9
Linear Elliptic Equations

9.1 Definitions

We will begin our study of linear elliptic PDEs by considering differential operators of the form \( L(x, D)u = 0 \) defined in (2.9). Note that by Definition 2.8 a differential operator of order \( m \) is elliptic at \( x_0 \) if and only if

\[
L^p(x_0, \xi) = \sum_{|\alpha|=m} a_\alpha(x)\xi^\alpha \neq 0 \quad \text{for every } \xi \in \mathbb{R}^n \setminus \{0\}. \quad (9.1)
\]

In fact, we can show the following.

**Lemma 9.1.** If a linear partial differential operator \( L \) of order \( m \) is elliptic at \( x_0 \in \mathbb{R}^n, n > 1 \), then \( m \) is an even integer \( (m = 2k) \) and \( \xi \mapsto L^p(x_0, \xi) \) takes on only one sign on \( \xi \neq 0 \).

**Proof.** By definition, \( \xi \mapsto L^p(x_0, \xi) \) is continuous and takes on the value 0 only at \( \xi = 0 \). Suppose \( L^p(x_0, \xi_1) < 0 \) and \( L^p(x_0, \xi_2) > 0 \), and then connect \( \xi_1 \) and \( \xi_2 \) using a path not going through 0. As noted, \( L^p(x_0, \xi) \) must vary continuously along the path, taking on the value 0. This is a contradiction.

It now follows that, for any \( \xi \in \mathbb{R}^n \),

\[
L^p(x_0, \xi) \quad \text{and} \quad L^p(x_0, -\xi) = (-1)^m L^p(x_0, \xi)
\]

must have the same sign. This implies that \( m \) is even. \( \square \)

In light of this result, we will use the following somewhat restricted definition of an elliptic operator for the remainder of the chapter.
Definition 9.2. Let \( \Omega \subseteq \mathbb{R}^n \) be a domain. We say that a linear partial differential operator
\[
L(x, D) = \sum_{|\alpha| \leq 2k} a_\alpha(x) D^\alpha
\]
is **elliptic** in \( \Omega \) if
\[
(-1)^k \sum_{|\alpha| = 2k} a_\alpha(x) \xi^\alpha > 0 \quad \text{for every } x \in \Omega, \ \xi \in \mathbb{R}^n \setminus \{0\}. \tag{9.3}
\]
We say that \( L \) is **uniformly elliptic** in \( \Omega \) if there exists a constant \( \theta > 0 \) such that
\[
(-1)^k \sum_{|\alpha| = 2k} a_\alpha(x) \xi^\alpha \geq \theta |\xi|^{2k} \quad \text{for every } x \in \Omega, \ \xi \in \mathbb{R}^n \setminus \{0\}. \tag{9.4}
\]

Example 9.3. The reader should recall the calculations of Chapter 2 which showed that the negative of the Laplacian \(-\Delta\) (which is of order 2) and the Biharmonic \(\Delta^2\) (order 4) are uniformly elliptic with \(\theta = 1\).

Example 9.4. A second-order operator in \( n \) space dimensions of the form
\[
Lu = a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + b_i(x) \frac{\partial u}{\partial x_i} + c(x) u \tag{9.5}
\]
is uniformly elliptic on a domain \( \Omega \) provided there exists a constant \( \theta \) such that
\[
\xi^T A(x) \xi > \theta |\xi|^2 \tag{9.6}
\]
for every \( x \in \Omega \). Here \( A(x) \) is the \( n \times n \) matrix with components \(-a_{ij}(x)\).

In our discussion of existence and regularity theory below, it is convenient to put our differential operators in a form which is amenable to integration by parts.

Definition 9.5. We say that an operator is in **divergence form** if there are functions \( a_{\sigma\gamma} : \Omega \to \mathbb{R} \) such that
\[
L(x, D)u = \sum_{0 \leq |\sigma|, |\gamma| \leq k} (-1)^{|\sigma|} D^\sigma (a_{\sigma\gamma}(x) D^\gamma u). \tag{9.7}
\]

Remark 9.6. Note that an operator in divergence form is elliptic if and only if
\[
\sum_{|\sigma|, |\gamma| = k} \xi^\sigma a_{\sigma\gamma}(x) \xi^\gamma > 0 \quad \text{for every } x \in \Omega, \ \xi \in \mathbb{R}^n \setminus \{0\}, \tag{9.8}
\]
and uniformly elliptic if and only if there exists \( \theta > 0 \) such that
\[
\sum_{|\sigma|, |\gamma| = k} \xi^\sigma a_{\sigma\gamma}(x) \xi^\gamma > \theta |\xi|^{2k} \quad \text{for every } x \in \Omega, \ \xi \in \mathbb{R}^n \setminus \{0\}. \tag{9.9}
\]
If our coefficients are smooth enough, we can put a general PDE into divergence form. We give conditions for doing so here which are sufficient, though by no means necessary.

**Lemma 9.7.** Let
\[ a_{\alpha} \in C^{|\alpha|-k}_{b}(\Omega) \quad \text{for } k < |\alpha| \leq 2k \]  
(9.10)
and
\[ a_{\alpha} \in C^{|\alpha|}_{b}(\Omega) \quad \text{for } |\alpha| \leq k. \]  
(9.11)
Then there exist \( a_{\sigma\gamma} \in C^{k|\sigma|}_{b}(\Omega) \) such that for every \( u \in C^{2k}(\Omega) \) we have
\[
L(x, D)u = \sum_{|\alpha| \leq 2k} a_{\alpha}(x) D^\alpha u = \sum_{0 \leq |\sigma|, |\gamma| \leq k} (-1)^{|\sigma|} D^\sigma (a_{\sigma\gamma}(x) D^\gamma u).
\]

**Proof.** We do the proof here for general \( k \), but the notation is rather cumbersome, so the reader should work through the details for the case \( k = 1 \).

For every \( |\alpha| \leq 2k \) we choose \( \sigma_{\alpha} \) and \( \gamma_{\alpha} \) satisfying \( |\sigma_{\alpha}|, |\gamma_{\alpha}| \leq k, \sigma_{\alpha} + \gamma_{\alpha} = \alpha \). This choice is, of course, not unique.

Now, for any \( u \in C^{2k}(\Omega) \) and \( \phi \in D(\Omega) \) we have
\[
\int_{\Omega} L(x, D)u \phi \, dx = \sum_{|\alpha| \leq 2k} \int_{\Omega} (D^\alpha u) a_{\alpha} \phi \, dx
\]
\[
= \sum_{|\alpha| \leq 2k} \int_{\Omega} (D^{\sigma_{\alpha}+\gamma_{\alpha}} u) a_{(\sigma_{\alpha}+\gamma_{\alpha})} \phi \, dx
\]
\[
= \sum_{|\alpha| \leq 2k} (-1)^{|\sigma_{\alpha}|} \int_{\Omega} (D^{\gamma_{\alpha}} u) (D^{\sigma_{\alpha}} a_{(\sigma_{\alpha}+\gamma_{\alpha})} \phi) \, dx
\]
\[
= \sum_{|\alpha| \leq 2k} (-1)^{|\sigma_{\alpha}|} \int_{\Omega} D^{\gamma_{\alpha}} u \sum_{\rho \leq \sigma_{\alpha}} \left( \begin{array}{c} \sigma_{\alpha} \\ \rho \end{array} \right) D^{\sigma_{\alpha}-\rho} a_{(\sigma_{\alpha}+\gamma_{\alpha})} D^{\rho} \phi \, dx
\]
\[
= \sum_{|\alpha| \leq 2k} (-1)^{|\sigma_{\alpha}|+|\rho|} \int_{\Omega} D^{\rho} \left[ \left( \begin{array}{c} \sigma_{\alpha} \\ \rho \end{array} \right) D^{\sigma_{\alpha}-\rho} a_{(\sigma_{\alpha}+\gamma_{\alpha})} D^{\gamma_{\alpha}} u \right] \phi \, dx
\]
\[
= \sum_{0 \leq |\sigma|, |\gamma| \leq k} \int_{\Omega} (-1)^{|\sigma|} D^{\sigma} (a_{\sigma\gamma}(x) D^{\gamma} u) \phi \, dx.
\]
(Note that the last equality is a definition.) Since this holds for all \( \phi \in D(\Omega) \) we have our result.

**Remark 9.8.** Unless explicitly stated otherwise, we shall assume that our coefficients satisfy the smoothness assumptions (9.10) and (9.11).
When dealing with systems of PDEs (as opposed to single equations) we will not, in general, focus on a more restrictive definition of ellipticity. That is, we will stick to the definition of an elliptic system as one with no real characteristics, and characteristics are to be determined by the principal part of the operator defined using appropriate weights. However, the reader should be aware of some particularly important examples of elliptic systems that arise in the calculus of variations. Let $\Omega$ be a domain in $\mathbb{R}^n$ and let $u : \Omega \to \mathbb{R}^N$. Consider the system of $N$ second-order differential equations in divergence form

$$
\frac{\partial}{\partial x_k} \left( A^{Ij}_{kl}(x) \frac{\partial u^I_j}{\partial x_l} \right) + c^{Ij}(x) u^I_j(x) = 0, \quad I = 1, \ldots, N. \tag{9.12}
$$

Here, the coefficients $A^{Ij}_{kl}$ and $c^{Ij}$, $I, J = 1, \ldots, N$, $k, l = 1, \ldots, n$, are assumed to be sufficiently smooth, and the summation convention is assumed to hold from 1 to $N$ for repeated uppercase indices and from 1 to $n$ for repeated lowercase indices.

Note that if

$$M^I_k A^{Ij}_{kl} M^J_l > 0 \tag{9.13}$$

for every $x \in \Omega$ and for every nonzero $N \times n$ matrix $M$, then we can show that the system is elliptic simply by taking the “obvious” principal part (giving weight 1 to each of the equations and dependent variables). Condition (9.13) or the uniform version

$$M^I_k A^{Ij}_{kl} M^J_l \geq \theta |M|^2 \tag{9.14}$$

for every $x \in \Omega$ and every nonzero $N \times n$ matrix $M$ for some $\theta > 0$ is often given as a definition of an elliptic system. However, such a definition does not fit such systems as the Stokes system.

Another important “ellipticity condition” is the **Legendre-Hadamard** condition

$$\eta^I \xi_k A^{Ij}_{kl} \eta^J \xi_l > 0 \tag{9.15}$$

for every $x \in \Omega$ and for every nonzero $\eta \in \mathbb{R}^N$ and $\xi \in \mathbb{R}^n$. The uniform version states that there exists $\theta > 0$ such that for every $x \in \Omega$

$$\eta^I \xi_k A^{Ij}_{kl} \eta^J \xi_l > \theta |\eta|^2 |\xi|^2 \tag{9.16}$$

for every nonzero $\eta \in \mathbb{R}^N$ and $\xi \in \mathbb{R}^n$. These conditions turn out to be more physically reasonable than (9.13) or (9.14) for many problems in elasticity.

Note that (9.15) and (9.16) are much weaker than the corresponding conditions (9.13) and (9.14). (The inequalities have to hold only for rank-1 $N \times n$ matrices.) Despite this, (9.15) and (9.16) are sometimes referred to as **strong ellipticity** conditions. As this example shows, the reader should be forewarned that the nomenclature surrounding elliptic systems does not necessarily make sense. More importantly, there is not universal agreement
9.2 Existence and Uniqueness of Solutions of the Dirichlet Problem

9.2.1 The Dirichlet Problem—Types of Solutions

We begin with a statement of the classical Dirichlet problem.

**Definition 9.9.** Let \( \Omega \subset \mathbb{R}^n \) be a bounded domain and suppose \( f \in C_b(\Omega) \) is given. A function 

\[
u \in C_b^{2k}(\Omega) \cap C_b^{2k-1}(\overline{\Omega})
\]

is a **classical solution** of the Dirichlet problem if 

\[
L(x, D)u = \sum_{0 \leq |\sigma|, |\gamma| \leq k} (-1)^{|\sigma|} D^\sigma (a_{\sigma \gamma}(x) D^\gamma u) = f
\]

in \( \Omega \); and 

\[
D^\alpha u = 0 \quad \text{for } |\alpha| \leq k - 1
\]

on \( \partial\Omega \).

