})$ is compact. Define $D = A \cap f^{-1}(\overline{B_\epsilon(p)})$. The continuous image of $f(D)$ is again compact. The set $B_\epsilon(p) \setminus f(D)$ is open and contains p and is therefore a neighborhood of p is disjoint of $f(A)$. This show that points $p \in (f(A))^c$ allow open neighborhoods and therefore $(f(A))^c$ is open and consequently $f(A)$ is closed.

1.2 We show that there exist constants $c_p, c'_p > 0$ such that $c_p|x|_2 \le |x|_p \le c'_p|x|_2$ for all $x \in \mathbb{R}^n$. The inequality holds for $x = 0$ and therefore assume that $x \ne 0$. Then $x/|x|$ is a point on the standard unit sphere $S^{n-1} := \{y \in \mathbb{R}^n \mid |y|_2 = 1\}$. Consider the postive continuous function $y \mapsto |y|_p$, $y \in S^{n-1}$. A continuous function on a compact space attains a minimum c_p and maximum c'_p and therefore

$$c_p \le |y|_p \le c'_p, \quad \forall\, y \in S^{n-1}.$$

For x this implies: $c_p \le |x|_p/|x|_2 \le c'_p$, which proves the statement.

1.6 Consider the function $f(x) = x^2 - 1$, $x \in \mathbb{R}$. Let $\Omega = (0,1)$ and let $p = 0$. In this case $f(\partial\Omega) = \{-1,0\}$ and therefore $0 \in f(\partial\Omega)$. The degree formula in Definition 1.5 given $\deg(f,\Omega,0) = 1$. If we take $p = \epsilon > 0$ sufficiently small, then $x^2 - 1 = \epsilon$ has no solutions in $[0,1]$ and thus $\deg(f,\Omega,\epsilon) = 0$. This demonstrates the instability of degree when $p \in f(\partial\Omega)$.

1.22 Under construction.

1.30 Let f^k be smooth functions on $\overline{\Omega}$ with $f^k \to f$ in $C^1(\overline{\Omega})$. For smooth functions we have $(f^k)^*\boldsymbol{\theta} = d((f^k)^*\boldsymbol{\theta})$, cf. [19]. Since the convergence is in C^1 we have that both $(f^k)^*\boldsymbol{\theta} \to f^*d\boldsymbol{\theta}$ and $d((f^k)^*\boldsymbol{\theta}) \to d(f^*\boldsymbol{\theta})$ in $C^0(\overline{\Omega})$, which completes the proof.

1.34 Let $c = \int_{\mathbb{R}^n} \boldsymbol{\omega}$ and normalize $\widehat{\boldsymbol{\omega}} := c^{-1}\boldsymbol{\omega}$. Since $\int_{\mathbb{R}^n} \widehat{\boldsymbol{\omega}} = 1$ we can apply Definition 1.33, which gives $\deg(f,\Omega,p) = \int_\Omega f^*\widehat{\boldsymbol{\omega}} = c^{-1}\int_\Omega f^*\boldsymbol{\omega} = \int_\Omega f^*\boldsymbol{\omega} / \int_{\mathbb{R}^n} \boldsymbol{\omega}$.

1.36 The mapping $[\boldsymbol{\omega}] \mapsto \int_{\mathbb{R}^n} \boldsymbol{\omega}$. Indeed, let $\boldsymbol{\omega}, \boldsymbol{\omega}' \in [\boldsymbol{\omega}]$, then $\boldsymbol{\omega}' - \boldsymbol{\omega} = d\boldsymbol{\theta}$ and thus $\int_{\mathbb{R}^n} \boldsymbol{\omega} = \int_{\mathbb{R}^n} \boldsymbol{\omega}'$ (see proof of Lemma 1.29). The integral is onto $\mathbb{R}$; replace $\boldsymbol{\omega}$ by $k\boldsymbol{\omega}$. Finally, infectivity follows from the Poincaré Lemma. Let $\int_{\mathbb{R}^n} \boldsymbol{\omega} = \int_{\mathbb{R}^n} \boldsymbol{\omega}'$, then $\boldsymbol{\omega}' - \boldsymbol{\omega} = d\boldsymbol{\theta}$, which shows that $[\boldsymbol{\omega}] = [\boldsymbol{\omega}']$.

