Problem I

1. Let $\Omega \subset \mathbb{R}^2$ be a bounded and open set with smooth boundary $\partial \Omega$ and let $X : \overline{\Omega} \rightarrow \mathbb{R}^2$ be a smooth vector field on $\overline{\Omega}$, which satisfies the boundary property that

$$X(x) \cdot n_x > 0, \quad \forall x \in \partial \Omega,$$

where $n_x$ is the outward pointing normal on $\partial \Omega$. Assume that $X$ only has isolated zeroes. If $x \in \Omega$ is an isolated zero of $X$, i.e. $X(x) = 0$, then define the index of $x$ by

$$i(x) = \deg(X, N_\epsilon, 0),$$

where $N_\epsilon$ is a any sufficiently small neighborhood of $x$ for which $x$ is the unique zero of $X$.

Show that if $\Omega$ is diffeomorphic to $B_1 \setminus \bigcup_{i=1}^g \overline{B}(x_i)$, i.e. a disc with $g$ holes, then for any vector field $X$ satisfying the above conditions it holds that

$$\sum_{x \in \Omega \atop X(x) = 0} i(x) = 1 - g.$$

2. Let $\Sigma \subset \mathbb{R}^3$ be a smooth, compact, orientable surface without boundary and let $X$ be a smooth vector field on $\Sigma$, i.e. $X : \Sigma \rightarrow \mathbb{R}^3$, with $X(x) \in T_x \Sigma$ for all $x \in \Sigma$. Assume that $X$ only has isolated zeroes. The index of a zero $x$ is again defined as the local degree of $X$ at $x$.

Show that for any vector field $X$ on $\Sigma$ satisfying the above conditions it holds that

$$\sum_{x \in \Sigma \atop X(x) = 0} i(x) = 2 - 2g,$$

where $g$ is the genus of the surface.