One of the most important ideas of the modern analysis is that if you want to guarantee the existence of a solution to a problem, it is usually easier to do so in a “bigger” space of functions. This is clearly the case with the classical Dirichlet problem. Although we might expect a solution to have all of the smoothness suggested in the statement of the problem, we must relax the conditions on the solution at first so that we can use the methods of the last three chapters. The first step in relaxing the conditions on the solution is to state the problem in terms of Sobolev spaces.

**Definition 9.10.** Let \( \Omega \subset \mathbb{R}^n \) be a bounded domain and suppose \( f \in L^2(\Omega) \) is given. A function 

\[
u \in H^{2k}(\Omega) \cap H_0^{k}(\Omega)
\]

is a **strong solution** of the Dirichlet problem if 

\[
L(x, D)u = \sum_{0 \leq |\sigma|, |\gamma| \leq k} (-1)^{|\sigma|} D^\sigma (a_{\sigma \gamma}(x) D^\gamma u) = f
\]

in \( \Omega \).

Note the following.

1. We have relaxed the conditions not only on the solution \( u \), but on the data \( f \). The space \( L^2(\Omega) \) is certainly the obvious space for \( f \).
once we have relaxed the conditions on $u$, so the additional generality (which includes such physically reasonable situations as discontinuous forcing functions) will come along “for free.” (In fact, we will be able to weaken the conditions on $f$ each time we relax the conditions on the solution, as we shall see below.)

2. For classical solutions, the differential equation (9.17) is taken to hold in a pointwise sense. For strong solutions, the differential equation (9.19) is understood either in terms of equivalence classes (the right and left sides of the equation represent the same equivalence class of sequences in the $L^2(\Omega)$ norm) or in an “almost everywhere” sense (for those who have studied measure theory.)

3. Instead of imposing boundary conditions (9.18) explicitly as we did in the classical problem, we have incorporated them into the space $H^k_0(\Omega)$ in the new problem.

4. By combining the previous observations we see that the new problem is indeed a generalization of the classical problem; i.e., any classical solution of the Dirichlet problem is also a strong solution.

We now take a further step in weakening the conditions on solutions of the Dirichlet problem: we state the problem in variational form. This is the same process which was used in discussing weak solutions of conservation laws in Chapter 3. The first step is to create a bilinear form from the differential operator $L$ using integration by parts. Let $\phi \in \mathcal{D}(\Omega)$ and $u \in H^{2k}(\Omega)$, then

$$
\int_{\Omega} \phi Lu \, dx = \sum_{0 \leq |\sigma|, |\gamma| \leq k} (-1)^{|\sigma|} \int_{\Omega} \phi D^\sigma (a_{\sigma \gamma}(x) D^\gamma u) \, dx
$$

$$
= \sum_{0 \leq |\sigma|, |\gamma| \leq k} \int_{\Omega} a_{\sigma \gamma}(x) D^\gamma u D^\sigma \phi \, dx.
$$

With this in mind we define

$$
B[v, u] := \sum_{0 \leq |\sigma|, |\gamma| \leq k} \int_{\Omega} a_{\sigma \gamma}(x) D^\gamma u D^\sigma v \, dx
$$

(9.21)

to be the bilinear form associated with the elliptic partial differential operator $L$. Note that $B[v, u]$ is well defined for $u$ and $v$ that are merely in $H^k(\Omega)$. With this in mind, we give the following definition of yet another type of solution of the Dirichlet problem.

**Definition 9.11.** Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and suppose $f \in H^{-k}(\Omega)$ is given. A function

$$
u \in H^k_0(\Omega)
$$


is a **weak solution** of the Dirichlet problem if

\[ B[v, u] = f(v) \]  

(9.22)

for every \( v \in H^k_0(\Omega) \).

**Remark 9.12.** We can extend the bilinear form \( B \) on the real Hilbert space \( H^k_0(\Omega) \) to be a **sesquilinear form** on the complex Hilbert space (also denoted \( H^k_0(\Omega) \)) by letting

\[ B[v, u] := \sum_{0 \leq |\sigma|, |\gamma| \leq k} \int_\Omega a_{\sigma\gamma}(x) D^{\gamma}u D^{\sigma}v \, dx. \]  

(9.23)

We will be rather sloppy about the distinction between complex-valued and real-valued functions, using the same notation for the spaces and the bilinear forms. Since we are mainly interested in discussing real solutions to partial differential equations, we will only go to complex spaces when forced to, such as when using Fourier transforms or discussing spectral theory.

Note that by using the calculations of (9.20) (though this time with a function \( v \in H^k_0(\Omega) \) in place of \( \phi \in D(\Omega) \)) we can see that any strong solution of the Dirichlet problem is automatically a weak solution. However, since we require so much less smoothness of weak solutions than strong ones, it will be far easier to show that weak solutions exist. Once we have done this, we will be able to show that if \( \Omega, f \) and the coefficients \( a_{\sigma\gamma} \) are sufficiently “nice,” the weak solution is, in fact, a strong solution or a classical solution.

**Example 9.13.** An important series of classical examples of Dirichlet problems come from electrostatics. Without going into any of the physics, let us assume that we wish to know a scalar quantity \( u \) called the **electrostatic potential** or more commonly the **voltage** in a domain \( \Omega \subset \mathbb{R}^3 \). We will assume that the boundary of the domain is grounded; i.e.,

\[ u(x) = 0 \quad \text{for} \ x \in \partial \Omega. \]  

(9.24)

Within \( \Omega \) there is a distribution of charge \( \rho : \Omega \to \mathbb{R} \), and this charge “generates” a voltage through the formula

\[ -\Delta u = \rho. \]  

(9.25)

The solution of this class of problems is the subject of classical potential theory, and many of the techniques of the classical theory (eigenfunction expansion, Green’s functions) are included in the modern theory as well. However, the classical and modern theory share the same approach only when we are looking for classical (\( \rho \in C_b(\Omega) \)) or strong (\( \rho \in L^2(\Omega) \) or piecewise continuous) solutions.

The classical and modern theories take a very different approach in dealing with charge distributions that occur on surfaces; i.e., situations where \( S \subset \Omega \) is a smooth surface and a charge distribution \( \omega : S \to \mathbb{R} \) is defined.
(We assume that either $\omega \in C_b(S)$ or $\omega \in L^2(S)$.) Of course, since $\omega$ is defined only on a surface (a set of measure zero to readers who have had measure theory) the differential equation $-\Delta u = \omega$ does not make sense either classically or as the identification of equivalence classes of sequences in $L^2(\Omega)$. In the modern theory the situation is very clear: although $\omega$ is not in $L^2(\Omega)$, we can use it to define a perfectly nice functional in $H^{-1}(\Omega)$ through the formula

$$
(\omega, \phi) = \int_S \omega(x) \phi(x) \, da(x),
$$

for every $\phi \in H^1_0(\Omega)$. (Recall that the trace theorem implies that $\phi \in L^2(S)$.) Here $da(x)$ indicates differential area at $x \in S$. Thus, the modern theory would simply have us look for a (weak) solution $u \in H^1_0(\Omega)$ of the variational problem

$$
B[v, u] := \int_{\Omega} \nabla v \cdot \nabla u \, dx = (\omega, v)
$$

for all $v \in H^1_0(\Omega)$. As we shall see below, this problem is well-posed. The classical theory solves this problem (and some other similar ones) using the theory of single and double layer potentials: essentially integral operators defined using singular surface integrals. As is so often the case, the classical theory lacks much of the conceptual unity of the modern theory, but provides much more detailed information in special (though often the most important) cases. We will not go into the results of classical potential theory in this book, but the reader is encouraged to read *Foundations of Potential Theory* by O.D. Kellogg [Ke] as a good starting point for more information on this subject.

### 9.2.2 The Lax-Milgram Lemma

The first tool that we will develop for deriving the existence theory for elliptic equations is commonly known as the Lax-Milgram lemma; though because of its importance we designate it as a theorem. The result is simply a generalization of the Riesz Representation Theorem to bilinear forms that need not be symmetric.

**Theorem 9.14 (Lax-Milgram).** Let $H$ be a Hilbert space and let

$$
B : H \times H \to \mathbb{R}
$$

be a bilinear mapping. Suppose there exist positive constants $c_1$ and $c_2$ such that

$$
|B[x, y]| \leq c_1 \|x\|_H \|y\|_H \quad \text{for all } x \text{ and } y \in H
$$

and

$$
B[x, x] \geq c_2 \|x\|_H^2 \quad \text{for all } x \in H.
$$
Then for every $f \in H^*$ there exists a unique $y \in H$ such that
\[ B[x, y] = f(x) \quad \text{for all } x \in H. \quad (9.31) \]
Furthermore, there exists a constant $C$, independent of $f$, such that
\[ \|y\|_H \leq C\|f\|_{H^*}. \quad (9.32) \]

Remark 9.15. A mapping $B$ satisfying (9.30) for some $c_2 > 0$ is called coercive. The inequality (9.30) can be thought of as an energy estimate. (The inequality says that the energy (the norm squared) can only blow up as fast as the bilinear form).

Remark 9.16. Note that by (9.29) and (9.30)
\[ \|x\|_B := \sqrt{B[x, x]} \quad (9.33) \]
is equivalent to the original norm on $H$. Furthermore, if $B$ is symmetric, i.e.,
\[ B[x, y] = B[y, x] \quad \text{for all } x \text{ and } y \text{ in } H, \quad (9.34) \]
then $B[x, y]$ defines a new inner product on $H$. Thus, in this case, the Riesz Representation Theorem directly implies that for every $f \in H^*$ there exists a unique $y \in H$ such that (9.31) holds. Therefore, the significance of the Lax-Milgram lemma is that it does not require $B$ to be symmetric.

We now prove the Lax-Milgram lemma.

Proof. For every fixed $y \in H$, the mapping
\[ H \ni x \mapsto B[x, y] \in \mathbb{R} \quad (9.35) \]
is bounded and linear, i.e., an element of $H^*$. Thus, by the Riesz Representation Theorem there exists a unique $z \in H$ such that
\[ B[x, y] = (x, z) \quad \text{for all } x \in H. \quad (9.36) \]
Since a unique $z \in H$ can be derived for each fixed $y \in H$ we can define a mapping $A : H \to H$ by
\[ z := A(y). \quad (9.37) \]
Thus the question of existence of a solution of (9.31) is now translated to the question of the invertibility of $A$. That is, for any $f \in H^*$ let $z \in H$ be the unique element such that
\[ (x, z) = f(x) \quad \text{for all } x \in H. \quad (9.38) \]
Then if for every $z \in H$ can we find a unique solution of $y \in H$ of $A(y) = z$, then $y$ is the unique solution of
\[ B[x, y] = (x, A(y)) = (x, z) = f(x) \quad \text{for all } x \in H. \quad (9.39) \]
We now note some basic properties of $A$. 

1. $A$ is linear. To see this note that
\[
(x, A(\alpha_1 y_1 + \alpha_2 y_2)) := B[x, \alpha_1 y_1 + \alpha_2 y_2] \\
= \alpha_1 B[x, y_1] + \alpha_2 B[x, y_2] \\
= \alpha_1 (x, A(y_1)) + \alpha_2 (x, A(y_2)) \\
= (x, \alpha_1 A(y_1) + \alpha_2 A(y_2)).
\]
Since this holds for arbitrary $x$, $\alpha_i$ and $y_i$ we have shown linearity.