3.33 By assumption $0 \in \Omega$ and since Ω is open there exists an open ball $B_\epsilon(0) \subset \Omega$. Let $x \in \partial\Omega$, then by convexity $tx \in \overline{\Omega}$ for all $t \in [0,1]$. Again by convexity $B_{t\epsilon}(tx) \subset \overline{\Omega}$ for all $0 \le t < 1$, which proves that tx is an interior point for all $0 \le t < 1$.

4.7 This result is trivially true if $S = \varnothing$. So assume $S \neq \varnothing$. If S is invariant the inclusion is trivially satisified for $t > 0$. For $t < 0$ the inclusion follows from Proposition **??** by choosing $s = -t$.

Now assume $S \subset \varphi(t,S)$ for all $t \in \mathbb{R}$. For all $t < 0$, the semi-group property and Proposition **??** implies that $\varphi(-t,S) \subset \varphi(-t,\varphi(t,S)) \subset S$. Thus,

$$\varphi(-t,S) \subset S \subset \varphi(t',S)$$

for any $t' > 0$. Observe that the result follows from choosing $t' = -t$.

4.9 Assume (i) that S is invariant. Since $\varphi(t,S) = S$ for $t \in \mathbb{R}^+$, we have $\bigcap_{t\in\mathbb{R}} \varphi(t,S) \subset S$. Conversely, since $S \subset \varphi(t,S)$ for all $t \in \mathbb{R}$, we have that $S \subset \bigcap_{t\in\mathbb{R}} \varphi(t,S)$, and $\varphi\|_S$ is surjective, establishing (ii).

Assume (ii). Given $x \in S$ there exists a unique forward orbit $\gamma_x^+ \subset S$. Consider $x_{-1} \in \varphi\|_S(-1,x) \neq \varnothing$. Then for $s \in [0,1]$, we have $\varphi\|_S(s-1,x) = \varphi\|_S(s,x_{-1})$, and $\varphi\|_S(1,x_{-1}) = x$. Inductively repeating this process by producing $x_{-k-1} \in \varphi\|_S(-1,x_{-k})$ yields an orbit γ_x in S, establishing (iii).

Assume (iii), then $\varphi(t,S) \subset S$ for all $t > 0$. Let $x \in S$. For each $t > 0$ there exists a $y \in \varphi(-t,x) \cap S$. Since $\varphi(t,y) \subset \varphi(t,S)$, we have $x \in \varphi(t,S)$, and thus $S \subset \varphi(t,S)$, establishing (i).

The equivalence between (i) and (iv) follows from the group property.

4.18 For $y \in \omega(U)$ it holds that $y \in \mathrm{cl}\big(\varphi([t,\infty),U))\big)$ for all $t \in \mathbb{T}^+$, and thus $\omega(U) \subset \bigcap_{t\ge0} \mathrm{cl}\big(\varphi\big([t,\infty),U\big)\big)$. On the other hand, if $y \in \bigcap_{t\ge0} \mathrm{cl}\big(\varphi\big([t,\infty),U\big)\big)$, then $y \in \mathrm{cl}\big(\varphi([t,\infty),U))\big)$ for all $t \in \mathbb{T}^+$. Choose an increasing sequence of $t_n \in \mathbb{T}^+$ and $x_n \in U$ such that $d(\varphi(t_n,x_n),y) < 1/n$. Since $d(\varphi(t_n,x_n),y) \to 0$ as $t_n \to \infty$, it follows that $y \in \omega(U)$ which proves the other inclusion.

In the case $U \subset X$ is forward invariant, the semi-group property implies, $\varphi(t+s,U) = \varphi\big(t,\varphi(s,U)\big) \subset \varphi(t,U)$ for all $s,t \geq 0$. Therefore $\varphi([t,\infty),U) = \varphi(t,U)$, which proves Equation (4.1.3).

The second identity in Equation (4.1.2) follows from the semi-group property, i.e. $\varphi([t,\infty),U) = \varphi(t,\Gamma^+(U))$, Equation (4.1.3) and the forward invariance of $\Gamma^+(U)$. Closedness follows immediately from the definition.

