Problem III

Consider the differential equation \( u'' = u^3 - u - u' \).

1. (i) Show that the set of all bounded solutions of the above equation is a compact invariant set \( S_{\text{bdd}} \).
(ii) Prove that the system is gradient-like, i.e. there exists a Lyapunov function \( V \) such that \( V \circ \varphi \) is strictly decreasing outside the set of equilibrium points \( E \).
(iii) Prove that \( E \) yields a Morse decomposition of \( S_{\text{bdd}} \).
(iv) Use Conley Theory to prove that the above differential equation has heteroclinic connections between the equilibrium points \(-1\) and \(0\) and between \(0\) and \(1\).
(v) Do the results change under small perturbations of the right hand side?

Consider the fourth order equation \( u''' = u - u^3 - \epsilon u' \), \( 0 < \epsilon \ll 1 \).

2. Show that the results in 1 remain unchanged for this fourth order equation.