2. $A$ is bounded. Using (9.29) we get
\[
\|A(y)\|^2 = (A(y), A(y)) = B[A(y), y] \leq c_1 \|A(y)\| \|y\|. \quad (9.40)
\]
Canceling, we get
\[
\|A(y)\| \leq c_1 \|y\|. \quad (9.41)
\]

3. The range of $A$ is dense in $H$. To see this we use (9.30) to note that if $y \in \mathcal{R}(A)^\perp$, then
\[
c_2 \|y\|^2 \leq B[y, y] = (y, A(y)) = 0. \quad (9.42)
\]

4. $A$ is bounded below. Using (9.30) again (now for arbitrary $y \in H$) we get
\[
c_2 \|y\|^2 \leq B[y, y] = (y, A(y)) \leq \|y\| \|A(y)\| \quad (9.43)
\]
or
\[
\|A(y)\| \geq c_2 \|y\|. \quad (9.44)
\]
5. Combining this with Problem 8.2 implies that the range of $A$, $\mathcal{R}(A)$, is closed. Since $\mathcal{R}(A)$ is dense, $A$ is surjective.

It follows that $A$ is invertible, which gives us the existence of a unique solution $y$. Finally, the estimate (9.32) follows from the Riesz representation theorem and the fact that $A$ is bounded below. \qed

9.2.3 Gårding’s Inequality

We now prove the basic energy or coercivity estimate for the elliptic Dirichlet problem.

**Theorem 9.17 (Gårding’s inequality).** Let $\Omega$ be a bounded domain with the $k$-extension property. Let $L(x, D)$ be a linear partial differential operator in divergence form of order $2k$ such that for some $\theta > 0$ the uniform ellipticity condition (9.9) holds. Also suppose that
\[
a_{\sigma \gamma} \in C_b(\overline{\Omega}) \quad \text{for all } |\sigma| = |\gamma| = k \quad (9.45)
\]
and
\[
a_{\sigma \gamma} \in L^\infty(\Omega) \quad \text{for all } |\sigma|, |\gamma| \leq k. \quad (9.46)
\]
Then there exist constants $c_3$ and $\lambda_G \geq 0$ such that
\[
B[u, u] + \lambda \|u\|_{L^2(\Omega)}^2 \geq c_3 \|u\|_{H^k(\Omega)}^2 \quad \text{for all } u \in H^k_0(\Omega).
\] (9.47)

**Proof.** Let $u \in H^k_0(\Omega)$. We begin by splitting $B[u, u]$ into principal part and lower-order terms; i.e., we let
\[
B[u, u] = I_1 + I_2,
\] (9.48)
where
\[
I_1 := \sum_{|\sigma| = |\gamma| = k} \int_{\Omega} a_{\sigma \gamma}(x) D^\gamma u D^\sigma u \, dx,
\] (9.49)
\[
I_2 := \sum_{0 \leq |\sigma|, |\gamma| \leq k} \sum_{|\sigma| + |\gamma| < 2k} \int_{\Omega} a_{\sigma \gamma}(x) D^\gamma u D^\sigma u \, dx.
\] (9.50)

We begin by estimating the lower-order terms. Let $\epsilon > 0$ be given.

\[
|I_2| \leq C \sum_{0 \leq |\sigma|, |\gamma| \leq k} \sum_{|\sigma| + |\gamma| < 2k} \int_{\Omega} |D^\gamma u||D^\sigma u| \, dx
\]
\[
\leq C \sum_{0 \leq |\sigma|, |\gamma| \leq k} \sum_{|\sigma| + |\gamma| < 2k} \|D^\gamma u\|_2 \|D^\sigma u\|_2
\]
\[
\leq C \|u\|_{k,2} \|u\|_{k-1,2}
\]
\[
\leq \frac{\epsilon}{2} \|u\|_{k,2}^2 + C(\epsilon) \|u\|_{k-1,2}^2.
\] (9.51)

Here we have used Hölder’s inequality and the elementary inequality
\[
ab \leq \varepsilon a^2 + \frac{1}{4\varepsilon} b^2.
\] (9.52)

Now let $\delta > 0$. We use the abstract version of Ehrling’s lemma (Theorem 7.30) and the previous estimate to get
\[
|I_2| \leq \frac{\epsilon}{2} \|u\|_{k,2}^2 + C(\epsilon) \left[\delta \|u\|_{k,2}^2 + c(\delta) \|u\|_2^2\right].
\] (9.53)

Now, for any $\epsilon > 0$, we can choose $\delta = \delta(\epsilon) > 0$ sufficiently small that
\[
C(\epsilon) \delta \leq \epsilon/2.
\] (9.54)

Combining this with the previous inequality gives us the estimate:
\[
|I_2| \leq \epsilon \|u\|_{k,2}^2 + C(\epsilon) \|u\|_2^2.
\] (9.55)

We now estimate the principal part. We assert the fact that each function $a_{\sigma \gamma}$ can be extended to be a continuous function on all of $\mathbb{R}^n$. (We already
know this to be true for Lipschitz domains since they have the $k$-extension property for any $k$. In fact, by the Tietze extension theorem (consult a topology text), it holds for any domain $\Omega$.

Now let $\Omega'$ be any bounded open domain such that $\Omega$ is compactly contained in $\Omega'$. Since each extended $a_{\sigma \gamma}$ ($|\sigma| = |\gamma| = k$) is uniformly continuous on $\Omega'$, there exists a nondecreasing modulus of continuity function $\omega : [0, \infty) \to [0, \infty)$ satisfying

$$0 = \omega(0) = \lim_{\delta \to 0^+} \omega(\delta)$$

and

$$|a_{\sigma \gamma}(x) - a_{\sigma \gamma}(y)| \leq \omega(|x - y|)$$

for every $|\sigma| = |\gamma| = k$ and every $x, y \in \Omega'$.

Now let $B = B(x_0, \delta)$ for some $x_0 \in \Omega$. We will choose $\delta > 0$ later, but for now we assume only that it is sufficiently small so that $B \subset \Omega'$. The first step in our estimate of $I_1$ is to do an estimate in the case where $u \in H^k_0(B)$. In this case we have

$$I_1 = I_{11} + I_{12},$$

where

$$I_{11} := \sum_{|\sigma| = |\gamma| = k} \int_{\mathbb{R}^n} a_{\sigma \gamma}(x_0) D^\gamma u D^\sigma u \, dx,$$

$$I_{12} := \sum_{|\sigma| = |\gamma| = k} \int_B [a_{\sigma \gamma}(x) - a_{\sigma \gamma}(x_0)] D^\gamma u D^\sigma u \, dx.$$
To estimate $I_{11}$ we use Fourier transforms

$$I_{11} = \sum_{|\sigma|=|\gamma|=k} a_{\sigma\gamma}(x_0) \int_{\mathbb{R}^n} D^\gamma u(x) \overline{D^\sigma u(x)} \, dx$$

$$= \sum_{|\sigma|=|\gamma|=k} a_{\sigma\gamma}(x_0) \int_{\mathbb{R}^n} \overline{D^\gamma u(\xi)} \overline{D^\sigma u(\xi)} \, d\xi$$

$$= \sum_{|\sigma|=|\gamma|=k} \int_{\mathbb{R}^n} a_{\sigma\gamma}(x_0)(i\xi)^\gamma (-i\xi)^\sigma |\hat{u}|^2 \, d\xi$$

$$\geq \theta \int_{\mathbb{R}^n} |\xi|^{2k} |\hat{u}|^2 \, d\xi.$$

In the last inequality we have used the uniform ellipticity condition. To continue, we use Theorem 7.12 to get

$$I_{11} \geq \theta \int_{\mathbb{R}^n} (1 + |\xi|^{2k}) |\hat{u}|^2 \, d\xi - \theta \int_{\mathbb{R}^n} |\hat{u}|^2 \, d\xi$$

for some $C > 0$ which depends only on $\Omega'$.

We now combine the estimates of $I_{11}$ and $I_{12}$ to get an estimate for $I_1$. At this time we assume that $\delta$ is sufficiently small so that $\omega(\delta) \leq \bar{C}/2$. Then we have

$$I_1 \geq I_{11} - |I_{12}|$$

$$\geq \bar{C} \|u\|_{k,2}^2 - \theta \|u\|_{l,2}^2 - \omega(\delta) \|u\|_{k,2}^2$$

$$\geq \bar{C} \frac{1}{2} \|u\|_{k,2}^2 - \theta \|u\|_{l,2}^2. \quad (9.62)$$

We now continue with our estimate of $I_1$ in the case of a general $u \in H^k_0(\Omega)$. The basic idea is to break up $u$ using a partition of unity, so that we can use the previous estimate.

We begin by covering $\overline{\Omega}$ with a finite collection of balls

$$B_i := B(x_i, \delta_i), \quad i = 1, \ldots, M, \quad (9.63)$$

with $x_i \in \Omega$ and $\delta_i > 0$, selected as in the previous estimate so that $B_i \subset \Omega'$. Now let $\psi_i$ be a partition of unity on $\overline{\Omega}$ subordinate to the covering $B_i$. We then set

$$\phi_i(x) := \left( \frac{\psi_i^2(x)}{\sum_{j=1}^M \psi_j^2(x)} \right)^{1/2}. \quad (9.64)$$

We then have
1. $0 \leq \phi_i(x) \leq 1$,
2. $\phi_i \in C^\infty(B_i \cap \overline{\Omega})$,
3. $\sum_{i=1}^{M} \phi_i^2(x) = 1$ for each $x \in \Omega$, and
4. $u_i := u \phi_i \in H^k_0(B_i)$.

This can be used to write
\[
I_1 = \sum_{|\sigma|=|\gamma|=k} \int_{\Omega} a_{\sigma \gamma}(x) D^\sigma u D^\gamma u \, dx
= \sum_{i=1}^{M} \sum_{|\sigma|=|\gamma|=k} \int_{\Omega} a_{\sigma \gamma}(x) \phi_i D^\sigma u \phi_i D^\gamma u \, dx
= \sum_{i=1}^{M} \sum_{|\sigma|=|\gamma|=k} \int_{\Omega} a_{\sigma \gamma}(x) (\phi_i D^\sigma u) (\phi_i D^\gamma u) \, dx
+ \sum_{i=1}^{M} \sum_{|\sigma|=|\gamma|=k} \int_{\Omega} a_{\sigma \gamma}(x) [\phi_i D^\sigma u - D^\sigma (\phi_i u)] (\phi_i D^\gamma u) \, dx
+ \sum_{i=1}^{M} \sum_{|\sigma|=|\gamma|=k} \int_{\Omega} a_{\sigma \gamma}(x) [\phi_i D^\gamma u - D^\gamma (\phi_i u)] (\phi_i D^\sigma u) \, dx
\geq \sum_{i=1}^{M} \sum_{|\sigma|=|\gamma|=k} \int_{\Omega} a_{\sigma \gamma}(x) D^\sigma u_i D^\gamma u_i \, dx
-C\|u\|_{k,2}^2 \|u\|_{k-1,2}
\]

We can now use the previous estimate for each $u_i \in H^k_0(B_i)$ to get
\[
I_1 \geq \frac{\bar{C}}{2} \sum_{i=1}^{M} \|u_i\|_{k,2}^2 - C\|u\|_{2}^2
- C\|u\|_{k,2} \|u\|_{k-1,2}
= \frac{\bar{C}}{2} \sum_{i=1}^{M} \sum_{|\alpha| \leq k} \int_{\Omega} |D^\alpha (\phi_i u)|^2 \, dx
- C[\|u\|_{2}^2 + \|u\|_{k,2} \|u\|_{k-1,2}]
\geq \frac{\bar{C}}{2} \sum_{|\alpha| \leq k} \sum_{i=1}^{M} (\phi_i^2) \|D^\alpha u\|_{2}^2 \, dx
- C[\|u\|_{2}^2 + \|u\|_{k,2} \|u\|_{k-1,2}]
= \frac{\bar{C}}{2} \|u\|_{k,2}^2 - C[\|u\|_{2}^2 + \|u\|_{k,2} \|u\|_{k-1,2}]
Thus, using (9.52) and (7.14) we get
\[ I_1 \geq \frac{C}{4} \|u\|_{k,2}^2 - C \|u\|_2^2. \quad (9.65) \]

Finally we combine this estimate with (9.55) to get
\[
B[u, u] = I_1 + I_2 \\
\geq \left( \frac{C}{4} - \epsilon \right) \|u\|_{k,2}^2 - C(\epsilon) \|u\|_2^2 \\
:= c_3 \|u\|_{k,2}^2 - \lambda_G \|u\|_2^2,
\]
where in defining \( c_3 \) and \( \lambda_G \) we have taken \( \epsilon \) to be sufficiently small, say \( \epsilon = C/8 \).