As for the forward invariance of $\omega(U)$ we argue as follows. We first show forward invariance of $\omega(U)$ when U is forward invariant. For any $t \in \mathbb{T}^+$ we have that

$$\begin{aligned}\varphi(t,\omega(U)) &= \varphi\Big(t,\bigcap_{s\geq 0}\mathrm{cl}\big(\varphi(s,U)\big)\Big) \subset \bigcap_{s\geq 0}\mathrm{cl}\Big(\varphi\big(t,\varphi(s,U)\big)\Big)\\ &= \bigcap_{s\geq 0}\mathrm{cl}\Big(\varphi\big(s,\varphi(t,U)\big)\Big) \subset \omega(U).\end{aligned}$$

For general U, use the fact that $\Gamma^+(U)$ is forward invariant. Therefore,

$$\varphi(t,\omega(U)) = \varphi\big(t,\omega(\Gamma^+(U))\big) \subset \omega(\Gamma^+(U)) = \omega(U),$$

which proves forward invariance for general U. The omega limit set is obviously contained in $\mathrm{cl}(\Gamma^+(U))$.

4.20 Property (i) follows from the characterization in Lemma 4.15. Define the truncation

$$\begin{aligned}\omega_N(U\cup V) &= \bigcap_{t\in[0,N]}\mathrm{cl}\left(\varphi([t,\infty),U\cup V)\right) = \mathrm{cl}\left(\varphi([N,\infty),U\cup V)\right)\\ &= \mathrm{cl}\left(\varphi([N,\infty),U)\right)\cup\mathrm{cl}\left(\varphi([N,\infty),V)\right)\\ &= \omega_N(U)\cup\omega_N(V),\end{aligned}$$

which proves the first part of (ii) by letting $N\to\infty$. As for the intersection we argue as follows. Note that $U\cap V\subset U$ and $U\cap V\subset V$, and by (i) $\omega(U\cap V)\subset\omega(U)$ and $\omega(U\cap V)\subset\omega(V)$. Combining this gives $\omega(U\cap V)\subset\omega(U)\cap\omega(V)$.

By forward invariance of $\omega(U)$, $\varphi(t,V)\subset\varphi(t,\omega(U))\subset\omega(U)$. Since the latter is closed it follows that $\omega(V)\subset\omega(U)$, proving (iii)

Since $U\subset\mathrm{cl}(U)$ it follows that $\omega(U)\subset\omega(\mathrm{cl}(U))$. As for the reversed inclusion we argue as follows. Since φ is continuous map from $\mathbb{T}^+\times X$ to X it follows that the image of $\mathrm{cl}\big([t,\infty)\times U\big)$ is contained in the closure of the image of $[t,\infty)\times U$, for all $t\geq 0$, see Lemma **??**(iii). Therefore, $\varphi([t,\infty),\mathrm{cl}(U))\subset\mathrm{cl}\big(\varphi([t,\infty),U)\big)$, and thus $\mathrm{cl}\big(\varphi([t,\infty),\mathrm{cl}(U))\big)\subset\mathrm{cl}\big(\varphi([t,\infty),U)\big)$. For the omega limit sets this implies that $\omega(\mathrm{cl}(U))\subset\omega(U)$, which proves (iv).

Since $\omega(\Gamma^+(U)) = \omega\big(\Gamma^+(\varphi(t,U))\big)$ and $\Gamma^+(\varphi(t,U)) = \varphi(t,\Gamma^+(U))$, Property (v) follows.

If there exists a backward orbit $\gamma_x^- \subset U$, then $y_n = \gamma_x^-(-t_n) \in U$, $t_n \to \infty$ has the property that $x \in \varphi(t_n, y_n)$ as $t_n \to \infty$, which shows that $x \in \omega(U)$ and proves Property (vi).

4.26 By definition any point $x \in W^s(S)$ has the property that $\Gamma^-(x) \subset W^s(S)$. Indeed, for any $s < 0$ it follows from Proposition **??** that $\varphi(t, \varphi(s,x)) = \varphi(t+s, x)$ for $t + s \geq 0$, and thus $\lim_{t\to\infty} d(\varphi(t,\varphi(s,x)), S) = \lim_{t\to\infty} d(\varphi(t+s,x), S) = 0$. Therefore $\Gamma(x) \subset W^s(S)$ for all $x \in W^s(S)$. Obviously, $\varphi(t, W^s(S)) \subset W^s(S)$, for $t \geq 0$, which shows that $W^s(S)$ is forward invariant. Also for any $x \in W^s(S)$, $\varphi(-t, x) \subset \Gamma^-(x) \subset W^s(S)$ which shows that $\varphi(t, W^s(S)) \subset W^s(S)$, for $t \leq 0$, proving backward invariance. If φ is surjective, then by Lemma **??**, $W^s(S)$ is strongly invariant.

