Problem II

In the following \( \varphi : \mathbb{R} \times X \to X \) is a flow and \( X \) a compact metric space.

1. Prove that the following statements are equivalent:
   (i) \( S \) is invariant;
   (ii) \( S = \bigcap_{t \in \mathbb{R}} \varphi(t, S) \) and \( \varphi|_S \) is surjective;
   (iii) for all \( x \in S \) there exists a complete orbit \( \gamma_x \subset S \);
   (iv) \( \varphi(t, S) = S \) for all \( t \in (0, \tau] \), for some \( \tau > 0 \).

2. Let \( S \subset X \) be an invariant set for \( \varphi \). Show that both \( W^s(S) \) and \( W^u(S) \) are invariant.

3. Given the equations \( x' = x(x-1) \), \( y' = y(y-1) \) and \( z' = z(z-1) \). Let \( X \) be given by \( X = [0,1]^3 \) and \( \varphi \) is the flow on \( X \) generated by the uncoupled system of differential equations.
   (i) Find the finest Morse decomposition.
   (ii) Describe the lattice of all attractors of the system.

4. Show that \( \text{sub}_{0,1}\text{Att}^F(X) \) is a lattice with \( \wedge \) set intersection and \( \vee \) defined by
   \[
   A \vee A' = \bigcap \{ A'' \in \text{sub}_{0,1}\text{Att}^F(X) \mid A \cup A' \subset A'' \}.
   \]
   Here \( \text{sub}_{0,1}\text{Att}^F(X) \) is the set of all \((0,1)\) sublattices of \( \text{Att}(X) \) (see also notes, pp. 112).

5. Let \( f : T^2 \to \mathbb{R} \) be a smooth function on the 2-torus \( T^2 \).
   (i) Prove that if \( f \) is a Morse function (\( f \) only has non-degenerate critical points), then the number of critical points of \( f \) is bounded from below by 4.
   (ii) Prove that in general that the number of critical points of \( f \) is bounded from below by 3.