Gårding’s inequality is much easier to prove for second-order equations; i.e., in the case where \( L(x, D) \) is a second-order differential operator of the form
\[
L(x, D)u := \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} a_{ij}(x) \frac{\partial u}{\partial x_j} + \sum_{i=1}^{n} b_i(x) \frac{\partial u}{\partial x_i} + c(x)u, \quad (9.66)
\]
with corresponding bilinear form
\[
B[v, u] := -\sum_{i,j=1}^{n} \int_{\Omega} a_{ij}(x) u_{x_j} v_{x_i} dx + \sum_{i=1}^{n} \int_{\Omega} b_i(x) u_{x_i} v dx + \int_{\Omega} c(x) uv dx. \quad (9.67)
\]
In this case we do not need to use either Fourier transforms or the partition of unity technique, and the proof can be carried out under weaker hypotheses on the higher-order coefficients.

**Theorem 9.18.** Let \( \Omega \) be a bounded domain. Let \( L(x, D) \) be a second-order linear partial differential operator in divergence form of the form described in (9.66) such that for some \( \theta > 0 \) the uniform ellipticity condition (9.6) holds. Also suppose that \( a_{ij}, b_k \in L^\infty(\Omega) \) for \( i, j = 1, \ldots, n \), \( k = 0, \ldots, n \). Then there exist constants \( c_3 \) and \( \lambda_G \geq 0 \) such that
\[
B[u, u] + \lambda_G \|u\|_{L^2(\Omega)}^2 \geq c_3 \|u\|_{H^1(\Omega)}^2 \quad \text{for all } u \in H^1_0(\Omega), \quad (9.68)
\]
where \( B \) is as defined in (9.67).
Proof. We start by using the uniform ellipticity condition and Hölder’s inequality to get

\[
B[u, u] := - \sum_{i,j=1}^{n} \int_{\Omega} a_{ij}(x) u_{x_i} u_{x_j} \, dx + \sum_{i=1}^{n} \int_{\Omega} b_{i}(x) u_{x_i} u \, dx + \int_{\Omega} c(x) u^2 \, dx \\
\geq \theta \int_{\Omega} |\nabla u|^2 \, dx - \max \|b_i\|_{L^\infty(\Omega)} \int_{\Omega} |\nabla u| \, dx \\
- \|c\|_{L^\infty(\Omega)} \int_{\Omega} |u|^2 \, dx.
\]

We now use (9.52) and Poincaré’s inequality (7.17) to get

\[
B[u, u] \geq \frac{\theta}{2} \int_{\Omega} |\nabla u|^2 \, dx - C\|u\|_2^2 \\
\geq C_1\|u\|_{1,2}^2 - \lambda_G \|u\|_2^2.
\]

This completes the proof. □

9.2.4 Existence of Weak Solutions

We are now in a position to prove our basic existence result for weak solutions.

Theorem 9.19. Let \( L(x, D) \) be a linear partial differential operator in divergence form of order \( 2k \), satisfying the hypotheses of Theorem 9.17 (Gårding’s inequality). Then there exists \( \lambda_G \geq 0 \) such that for any \( \tilde{\lambda} \geq \lambda_G \), and any \( f \in H^{-k}(\Omega) \), the Dirichlet problem for the operator

\[
\tilde{L}(x, D) := L(x, D) + \tilde{\lambda}
\]

has a unique weak solution \( u \in H^k_0(\Omega) \). Furthermore, this solution satisfies

\[
\|u\|_{k,2} \leq C\|f\|_{-k,2}.
\]

Proof. Theorem 9.17 guarantees the existence of \( \lambda_G \geq 0 \) such that (9.47) holds. Let \( \tilde{\lambda} \geq \lambda_G \). Note that

\[
\tilde{B}[u, v] := B[u, v] + \tilde{\lambda}(u, v)_{L^2(\Omega)}
\]

is the bilinear form associated with the operator \( \tilde{L} \) defined in (9.69). We now show that \( \tilde{B} \) satisfies the hypotheses of the Lax-Milgram lemma.
Let $H = H^k_0(\Omega)$, and let $u, v \in H$. Then

$$|\tilde{B}[v, u]| \leq |B[v, u]| + |\tilde{\lambda}|(u, v)$$

$$\leq \sum_{0 \leq |\sigma|, |\gamma| \leq k} \int_{\Omega} |a_{\sigma \gamma}(x)| |D^\gamma u| |D^\sigma v| \, dx + |\tilde{\lambda}|(u, v)$$

$$\leq \max_{|\sigma|, |\gamma| \leq k} \|a_{\sigma \gamma}\|_{L^\infty(\Omega)} \sum_{0 \leq |\sigma|, |\gamma| \leq k} \int_{\Omega} |D^\gamma u| |D^\sigma v| \, dx + |\tilde{\lambda}|(u, v)$$

$$\leq C \|v\|_H \|u\|_H.$$

Thus, $\tilde{B}$ satisfies (9.29).

Now by Gårding’s inequality (9.47) we have

$$\tilde{B}[u, u] = \tilde{\lambda} \|u\|_2^2 + B[u, u] \geq c_3 \|u\|_H^2.$$  \hspace{1cm} (9.72)

Thus, $\tilde{B}$ satisfies (9.30).

Thus, Lax-Milgram guarantees that for every $f \in H^{-k} = H^*$ there is a unique weak solution $u \in H$ of the Dirichlet problem, and that the solution satisfies the estimate (9.70).

Problems

9.1. Let $D$ be the unit disk in the plane and let $\Omega = D\setminus\{0\}$. It is well-known that the Dirichlet problem $\Delta u = 1$ with $u = 0$ on $\partial\Omega$ has no classical solution. What is the weak “solution” given by Theorem 9.19? Hint: First characterize $H^1_0(\Omega)$.

9.2. Consider the ODE boundary-value problem $y'' + p(x)y' + q(x)y = f(x)$, $y(0) = y(1) = 0$. Here $p \in C^1[0, 1]$, $q \in C[0, 1]$. Prove that a unique solution exists if $p' - 2q \geq 0$.

9.3. Let the double sequence $a_{ij}$ be such that $\sum_{i,j=1}^{\infty} |a_{ij}|^2 < \infty$. Assume, moreover, that the matrix $a_{ij}$, $i, j = 1, \ldots, N$, is positive definite for every $N$. Prove that the equation

$$u_n + \sum_{j=1}^{\infty} a_{nj} u_j = f_n$$  \hspace{1cm} (9.73)

has a unique solution $u \in \ell^2$ for every $f \in \ell^2$.

9.4. Consider a “weak” solution of the Dirichlet problem for the differential operator defined in (9.66) in a situation where the coefficients $a_{ij}$, $b_i$ and $c$ have discontinuities across a smooth surface. Assume you know that the solution is smooth on both sides of this interface. Determine the “matching conditions” which are satisfied across the interface.
9.3 Eigenfunction Expansions

Under suitable hypotheses on the elliptic operator $L$, Theorem 9.19 guarantees that there exists $\lambda_G$ such that if $\tilde{\lambda} > \lambda_G$, then for any $f \in H^{-k}(\Omega)$ there exists a unique (weak) solution $u \in H^k_0(\Omega)$ of the Dirichlet problem for

$$\tilde{L}(x, D)u := L(x, D)u + \tilde{\lambda}u = f.$$ 

In this section we will apply some of the operator techniques developed in the previous chapter to this problem. This investigation will give us two basic improvements over the present existence theory. First, the Fredholm theorems will give us information on the existence and uniqueness of solutions for values of $\tilde{\lambda} < \lambda_G$. Second, if the operator $L$ satisfies a symmetry condition, we can use the method of eigenfunction expansion to construct (or in real life approximate) solutions.

9.3.1 Fredholm Theory

In this section we consider the nonhomogeneous eigenvalue problem

$$L(x, D)u + \lambda u = f$$

for $f \in L^2(\Omega)$, where $L(x, D)$ is the operator

$$L(x, D)u = \sum_{0 \leq |\sigma|,|\gamma| \leq k} (-1)^{|\sigma|} D^\sigma (a_{\sigma\gamma}(x) D^\gamma u),$$

and the bilinear form associated with $L$ is

$$B[v, u] = \sum_{0 \leq |\sigma|,|\gamma| \leq k} \int_\Omega a_{\sigma\gamma}(x) D^\gamma u D^\sigma v \, dx.$$ 

Let us assume the hypotheses of Theorem 9.19 are satisfied and fix $\tilde{\lambda} > \lambda_G$ with $\tilde{\lambda} > 0$. Then for any $f \in L^2(\Omega)$ there is a unique solution $u \in H^k_0(\Omega)$ to the problem

$$B_{\tilde{\lambda}}[v, u] := B[v, u] + \tilde{\lambda}(v, u) = (v, f)_{L^2(\Omega)} \quad \text{for every } v \in H^k_0(\Omega). \quad (9.75)$$

We now define an operator $\overline{G} : L^2(\Omega) \to H^k_0(\Omega)$ as follows: for every $f \in L^2(\Omega)$ we define

$$\overline{G}(f) := \tilde{\lambda}u,$$ 

where $u$ is the unique (weak) solution of the Dirichlet problem for

$$L(x, D)u + \tilde{\lambda}u = f; \quad (9.77)$$

i.e., $u$ solves (9.75). In other words, for every $f \in L^2(\Omega)$ and $v \in H^k_0(\Omega)$ we have

$$B_{\tilde{\lambda}}[v, \overline{G}(f)] = \tilde{\lambda}(v, f)_{L^2(\Omega)}. \quad (9.78)$$
Formally, we have
\[ G = \tilde{\lambda}(L + \tilde{\lambda})^{-1}. \]  
(9.79)

By (9.70), the operator \( \overline{G} \) is bounded. We now define the operator \( G : L^2(\Omega) \to L^2(\Omega) \) by the composition of \( \overline{G} \) and \( \overline{I} \),
\[ G := \overline{I}G, \]  
(9.80)
where \( \overline{I} \) is the identity mapping from \( H^k(\Omega) \) to \( L^2(\Omega) \). We know from Theorem 7.29 that this operator is compact. Since the composition of a bounded operator and a compact operator is compact (cf. Problem 8.39) we have the following.

**Lemma 9.20.** The solution operator \( G : L^2(\Omega) \to L^2(\Omega) \) is compact.

We now apply the Fredholm alternative theorem (Theorem 8.93) to the operator \( G \) to get the following.

**Theorem 9.21.** Let \( L(x, D) \) be a uniformly elliptic differential operator of order \( 2k \) satisfying the hypotheses of Theorem 9.19. Then for every \( \mu \in \mathbb{C} \) the Fredholm alternative holds; i.e., either
1. for every \( f \in L^2(\Omega) \) there exists a unique weak solution \( u \in H^k_0(\Omega) \) of the Dirichlet problem for the equation
\[ L(x, D)u - \mu u = f, \]  
(9.81)
i.e.,
\[ B[v, u] - \mu(v, u)_{L^2(\Omega)} = (v, f)_{L^2(\Omega)} \]  
(9.82)
for all \( v \in H^k_0(\Omega) \), or
2. there exists at most a finite linearly independent collection of functions \( u_i \in H^k_0(\Omega) \), \( i = 1, \ldots, N \), such that
\[ B[v, u_i] - \mu(v, u_i) = 0, \]  
(9.83)
for all \( v \in H^k_0(\Omega) \).

Furthermore, the set of values at which the second alternative holds forms an infinite discrete set with no finite accumulation point.