The invariance of $W^u(S)$ is established as follows. Let $x \in W^u(S)$ and let γ_x^+ be the forward orbit. Choose any point $y = \varphi(s,x) \in \gamma_x^+$, then

$$\gamma_y^-(t) = \begin{cases} \gamma_x^+(t+s) & \text{for } -s \leq t \leq 0 \\ \gamma_x^-(t+s) & \text{for } t \leq -s, \end{cases}$$

is a backward orbit through y and $d(\gamma_y^-(t), S) \to 0$ as $t \to -\infty$. Indeed, since for any $\epsilon > 0$ there exists a $T_\epsilon < 0$ such that $d(\gamma_x^-(t), S) < \epsilon$ for all $t \leq T_\epsilon$, it holds that $d(\gamma_y^-(t), S) < \epsilon$ for all $t \leq T_\epsilon - s$. We have now proved that for any $y = \varphi(s,x) \in \gamma_x^+$ there exists a backward orbit γ_y^- which converges to S in backward time. The same holds of course for points $y \in \gamma_x^-$. Therefore, for each $x \in W^u(S)$ we have that $\gamma_x = \gamma_x^+ \cup \gamma_x^- \subset W^u(S)$ which proves the invariance of $W^u(S)$ by Proposition 4.8(iii).

4.29 Invariance of $C(S', S)$ follows from Proposition **??** and Exercise **??** that the intersection of forward and backward invariant set with an invariant set is invariant.

Assume $x \in C(S', S) \cap S'$. Then, for points $x \in S'$ it holds that $\varphi(t,x) \in S'$ for $t \geq 0$ (invariance) and for points $x \in C(S', S)$ it holds that $d(\varphi(t,x), S) \to 0$ as $t \to \infty$. Since $S \cap S' = \varnothing$, such points do not exist. Similarly, assume that $x \in C(S', S) \cap S$. Then, for points $x \in S$ it holds that $\varphi(t,x) \in S$ for $t \geq 0$ (invariance) and for points $x \in C(S', S)$ it holds that $d(\varphi(t,x), S) \to 0$ as $t \to \infty$. There may be points that satisfy these properties unless φ is invertible.

From the arguments in Lemma **??** it follows that $\omega(x) \subset S$ and similarly $\alpha_o(\gamma_x^-) \subset S$ for some orbit γ_x.

Answers Problems

8.23 We have

$$\|t\mathbf{e}^- + s\mathbf{e}^+\|^2_{L^2\times L^2} = t^2\|\mathbf{e}^-\|^2_{L^2\times L^2} + s^2\|\mathbf{e}^+\|^2_{L^2\times L^2} + 2st(\mathbf{e}^-, \mathbf{e}^+)_{L^2\times L^2}.$$

For the inner product we have

$$(\mathbf{e}^-, \mathbf{e}^+)_{L^2 \times L^2} = \|\mathbf{e}^-\|_{L^2 \times L^2} \|\mathbf{e}^+\|_{L^2 \times L^2} \cos(\chi),$$

which implies $2st(\mathbf{e}^-, \mathbf{e}^+)_{L^2 \times L^2} \leq t^2 \|\mathbf{e}^-\|^2_{L^2 \times L^2} + s^2 \cos^2(\chi) \|\mathbf{e}^+\|^2_{L^2 \times L^2}$. Combining this we obtain

$$\begin{aligned} \|t\mathbf{e}^- + s\mathbf{e}^+\|^2_{L^2 \times L^2} &= t^2 \|\mathbf{e}^-\|^2_{L^2 \times L^2} + s^2 \|\mathbf{e}^+\|^2_{L^2 \times L^2} + 2st(\mathbf{e}^-, \mathbf{e}^+)_{L^2 \times L^2} \\ &\geq s^2 \|\mathbf{e}^+\|^2_{L^2 \times L^2} - s^2 \cos^2(\chi) \|\mathbf{e}^+\|^2_{L^2 \times L^2} \\ &= s^2 \sin^2(\chi) \|\mathbf{e}^+\|^2_{L^2 \times L^2}, \end{aligned}$$

which completes the proof.

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