1. Given the cylinder

\[ M = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = 1, \quad 0 \leq z \leq 1\}, \]

with the standard orientation \( \mathcal{O} \).

(a) Determine the induced, or Stokes orientation \( \partial \mathcal{O} \) on the boundary \( \partial M \) (Hint: provide 1-forms on \( \partial M \)).

(b) Given the 2-form

\[ \sigma = dx \wedge dy + dy \wedge dz + dx \wedge dz, \]

on \( M \), show that \( \sigma \) is exact.

(c) Use Stokes’s Theorem to compute the integral

\[ \int_M \sigma. \]

2. Consider the point set \( E \) defined as

\[ E = \{(p_1, p_2, \xi_1, \xi_2) \in \mathbb{R}^4 \mid p_1 = \cos(\theta), p_2 = \sin(\theta), \quad \cos(\theta/2)\xi_1 + \sin(\theta/2)\xi_2 = 0\}, \quad \theta \in [0, 2\pi]. \]

(a) Show that \( E \) is a smooth rank 1 vector bundle over \( S^1 \).

(b) Prove that for the zero section \( s : S^1 \to E \), given by \( p \mapsto (p, 0) \), it holds that \( \Sigma = s(S^1) \) is diffeomorphic to \( S^1 \).

(c) Consider the set \( E \setminus \Sigma \) and show that \( E \setminus \Sigma \) is diffeomorphic to \( \mathbb{R} \times S^1 \).

3. Consider the standard ellipse

\[ \left\{(x, y) \in \mathbb{R}^2 \mid \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\}, \quad a, b > 0, \right\} \]

which can be parametrized by \( (x, y) = (a \cos(\theta), b \sin(\theta)), \ \theta \in [0, 2\pi] \). Use the 1-form \( \omega = xdy \) and Stokes’s Theorem to compute the area enclosed by the ellipse.
4. Let \( M = \mathbb{R}^3 \setminus \{(0,0,0)\} \), and define the following 2-form on \( M \):
\[
\omega = \frac{1}{(x^2 + y^2 + z^2)^{3/2}}(x\,dy \wedge dz + y\,dz \wedge dx + z\,dx \wedge dy).
\]
(a) Show that \( \omega \) is closed (\( d\omega = 0 \)).
(b) Prove that \( \omega \) is not exact on \( M \).

On \( N = \mathbb{R}^3 \setminus \{z - \text{axis}\} \subset M \) consider the 1-form:
\[
\eta = -z \frac{x\,dy - y\,dx}{(x^2 + y^2 + z^2)^{1/2}}.
\]
(a) Show that \( \omega \) is exact as 2-form on \( N \), and verify that \( d\eta = \omega \).

5. Let \( S^n \subset \mathbb{R}^{n+1} \) be the standard unit sphere given by
\[
S^n = \{p \in \mathbb{R}^{n+1} \mid ||p||^2 = 1\},
\]
with \( ||\cdot|| \) the Euclidean norm on \( \mathbb{R}^{n+1} \). A vector field \( X \) on \( S^n \) is a smooth section in \( TS^n \). A vector field is non-vanishing if \( X(p) \neq 0 \) for all \( p \in S^n \). The Hairy Ball Theorem states: There exists a non-vanishing vector field on \( S^n \) if and only \( n \) is odd.
(a) Let \( n \) be odd. Give a smooth non-vanishing vector field \( X \) on \( S^n \).
(b) Let \( n \) be arbitrary and let \( X \) be any smooth non-vanishing vector field on \( S^n \).
Show that there exists a smooth mapping \( H : [0,1] \times S^n \to S^n \), with the property that \( H(0,p) = p \) and \( H(1,p) = -p \).
The map \( H \) above is called a smooth homotopy between the identity map and the antipodal map \( p \mapsto -p \).
(c) Prove that if the maps \( p \mapsto p \) and \( p \mapsto -p \) are smoothly homotopic, then the antipodal map is orientation-preserving (Hint: Use Stokes’s Theorem on \([0,1] \times S^n\) with \( H^*\omega \), where \( \omega \) is an arbitrary \( n \)-form on \( S^n \)).
(d) Show that the antipodal map \( p \mapsto -p \) is an orientation-preserving map \( S^n \to S^n \), then \( n \) is odd.
(e) Combine the statements (a)-(d) in order to prove the Hairy Ball Theorem.

6*. Let \( D \subset \mathbb{R}^2 \) be an open, simply connected subset. A differential equation
\[
F(x,y)dx + G(x,y)dy = 0, \quad F, G \in C^\infty(D),
\]
is exact if there is a potential function \( \varphi : D \to \mathbb{R} \) (smooth) such that \( F = \partial_x \varphi \) and \( G = \partial_y \varphi \).
(a) Compute the general solution for an exact differential equation as given above.
(b) Prove that a differential equation as described above is exact if and only if
\[ \frac{\partial F}{\partial y} = \frac{\partial G}{\partial x} \]
(Hint: the De Rham cohomology of $D$ is given by $H^0_{dR}(D) \cong \mathbb{R}$ and $H^k_{dR}(D) = 0$ for $k \geq 1$).

*Good luck*

Problem 6* is for extra credit and not mandatory.
The exam needs to be submitted by email before midnight, December 21 (scanned or typed)! Email: r.c.a.m.vander.vorst@vu.nl, or vdvorst@few.vu.nl