**Proof.** We first write the equation
\[ Lu = \mu u \]  
(9.84)
as
\[ (L + \tilde{\lambda})u = (\tilde{\lambda} + \mu)u. \]  
(9.85)
Then by a formal calculation in which we act on both sides of (9.85) with \( G/\tilde{\lambda} = (L + \tilde{\lambda})^{-1} \) we see that (9.84) has a nontrivial solution \( u \) if and only
if \( u \) solves

\[
    u = (L + \bar{\lambda})^{-1}(L + \bar{\lambda})u = \frac{\bar{\lambda} + \mu}{\bar{\lambda}} Gu = \frac{1}{\sigma} Gu.
\]  

(9.86)

Thus, we see that \( u \in L^2(\Omega) \) is an eigenfunction of \( G \) corresponding to the eigenvalue \( \sigma \) if and only if \( u \) is an eigenfunction of \( L \) corresponding to the eigenvalue \( \mu \) where

\[
    \sigma = \frac{\bar{\lambda}}{\bar{\lambda} + \mu},
\]  

(9.87)

\[
    \mu = -\bar{\lambda} + \frac{\bar{\lambda}}{\sigma}.
\]  

(9.88)

By the Fredholm alternative theorem, the nonzero eigenvalues of \( G \) are of finite multiplicity and thus the eigenvalues of \( L \) are as well. Also, the eigenvalues of \( G \) form a discrete set whose only possible accumulation point is zero, and since we have arranged it so that 0 is not an eigenvalue of \( G \), \( G \) must have an infinite collection of eigenvalues. Thus, there must be an infinite collection of eigenvalues of \( L \) with no finite accumulation point.

When \( \mu \neq -\bar{\lambda} \) is not an eigenvalue of \( L \), we note that \( u \in H^k_0(\Omega) \) is a solution of (9.81) if and only if \( u \) is a solution of

\[
    G(u) - \frac{\bar{\lambda}}{\mu + \lambda} u = -\frac{1}{\mu + \lambda} G(f).
\]  

(9.89)

We leave it to the reader to supply the rigor necessary to shore up this formal argument. The only delicate points involve showing that functions \( u \) that are solutions of equations involving \( G \) (and are thus naturally thought of as being only in \( L^2(\Omega) \)) must actually be functions in \( H^k_0(\Omega) \) imbedded into \( L^2(\Omega) \) (and can thus work as weak solutions of equations involving \( L \)).

\( 9.3.2 \) Eigenfunction Expansions

When the coefficients of \( L(x, D) \) satisfy the symmetry condition

\[
    a_{\sigma \gamma} = a_{\gamma \sigma},
\]  

(9.90)

then it is easy to show that \( L \) is symmetric. Moreover, by direct calculation we see that for every \( u, v \in H^k_0(\Omega) \) we have

\[
    B[u, v] = B[v, u].
\]  

(9.91)
For any $f, g \in L^2(\Omega)$ this gives us
\[
(G(f), g)_{L^2(\Omega)} = (\bar{G}(f), g)_{L^2(\Omega)} = \frac{1}{\lambda} B_\lambda [\bar{G}(f), G(g)] = \frac{1}{\lambda} B_\lambda [\bar{G}(g), G(f)] = (\bar{G}(g), f)_{L^2(\Omega)} = (f, G(g))_{L^2(\Omega)}.
\]

So $G$ is self-adjoint. Thus, we can use the Hilbert-Schmidt theorem to get the following.

Theorem 9.22. If $L$ is symmetric, then there is a sequence of real eigenvalues
\[
\bar{\lambda} \leq \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \leq \cdots
\] (9.92)
with no finite accumulation point and $\lim_{i \to \infty} \lambda_i = \infty$, and an orthonormal set of eigenfunctions $\{\phi_i\}_{i=1}^\infty$ such that
\[
L\phi_i = \lambda_i \phi_i
\] (9.93)
(in the weak sense). Furthermore, if $\mu \neq \lambda_i$, $i = 1, 2, \ldots, \infty$, then for any $f \in L^2(\Omega)$ the unique weak solution of
\[
L(x, D)u - \mu u = f
\] (9.94)
is given by
\[
u = \sum_{i=1}^{\infty} \frac{(\phi_i, f)}{\lambda_i - \mu} \phi_i.
\] (9.95)
If $\mu$ is an eigenvalue; i.e., $\mu = \lambda_j$ for $j$ in some index set $J \subset \mathbb{N}$, then (9.94) is solvable if and only if
\[
(\phi_j, f) = 0, \quad j \in J.
\] (9.96)
If so, there is a family of solutions given by
\[
u = \sum_{j \in J} \alpha_j \phi_j + \sum_{\mathbb{N} \setminus J} \frac{(\phi_i, f)}{\lambda_i - \mu} \phi_i.
\] (9.97)
(Here the series (9.95) and (9.97) converge in $L^2(\Omega)$.)

The proof is left to the reader.

9.4 General Linear Elliptic Problems

So far in this chapter, we have discussed only Dirichlet boundary conditions for elliptic problems. In this section we shall discuss a few of the
other boundary conditions that arise in physical and mathematical problems. Since physical problems present us with a wide variety of boundary conditions for consideration our discussion will not be exhaustive.

9.4.1 The Neumann Problem

After the Dirichlet problem, the second most common and important elliptic boundary-value problem is the Neumann problem.

**Definition 9.23.** Let \( \Omega \subset \mathbb{R}^n \) be a bounded domain with \( C^1 \) boundary and suppose \( f \in C_b(\Omega) \) is given. A function \( u \in C^2_b(\Omega) \cap C^1_b(\Omega) \) is a **classical solution** of the Neumann problem if

\[
L(x, D)u := \sum_{i,j=1}^n a_{ij}(x) \frac{\partial u}{\partial x_j} \bigg|_{x} + \sum_{i=1}^n b_i(x) \frac{\partial u}{\partial x_i} + c(x)u = f \quad (9.98)
\]

in \( \Omega \); and

\[
\sum_{i,j=1}^n a_{ij}(x) \frac{\partial u}{\partial x_j} \eta_i(x) = 0 \quad (9.99)
\]
on \( \partial \Omega \) where \( \eta(x) \) is the unit outward normal to \( \partial \Omega \) at \( x \).

As with the Dirichlet problem, we can define a strong solution of the Neumann problem.

**Definition 9.24.** Let \( \Omega \subset \mathbb{R}^n \) be a bounded domain with a \( C^1 \) boundary and suppose \( f \in L^2(\Omega) \) is given. A function \( u \in H^2(\Omega) \) is a **strong solution** of the Neumann problem if (9.98) holds in \( L^2(\Omega) \) and (9.99) holds in the sense of trace on \( \partial \Omega \).

In order to state the Neumann problem in weak form we proceed as before and use integration by parts to create a bilinear form from the differential operator \( L \) and the boundary conditions. Note that for any \( \phi \) and \( u \) in \( H^2(\Omega) \) we have

\[
\int_{\Omega} \phi Lu \, dx = B[\phi, u] + \sum_{i,j=1}^n \int_{\partial \Omega} a_{ij}(x) \frac{\partial u}{\partial x_j} (x) \eta_i(x) \phi(x) \, dx, \quad (9.100)
\]

where the bilinear form \( B \) is defined in (9.67). Thus, if \( u \) satisfies the boundary condition (9.99), then we have

\[
\int_{\Omega} \phi Lu \, dx = B[\phi, u] \quad (9.101)
\]
for every \( \phi \in H^1(\Omega) \). In fact, we take this as the definition of a weak solution of the Neumann problem.

**Definition 9.25.** Let \( \Omega \subset \mathbb{R}^n \) be a bounded domain and suppose \( f \in L^2(\Omega) \) is given. A function 

\[
u \in H^1(\Omega)
\]

is a **weak solution** of the Neumann problem if 

\[
B[v,u] = (f,v) \quad (9.102)
\]

for every \( v \in H^1(\Omega) \).

A few comments are in order.

1. Of course, we have constructed things so that every strong solution of the Neumann problem is also a weak solution.

2. In the construction of the weak form, the boundary conditions “disappear”; i.e., condition (9.99) does not appear explicitly in either the bilinear form or the space of admissible functions. Because of this, Neumann conditions are referred to as **natural boundary conditions**. In fact, since the trace theorem does not guarantee the existence of normal derivatives of \( H^1(\Omega) \) functions, it does not necessarily make sense to evaluate the boundary condition (9.99) on a weak solution of (9.102).

3. We have assumed that the data \( f \) is in the space \( L^2(\Omega) \). This can be weakened, for instance by taking \( f \in L^2(S) \) where \( S \) is a smooth surface contained in \( \Omega \). However, we cannot take arbitrary data in \( H^{-1}(\Omega) \) as we did for the Dirichlet problem.

As we have indicated above, the key to obtaining an existence theory is proving an energy estimate analogous to Gårding’s inequality.

**Theorem 9.26.** Let \( L(x,D) \) be a second-order linear partial differential operator in divergence form of the form described in (9.66) such that for some \( \theta > 0 \) the uniform ellipticity condition (9.6) holds. Also suppose that \( a_{ij}, b_k \in L^\infty(\Omega) \) for \( i,j = 1, \ldots, n, k = 0, \ldots, n \). Then there exist constants \( \bar{c} \) and \( \lambda_N \geq 0 \) such that 

\[
B[u,u] + \lambda_N \| u \|^2_{L^2(\Omega)} \geq \bar{c} \| u \|^2_{H^1(\Omega)} \quad \text{for all } u \in H^1(\Omega),
\]

(9.103)

where \( B \) is as defined in (9.67).

**Proof.** The statement of this theorem is identical to that of Theorem 9.18 except that now we are trying to prove the theorem over the space \( H^1(\Omega) \) rather than just \( H^1_0(\Omega) \). Thus, the only difference in the proof of this result is that we no longer have Poincaré’s inequality. However, as before we can
get

\[ B[u, u] \geq \frac{\theta}{2} \int_{\Omega} |\nabla u|^2 \, dx - C\|u\|_2^2. \]  
(9.104)

And instead of using Poincaré’s inequality at this point we simply write

\[ B[u, u] \geq \frac{\theta}{2} \|u\|_{1,2}^2 - (C + \frac{\theta}{2})\|u\|_2^2, \]  
(9.105)

which completes the proof.

It is worth noting that the only real reason for using Poincaré in the proof of Theorem 9.18 was to get a sharper estimate on the constant \( \lambda_G \). However, since we haven’t been trying to specify optimal constants anyway, this effort was sort of wasted.

With our energy estimate in place to take care of the coercivity condition in the Lax-Milgram lemma, the existence of a weak solution of the Neumann problems follows with only minor modifications of the proof of Theorem 9.19.

**Theorem 9.27.** Let \( \tilde{L}(x, D) \) be a second-order linear partial differential operator in divergence form satisfying the hypotheses of Theorem 9.26. Then there exists \( \lambda_N \geq 0 \) such that for any \( \tilde{\lambda} \geq \lambda_N \) and for any \( f \in L^2(\Omega) \), the Neumann problem for the operator

\[ \tilde{L}(x, D) := L(x, D) + \tilde{\lambda} \]  
(9.106)

has a unique weak solution \( u \in H^1(\Omega) \). Furthermore, this solution satisfies

\[ \|u\|_{1,2} \leq C\|f\|_2. \]  
(9.107)

**9.4.2 The Complementing Condition for Elliptic Systems**

For ODE boundary-value problems, it is well known that a “reasonable” boundary-value problem is obtained if the number of boundary conditions at each point equals half the order of the differential equation and the boundary conditions at each point are linearly independent. For elliptic PDEs, the picture is more complicated. Consider the problem

\[ \Delta \Delta u = 0 \]  
(9.108)

with boundary conditions

\[ \Delta u = \frac{\partial}{\partial n} \Delta u = 0. \]  
(9.109)

Although the two boundary conditions are independent of each other, we see that every harmonic function satisfies both the differential equation and the boundary conditions. There are infinitely many linearly independent harmonic functions even within the class of polynomials. This illustrates the need to formulate hypotheses which classify those boundary conditions
leading to “good” problems. These hypotheses will not just express independence of the boundary conditions from each other, but also involve a relationship between the boundary conditions and the differential equation.

The complementing condition provides such a characterization of “good” boundary conditions. We shall state it for general elliptic systems. As in Chapter 2, let us consider a $k \times k$ system of equations

$$L_{ij}(x, D)u_j = f_i(x), \ x \in \Omega, \ i = 1, \ldots, k. \quad (9.110)$$

As before, we assign the “weights” $s_i$ to the $i$th equation and $t_j$ to the $j$th independent variable in such a way that $L_{ij}$ is at most of order $s_i + t_j$, and we denote the terms that are exactly of order $s_i + t_j$ by $L^p_{ij}$. The condition of ellipticity is that

$$\det L^p(x, \xi) \neq 0 \ \forall \xi \in \mathbb{R}^n \setminus \{0\}. \quad (9.111)$$

As we remarked in Chapter 2, this condition can be interpreted as follows: Take the values of the coefficients at any fixed point $x_0$ in $\Omega$ and consider the system $L^p(x_0, D)u = 0$ on all of $\mathbb{R}^n$. Ellipticity means that this constant coefficient system has no nonconstant periodic solutions. The complementing condition will be an analogue of this for points at the boundary.

Let the order of the system, i.e., the order of the polynomial $\det L^p(x, \xi)$, be $2m$. We impose $m$ boundary conditions

$$B_{lj}(x, D)u_j = g_l(x), \ x \in \partial \Omega, \ l = 1, \ldots, m. \quad (9.112)$$

We now define weights $r_l$ to each boundary condition $l = 1, \ldots, m$ so that the order of $B_{lj}$ be bounded by $r_l + t_j$. Again the terms which are precisely of order $r_l + t_j$ will be considered the principal part. (If $r_l + t_j$ is negative, it is of course understood that $B_{lj} = 0$.) Let now $x_0$ be a point on $\partial \Omega$ and let $n$ be the outer normal to $\Omega$. We consider the constant coefficient problem

$$L^p_{ij}(x_0, D)u_j = 0, \ i = 1, \ldots, k, \quad (9.113)$$

on the half-space $(x - x_0) \cdot n < 0$, with boundary conditions

$$B^p_{lj}(x_0, D)u_j = 0, \ l = 1, \ldots, m, \quad (9.114)$$

on the boundary $(x - x_0) \cdot n = 0$.

**Definition 9.28.** We say that the complementing conditions holds at $x_0$, if there are no nontrivial solutions of (9.113), (9.114) of the following form:

$$u(x) = \exp(i\xi \cdot (x - x_0))v(\eta), \quad (9.115)$$

where $\xi$ is a nonzero real vector perpendicular to $n$, $\eta = (x - x_0) \cdot n$, and $v(\eta)$ tends to $0$ exponentially as $\eta \to -\infty$.

**Example 9.29.** Consider the Stokes system $\Delta u - \nabla p = 0$, $\text{div} \ u = 0$, where $u \in \mathbb{R}^3$, $p \in \mathbb{R}$, in the half-space $z > 0$, with the Dirichlet boundary condition $u = 0$ on $z = 0$. As we showed in Chapter 2, the system is
elliptic, and of order 6. Assume now that we have a solution $u(x, y, z) = \exp(i\zeta_1 x + i\zeta_2 y)v(z)$, $p(x, y, z) = \exp(i\zeta_1 x + i\zeta_2 y)q(z)$, where $v$ and $q$ tend to zero exponentially as $z \to \infty$. Let $\Sigma = (0, L_1) \times (0, L_2)$, where $L_i$ is a multiple of $2\pi/\zeta_i$ if $\zeta_i \neq 0$ and arbitrary if $\zeta_i = 0$. We find

$$0 = \int_{\Sigma \times \mathbb{R}^+} (\Delta u - \nabla p) \cdot u \, dx \, dy \, dz = -\int_{\Sigma \times \mathbb{R}^+} |\nabla u|^2 \, dx \, dy \, dz. \quad (9.116)$$

This implies that $u$ is constant, which is compatible with our assumptions only if $u = 0$. It easily follows that $p$ is also zero. Hence the Stokes system with Dirichlet boundary conditions satisfies the complementing condition.

**Example 9.30.** Consider the biharmonic equation $\Delta \Delta u = 0$ in the half-plane $y > 0$ with boundary conditions $\Delta u = \frac{\partial}{\partial y} \Delta u = 0$ on the line $y = 0$. For every $\xi \in \mathbb{R}$, the function $u(x, y) = \exp(i\xi x - |\xi|y)$ is a solution. Hence the complementing condition does not hold.

The complementing condition or its failure is not always as easy to verify as in the preceding examples. However, it can always be reduced to a purely algebraic problem. If we insert the ansatz (9.115) into (9.113), we obtain a system of ODEs for $v(\eta)$, which can as usual be solved by the ansatz $v(\eta) = \exp(\lambda \eta)v_0$. Ellipticity means that no roots $\lambda$ are imaginary, and if the coefficients of our system are real, then an equal number of roots must have positive and negative real parts. Let $\lambda^+ (x_0, \xi)$ denote the roots with positive real part. Then one obtains $m$ linearly independent solutions of (9.113) in the form $\exp(i\xi \cdot (x - x_0) + \lambda^+ \eta)u_l$ (in the usual way, this may need to be modified by including powers of $\eta$ if there are repeated roots). It remains to be checked if any linear combination of these solutions satisfies the boundary conditions, which is a purely algebraic problem. For equivalent algebraic characterizations of the complementing conditions we refer to the literature, see [ADN2].

What can we get out of the complementing condition? This question was answered in the work of Agmon, Douglis and Nirenberg [ADN2]. Before we state their results, let us introduce some notation. For $M \in \mathbb{N}$, let

$$X_M = \prod_{j=1}^k H^{M+t_j}(\Omega), \quad Y_M = \prod_{i=1}^k H^{M-s_i}(\Omega), \quad Z_M = \prod_{l=1}^m H^{M-r_l-1/2}(\partial \Omega). \quad (9.117)$$

We now consider the problem (9.110) with boundary condition (9.112). We write the equations in the compact form $Lu = f$ and $Bu = g$ and we denote by $A$ the operator which maps $u$ to $(Lu, Bu_{\partial \Omega})$. In choosing weights, we shall now make the convention that $s_i \leq 0$ and $t_j \geq 0$ for all $i$ and $j$; this can always be achieved by subtracting a constant from the $s_i$ and $r_l$ and adding the same constant to the $t_j$. Let $t' = \max_j t_j$, $M_1 = \max(0, \max_i r_l + 1)$. Then the following result holds.
Theorem 9.31 (Agmon, Douglis, Nirenberg). Let $M \geq M_1$ be an integer. Assume that $\Omega$ is a bounded domain of class $C^{M+\ell'}$, that the coefficients of $L_{ij}$ are of class $C^{M-s_i}(\overline{\Omega})$ and that the coefficients of $B_{lj}$ are of class $C^{M-r_l}(\partial \Omega)$. Moreover, assume that ellipticity holds throughout $\Omega$ and that the complementing condition holds everywhere on $\partial \Omega$. Assume that $f \in Y_M$ and $g \in Z_M$. Then the following hold:

1. Every solution $u \in X_M$ is in fact in $X_M$.

2. There is a universal constant $K$, independent of $u$, $f$ and $g$, such that, for every solution $u \in X_M$, we have

$$
\|u\|_{X_M} \leq K \left( \|f\|_{Y_M} + \|g\|_{Z_M} + \sum_{j=1}^{k} \|u_j\|_{L^2(\Omega)} \right).
$$

(9.118)

If $u$ is a unique solution, then the last term in (9.118) can be omitted.

The result thus consists of a regularity statement and an a priori estimate. Agmon, Douglis and Nirenberg actually prove more than we have stated; they establish similar results in $L^p$-based Sobolev spaces and also in Hölder spaces. We also note that some of the smoothness hypotheses on $\Omega$ and the coefficients can be weakened. We shall not pursue this point here. A proof of the theorem is beyond the scope of this introductory text. However, we refer to Sections 9.5 and 9.6 for a proof of a special case, namely, second-order elliptic PDEs with Dirichlet boundary condition.

We next derive an interesting corollary.

Corollary 9.32. Let all assumptions be as in the preceding theorem. Assume in addition that $M + \ell_j > 0$ for every $j$. Then the operator $A : X_M \to Y_M \times Z_M$ is semi-Fredholm.

Proof. It easily follows from the smoothness hypotheses on the coefficients that $A$ does indeed map $X_M$ to $Y_M \times Z_M$. Let $N(A)$ be the nullspace of $A$, and let $B$ be the intersection of $N(A)$ with the unit ball in $(L^2(\Omega))^k$. By the theorem, $B$ is bounded in the norm of $X_M$, hence precompact in $(L^2(\Omega))^k$. Since the unit ball in an infinite-dimensional space is never precompact, $N(A)$ must be finite-dimensional.

Next, we shall show that the range of $A$ is closed. For that purpose, assume that $u_N$ is a solution of $Lu_N = f_N$ with boundary conditions $Bu_N = g_N$, and that $f_N$ and $g_N$ converge in $Y_M$ and $Z_M$ to $f$ and $g$, respectively. Without loss of generality, we may assume that $u_N$ is perpendicular to $N(A)$ in $(L^2(\Omega))^k$. We claim that $u_N$ is then bounded in $(L^2(\Omega))^k$. Suppose not. After taking a subsequence, we may assume $\|u_N\|_2 \to \infty$. Let $v_N = u_N/\|u_N\|_2$. Then $v_N$ solves the problem $Lv_N = f_N/\|u_N\|_2$ with boundary conditions $Bv_N = g_N/\|u_N\|_2$. It follows from (9.118) that the sequence $v_N$ is bounded in $X_M$. Hence it has a subsequence which converges weakly in $X_M$, hence strongly in $(L^2(\Omega))^k$. Let $v$ be the limit. Then $v$ is in
the nullspace of $A$ and in its orthogonal complement, hence zero. But this is a contradiction, since $\|v\|_2 = \lim_{n \to \infty} \|v_N\|_2 = 1$. Since $u_N$ is bounded in $(L^2(\Omega))^k$, (9.118) implies that it is also bounded in $X_M$. Hence, after taking a subsequence, $u_N$ converges weakly in $X_M$ and strongly in $(L^2(\Omega))^k$. Applying (9.118) again, we see that $u_N$ actually converges strongly in $X_M$. The limit $u$ is a solution of $Lu = f$ with boundary condition $Bu = g$. □

The next interesting question is of course if the index of $A$ is finite, and, more particularly, when it is zero. One of the standard methods in answering this question is to exploit the homotopy invariance of the Fredholm index. Consider for example a second-order elliptic operator

$$L(x, D)u = a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + b_i(x) \frac{\partial u}{\partial x_i} + c(x)u$$  \hspace{1cm} (9.119)

with Dirichlet boundary condition $B(x, D)u = u$. We assume the matrix $a_{ij}$ is symmetric and positive definite. We may then consider the one-parameter family of operators

$$L_t = (1 - t)\Delta + tL, \quad B_t = B.$$  \hspace{1cm} (9.120)

If $\Omega$ and the coefficients satisfy the relevant smoothness assumptions, then the assumptions of Theorem 9.31 apply for every $t \in [0, 1]$; hence the Fredholm index for $(L, B)$ is the same as for Laplace’s equation. In Section 9.2, we proved that the problem $\Delta u = f$ with boundary condition $u = 0$ has a unique solution $u \in H^1(\Omega)$ for every $f \in H^{-1}(\Omega)$. Using the inverse trace theorem, we can trivially conclude that there is a unique solution $u \in H^1(\Omega)$ of the problem $\Delta u = f, \quad u|_{\partial \Omega} = g$ for every $f \in H^{-1}(\Omega), \quad g \in H^{1/2}(\Omega)$. What we would now like to know is that if $f \in L^2(\Omega)$ and $g \in H^{3/2}(\Omega)$, then $u \in H^2(\Omega)$. This is a statement much along the lines of the first assertion of Theorem 9.31, but is not actually implied by Theorem 9.31. The reason is that for the Dirichlet problem of Laplace’s equation, we would choose $s_1 = 0, \quad t_1 = 2$ and $r_1 = -2$, making $M_1 = 0$ and $X_{M_1} = H^2(\Omega)$. Hence the theorem asserts higher regularity of $H^2$ solutions if the data are appropriate, but not $H^2$ regularity of $H^1$ solutions. Nevertheless, the regularity of weak solutions can be proved along very similar lines as Theorem 9.31 and Agmon, Douglis and Nirenberg actually state such results for scalar elliptic equations. For second-order equations with Dirichlet conditions, see Sections 9.5 and 9.6.

A natural question is now to ask for a class of problems to which the approach of Section 9.2, based on the Lax-Milgram lemma, can be extended. This will lead us to Agmon’s condition, to be discussed in Subsection 9.4.4. The Lax-Milgram lemma will imply existence of a “weak” solution, and again the regularity of weak solutions has to be addressed before Theorem 9.31 is applicable.

Another interesting question is to characterize the orthogonal complement of the range of $A$; i.e., what conditions must $f$ and $g$ satisfy so...
that the problem $Lu = f$ with boundary conditions $Bu = g$ is solvable? Usually, one can find a $u$ satisfying $Bu = g$ by an application of the inverse trace theorem (see next subsection); hence we are reduced to the case $g = 0$. This leaves us with the question of characterizing those $v$ for which $(v, Lu) = 0$ for every $u$ satisfying $Bu = 0$. By formally integrating by parts, one can obtain an elliptic boundary-value problem for $v$, known as the adjoint boundary-value problem. We shall study adjoint boundary-value problems for scalar elliptic equations in the next subsection. Of course, a priori $v$ will satisfy the adjoint boundary-value problem only in a “weak” or “generalized” sense. Hence the regularity of weak solutions becomes again an important issue. In particular, in order to show that the operator $\mathcal{A}$ is Fredholm, one has to show that the nullspace of the adjoint is finite-dimensional. Of course, one has to show this for weak solutions of the adjoint problem, not just for strong solutions. Indeed, it is possible to prove this. If the coefficients are smooth enough, it turns out that weak solutions of the adjoint problem are actually smooth.

9.4.3 The Adjoint Boundary-Value Problem

Throughout this subsection, let $L(x, D)$ be a scalar elliptic differential operator of order $2m$ and let $B_j(x, D), j = 1, \ldots, m$, be $m$ boundary operators which satisfy the complementing conditions. The general theory of adjoints requires rather stringent regularity assumptions on $\Omega$ and the coefficients; for simplicity we shall assume they are of class $C^\infty$ and that $\Omega$ is bounded. We make these assumptions throughout. We shall make the additional assumption that the $B_j$ are normal. This property is defined as follows.

**Definition 9.33.** The boundary operators $B_j(x, D)$ are called normal, if their orders $m_j$ are different from each other and less than or equal to $2m - 1$ and if, moreover, the leading-order term in $B_j$ contains a purely normal derivative, i.e., $B_j^p(x, n) \neq 0$ for every $x \in \partial \Omega$ (here $n$ is the unit normal to $\partial \Omega$).

The orders of the $B_j$ cover only half the values from 0 to $2m - 1$. We can add additional boundary operators $S_j, j = 1, \ldots, m$, to fill in the missing orders. Obviously, we can do this in such a way that the extended set of boundary operators still satisfies the conditions of normality; we merely have to take $S_j$ to be the appropriate powers of $\partial/\partial n$. We make the following definition.

**Definition 9.34.** The boundary operators $F_j(x, D), j = 1, \ldots, p$, are called a Dirichlet system of order $p$, if their orders $m_j$ cover all values from zero to $p - 1$ and if, moreover, the leading-order term in $F_j$ contains a purely normal derivative, i.e., $F_j^p(x, n) \neq 0$ for every $x \in \partial \Omega$ (here $n$ is the unit normal to $\partial \Omega$).

We have the following lemma.
Lemma 9.35. Let $F_i(x, D)$, $i = 1, \ldots, p$, be a Dirichlet system, and suppose the order of $F_i$ is $i-1$. Then there exist tangential differential operators $\Phi_{ij}(x, D)$ and $\Psi_{ij}(x, D)$, of order $i - j$, such that

$$F_i(x, D) = \sum_{j=0}^{i} \Phi_{ij}(x, D) \frac{\partial^{j-1}}{\partial n^{j-1}}, \quad \frac{\partial^{i-1}}{\partial n^{i-1}} = \sum_{j=0}^{i} \Psi_{ij}(x, D) F_j(x, D). \quad (9.121)$$

The existence of the $\Phi_{ij}$ is obvious from the definition. The $\Psi_{ij}$ are then obtained by inverting the triangular matrix of the $\Phi_{ij}$. We leave the details of the proof as an exercise; see Problem 9.7.

Corollary 9.36. Let $F_i$, $i = 1, \ldots, 2m$, be a Dirichlet system, and let $m_i$ denote the order of $F_i$. Let $g_i \in H^{2m+k-m_i-1/2}(\partial \Omega)$ be given. Then there exists $u \in H^{2m+k}(\Omega)$ such that $F_i u = g_i$ on $\partial \Omega$.

The proof follows immediately from the previous lemma and Theorem 7.40.

We are now ready to state Green’s formula.

Theorem 9.37. Let $L(x, D)$ be an elliptic operator of order $2m$ on $\overline{\Omega}$ and let $B_j(x, D)$, $j = 1, \ldots, m$, be a set of normal boundary operators. Let $S_j(x, D)$, $j = 1, \ldots, m$, be a set of boundary operators which complements the $B_j$ to form a Dirichlet system. Then there exist boundary operators $C_j(x, D)$, $T_j(x, D)$, $j = 1, \ldots, m$, with the following properties:

1. $\text{ord } C_j = 2m - 1 - \text{ord } S_j$, $\text{ord } T_j = 2m - 1 - \text{ord } B_j$. (ord stands for the order of the operator.)

2. The $C_j$ and $T_j$ form a Dirichlet system.

3. For every $u, v \in H^{2m}(\Omega)$, we have

$$\int_{\Omega} (Lu)v - u(L^*v) \, dx = \sum_{j=1}^{m} \int_{\partial \Omega} (S_j u)(C_j v) - (B_j u)(T_j v) \, dS. \quad (9.122)$$

Here $L^*$ is the formal adjoint of $L$; see Definition 5.53.

4. If the $B_j$ satisfy the complementing condition for $L$, the $C_j$ satisfy the complementing condition for $L^*$.

Proof. Integration by parts yields a formula of the form

$$\int_{\Omega} (Lu)v - u(L^*v) \, dx = \sum_{\alpha, \beta} \int_{\partial \Omega} a_{\alpha \beta}(x) D^\alpha u D^\beta v \, dS, \quad (9.123)$$

where the sum extends over $\alpha$ and $\beta$ with $|\alpha| + |\beta| \leq 2m - 1$. We next integrate by parts on $\partial \Omega$ and move all tangential derivatives from $u$ to $v$, ...
so that only purely normal derivatives of $u$ are left (carrying out this step requires a partition of unity and local coordinate charts on $\partial \Omega$). This leads to a formula of the form
\[
\int_{\Omega} (Lu)v - u(L^*v) \, dx = \sum_{j=0}^{2m-1} \int_{\partial \Omega} a_j(x) \frac{\partial^j u}{\partial n^j} E_j(x, D)v \, dS, \tag{9.124}
\]
where $E_j$ is a differential operator of order $2m - j - 1$. If $L$ is elliptic (or, even more generally, if $\partial \Omega$ is noncharacteristic), then $E_j$ contains terms proportional to $\frac{\partial^{2m-j-1}}{\partial n^{2m-j-1}}$ with a nonzero coefficient; in other words, the $E_j$ form a Dirichlet system. We next use Lemma 9.35 to find
\[
\frac{\partial^j u}{\partial n^j} = \sum_{k=1}^{m} \Psi_{jk}(x, D)B_k(x, D)u + \sum_{k=1}^{m} \Psi'_{jk}(x, D)S_k(x, D)u. \tag{9.125}
\]
We substitute this into (9.124) and then integrate by parts on $\partial \Omega$ to move the tangential differential operators $\Psi_{jk}$ and $\Psi'_{jk}$ from $u$ to $v$. This yields (9.122).

To verify the complementing condition, let $\Omega$ be the half-space $\{x_n > 0\}$ and let $L$ have constant coefficients. Consider solutions of the form $\exp(i\xi \cdot x)v(x_n)$, where $\xi_n = 0$ and $v(x_n) \to 0$ as $x_n \to \infty$. Green’s formula (9.122) holds with $\Omega$ replaced by $\Sigma \times \mathbb{R}^+$, where $\Sigma$ is a parallelepiped in $\mathbb{R}^{n-1}$ corresponding to one period. Moreover, $L$ now becomes an ordinary differential operator. For such operators, a Fredholm alternative holds, i.e., the initial-value problem
\[
L(x, D)u = 0, \quad B_j(x, D)u = 0 \quad \text{for } x_n = 0 \tag{9.126}
\]
has only the trivial solution if and only if the problem
\[
L(x, D)u = f, \quad B_j(x, D)u = g_j \quad \text{for } x_n = 0 \tag{9.127}
\]
is solvable for all $f$ and $g_j$. By Green’s formula, the latter condition implies that the initial-value problem
\[
L^*(x, D)v = 0, \quad C_j(x, D)v = 0 \quad \text{for } x_n = 0 \tag{9.128}
\]
has only the trivial solution, i.e., that the $C_j$ satisfy the complementing condition for $L^*$. Note that if $v$ were a nontrivial solution of (9.128), then $f$ and $g_j$ in (9.127) would have to satisfy
\[
\int_{\Omega} f v \, dx = - \sum_{j=1}^{m} \int_{\partial \Omega} g_j T_j v \, dS. \tag{9.129}
\]
This completes the proof. \hfill \Box

Suppose now that $v \in L^2(\Omega)$ is such that
\[
\int_{\Omega} (Lu)v \, dx = 0 \tag{9.130}
\]
for every \( u \in H^{2m}(\Omega) \) such that \( B_j u = 0 \) on \( \partial \Omega \) for \( j = 1, \ldots, m \). If we actually knew that \( v \in H^{2m}(\Omega) \), then we could use Green’s formula to conclude that
\[
\int_{\Omega} u(L^* v) \, dx = - \sum_{j=1}^{m} \int_{\partial \Omega} S_j u C_j v \, dS. \tag{9.131}
\]
Since the \( B_j u \) and \( S_j u \) can be chosen arbitrarily and independently, (9.131) implies that \( L^* v = 0 \) and \( C_j v = 0 \) on the boundary. Even without the assumption that \( v \in H^{2m}(\Omega) \), we find \( L^* v = 0 \) in the sense of distributions by restricting \( u \) to \( D(\Omega) \).

Let
\[
N^* = \{ v \in H^{2m}(\Omega) \mid L^* v = 0, \, C_1 v = \cdots = C_m v = 0 \text{ on } \partial \Omega \}, \tag{9.132}
\]
let \( (N^*)^\perp \) be the orthogonal complement of \( N^* \) in \( L^2(\Omega) \) and let \( M^* = (N^*)^\perp \cap H^{2m}(\Omega) \). On \( M^* \), we consider the quadratic form
\[
[u, v] = \int_{\Omega} L^* u L^* v \, dx + \sum_{j=1}^{m} (C_j u, C_j v)_{2m-m_j-1/2}, \tag{9.133}
\]
where \( m_j \) is the order of \( C_j \) and \((\cdot, \cdot)_s\) denotes the inner product in \( H^s(\partial \Omega) \).

**Lemma 9.38.** There exists a constant \( C \) such that, for all \( u \in M^* \), we have
\[
\|u\|_{2m}^2 \leq C[u, u]. \tag{9.134}
\]

**Proof.** Since the \( C_j \) boundary operators satisfy the complementing condition for \( L^* \), we can use the Agmon-Douglis-Nirenberg theorem to get
\[
\|u\|_{2m}^2 \leq C([u, u] + \|u\|_{0}^2) \tag{9.135}
\]
for any \( u \in H^{2m}(\Omega) \).

Suppose now for the sake of contradiction that \( u_n \in M^* \) and \([u_n, u_n] \to 0\), whereas \( \|u_n\|_{2m} = 1 \). Then a subsequence of the \( u_n \) converges weakly in \( H^{2m}(\Omega) \) and strongly in \( L^2(\Omega) \); let \( u \) be the limit. It now follows from (9.135) that the subsequence actually converges to \( u \) strongly in \( H^{2m}(\Omega) \). Since \([u, u] = 0\), we have \( u \in N^* \), and since \( u_n \in M^* \), we also have \( u \in M^* \). Since \( N^* \) and \( M^* \) are orthogonal in \( L^2(\Omega) \), this implies \( u = 0 \). But that is a contradiction, since \( \|u\|_{2m} = 1 \). \( \square \)

The Lax-Milgram lemma now implies that the equation \([u, v] = (f, v)\) for all \( v \in M^* \) has a unique solution \( u \in M^* \). If \( f \in (N^*)^\perp \), we actually have \([u, v] = (f, v)\) for all \( v \in H^{2m}(\Omega) \). We can summarize this as follows.
Lemma 9.39. The equation
\[ [u, v] = \int_{\Omega} f v \, dx = (f, v) \quad \forall v \in H^{2m}(\Omega) \] (9.136)
has a solution \( u \in H^{2m}(\Omega) \) if and only if \( f \in (N^*)^\perp \). \( u \) is unique up to addition of an arbitrary element of \( N^* \).

The following is a regularity theorem along the lines of the Agmon-Douglis-Nirenberg result.

Theorem 9.40. Let \( u \) be a solution of (9.136). Then \( u \in H^{4m}(\Omega) \).

The proof is hard and tedious and will not be given. We are now ready to state the main result of this subsection.

Theorem 9.41. The boundary-value problem
\[ Lu = f, \quad B_j u = 0 \quad \text{on} \quad \partial \Omega \quad \text{for} \quad j = 1, \ldots, m, \] (9.137)
with \( f \in L^2(\Omega) \) has a solution \( u \in H^{2m}(\Omega) \) if and only if \( f \in (N^*)^\perp \).

Proof. It is obvious from Green’s formula that the condition \( f \in (N^*)^\perp \) is necessary. Let now \( f \in (N^*)^\perp \). Then we can find \( g \in M^* \cap H^{4m}(\Omega) \) such that \([g, v] = (f, v)\) for every \( v \in H^{2m}(\Omega) \). Now let \( u = L^* g \in H^{2m}(\Omega) \). For every \( v \in H^{2m}(\Omega) \) which satisfies the boundary conditions \( C_j v = 0 \), we conclude
\[ \int_{\Omega} u(L^* v) \, dx = \int_{\Omega} f v \, dx \] (9.138)
and, after integration by parts,
\[ \int_{\Omega} (Lu)v \, dx + \sum_{j=1}^{m} \int_{\partial \Omega} B_j u T_j v \, dS = \int_{\Omega} f v \, dx. \] (9.139)
It follows readily that \( u \) satisfies \( Lu = f \) and \( B_j u = 0 \).

Remark 9.42. The adjoint boundary operators \( C_j \) are generally not unique. However, it is clear from the last theorem that the space \( N^* \) is uniquely determined. Hence different sets of adjoint boundary conditions are equivalent in the sense that they determine the same nullspace.

9.4.4 Agmon’s Condition and Coercive Problems

We consider a scalar elliptic operator of order \( 2m \), given in divergence form as in (9.7):
\[ L(x, D)u = \sum_{|\alpha|,|\beta| \leq m} (-1)^{|\alpha|} D^\alpha(a_{\alpha\beta}(x)D^\beta u), \] (9.140)
where the $a_{\alpha \beta}$ are continuous on $\overline{\Omega}$ and the ellipticity condition
\[ \sum_{|\alpha| = |\beta| = m} \xi^\alpha a_{\alpha \beta}(x) \xi^\beta > 0 \] (9.141)
holds throughout $\overline{\Omega}$. Moreover, we consider $p$ normal boundary-value operators $B_j(x, D)$, with coefficients of class $C^{m_j - m_j}(\partial \Omega)$, where $m_j < m$ is the order of $B_j$. In general, $p$ can take any value between 0 and $m$. We define
\[ V = \{ u \in H^m(\Omega) \mid B_j(x, D)u = 0 \text{ on } \partial \Omega, \ j = 1, \ldots, p \}. \] (9.142)

We consider the quadratic form
\[ a(u, v) = \int_{\Omega} \sum_{|\alpha| \leq m, |\beta| \leq m} a_{\alpha \beta}(x) D^\beta u D^\alpha v \, dx, \] (9.143)
and we ask for conditions under which this form is coercive on $V$:
\[ a(u, u) \geq c_1 \| u \|_m^2 - c_2 \| u \|_0^2 \quad \forall \ u \in V. \] (9.144)
If the form is coercive, we can apply the Lax-Milgram lemma to conclude that, for $\lambda$ large enough, the equation
\[ a(u, v) + \lambda(u, v) = (f, v) \quad \forall v \in V \] (9.145)
has a unique solution $u \in V$ for every $f \in V'$. It is then clear that $L(x, D)u + \lambda u = f$ in the sense of distributions, and that $B_j(x, D)u = 0$ on the boundary. In addition $u$ will satisfy $m - p$ “natural” boundary conditions, which arise in a similar way as the Neumann boundary condition in Section 9.4.1.

The condition guaranteeing coercivity is known as Agmon’s condition. Consider a point $x_0 \in \partial \Omega$; we may orient our coordinate system in such a way that $x_0$ is the origin and the inner normal points in the $x_n$ direction. We then consider the constant coefficient problem $L^p(0, D)u = 0$ in the half-space $x_n > 0$ with boundary conditions $B^p_j(0, D)u = 0$ for $j = 1, \ldots, p$. We shall use the notation $x = (x', x_n)$, where $x' \in \mathbb{R}^{n-1}$, and correspondingly we write $\alpha = (\alpha', \alpha_n)$ for a multi-index $\alpha$. We now pick any $\xi' \in \mathbb{R}^{n-1} \setminus \{0\}$ and consider the ODE
\[ L^p(0, i\xi', \frac{d}{dt})v(t) = 0, \ t > 0, \] (9.146)
with initial conditions
\[ B^p_j \left( 0, i\xi', \frac{d}{dt} \right) v(0) = 0, \ j = 1, \ldots, p. \] (9.147)

**Definition 9.43.** We say that **Agmon’s condition** holds if for any $\xi' \in \mathbb{R}^{n-1} \setminus \{0\}$, and any nonzero solution $v(t)$ of (9.146) and (9.147) such that
v tends to zero exponentially as $t \to \infty$, we have the inequality
\[
\int_0^\infty \sum_{|\alpha'|+k=m} a_{(\alpha',k)}(\beta',l)(0)(\xi')\alpha'(\xi')^\beta' \frac{d^k v(t)}{dt^k} \frac{dt^l}{dt} \ dt > 0. \quad (9.148)
\]

**Remark 9.44.** If $p = m$ and the complementing condition holds, then Agmon’s condition is vacuously true. Indeed, if $p = m$, then, by Lemma 9.35, the boundary conditions are equivalent to Dirichlet conditions. In fact, Dirichlet conditions always satisfy the complementing condition; see Problem 9.6.

The following result generalizes Gårding’s inequality.

**Theorem 9.45.** Let $L$, $B_j$ and $a$ be as above. Assume that Agmon’s condition holds at each point of $\partial \Omega$. Then there exist constants $c_1$ and $c_2$ such that (9.144) holds.

We now address the question how (9.145) is to be interpreted as an elliptic boundary-value problem. For this, we first need a regularity statement.

**Theorem 9.46.** Assume that $\Omega$ and the coefficients of $L$ and the $B_j$ are sufficiently smooth. Assume in addition that $f \in L^2(\Omega)$. Then the solution $u$ of (9.145) lies in $H^{2m}(\Omega)$.

Next, we need a Green’s formula.

**Theorem 9.47.** Let $L$ and $a$ be as above. Let $B_i(x,D)$, $i = 1, \ldots, m$, be a Dirichlet system of order $m$. Assume that $\Omega$ and the coefficients of the operators involved are sufficiently smooth. Then there exist normal boundary-value operators $C_j$, of order $2m - 1 - \text{ord } B_j$, such that, for all $u, v \in H^{2m}(\Omega)$, we have
\[
a(u, v) = \int_\Omega (Lu)v\ dx - \sum_{i=1}^m \int_\Omega (C_i u)(B_i v)\ dS. \quad (9.149)
\]

The proof is completely analogous to that of Theorem 9.37. For $u \in H^{2m}(\Omega)$ and $f \in L^2(\Omega)$, equation (9.145) now assumes the form
\[
\int_\Omega (Lu + \lambda u)v\ dx - \sum_{j=p+1}^m \int_{\partial \Omega} (C_j u)(B_j v)\ dS = \int_\Omega f v\ dx. \quad (9.150)
\]

This identifies (9.145) as the weak form of the elliptic boundary-value problem
\[
Lu + \lambda u = f, \quad B_j u = 0, \quad j = 1, \ldots, p, \quad C_j u = 0, \quad j = p + 1, \ldots, m. \quad (9.151)
\]

The first set of boundary conditions is called essential; they are directly imposed on $u$ in the weak formulation of the problem. The second set of boundary conditions is called “natural”; they are not imposed explicitly, but arise from an integration by parts just like Neumann’s condition in Section 9.4.1.
9.5. Assume that \( \Omega \) is bounded, connected, and has the 1-extension property. Let

\[
V = \left\{ u \in H^1(\Omega) \mid \int_{\Omega} u(x) \, dx = 0 \right\}.
\]

(a) Show that for each \( f \in L^2(\Omega) \) there is a unique \( u \in V \) such that

\[
\int_{\Omega} \nabla u \cdot \nabla v = \int_{\Omega} fv \quad \text{for all } v \in V.
\] (9.152)

(See Problem 7.15.)

(b) Explain why it is appropriate to regard (9.152) as a weak form of the Neumann problem

\[
-\Delta u = f \quad \text{in } \Omega
\]

\[
\frac{\partial u}{\partial n} = 0 \quad \text{on } \partial \Omega
\] (9.153)

if \( \int_{\Omega} f = 0 \).

(c) If \( \int_{\Omega} f \neq 0 \), is it still reasonable to call the solution of (9.152) a solution of (9.153)? Explain.

9.6. Show that Dirichlet boundary conditions for scalar elliptic PDEs always satisfy the complementing condition.

9.7. Fill in the details for the proof of Lemma 9.35.

9.8. Suppose that Agmon’s condition holds. Show that the complementing condition is satisfied for (9.151). Hint: Apply (9.149) on a half-space.

9.9. Formulate a weak form of (9.151) when the boundary conditions are allowed to be inhomogeneous.

9.10. Show that the “traction boundary conditions” \( (\nabla u + (\nabla u)^T) \cdot n - pn = 0 \) satisfy the complementing condition for the Stokes system.

9.11. Show that a scalar elliptic operator with Dirichlet conditions has Fredholm index 0. Hint: Show that the adjoint problem also has Dirichlet conditions.

9.5 Interior Regularity

In Section 9.2, we have shown the existence of weak solutions \( u \in H^k(\Omega) \) of the Dirichlet problem for elliptic operators of order \( 2k \). We now wish to show that under suitable hypotheses on the smoothness of the coefficients \( a_{\sigma\gamma} \), the forcing function \( f \) and the boundary of \( \Omega \), our weak solution is, in
fact, a strong solution or a classical solution. In order to give some idea of how we plan to go about this, we make a couple of formal calculations.

For our first calculation let us assume that \( \Omega \) has a smooth boundary \( \partial \Omega \) with unit outward normal \( \eta = (\eta_1, \ldots, \eta_n) \) and that \( u \) is a classical solution of

\[
-\Delta u = f
\]

in \( \Omega \), and

\[
u = 0
\]
on \( \partial \Omega \). Our goal is to show that (weak) solutions of elliptic problems such as the one above are actually in a “better” space than \( H^1_0(\Omega) \). In order to prepare for this, we will now estimate the \( L^2(\Omega) \) norm of the matrix of second partials of \( u \) in terms of the \( H^1(\Omega) \) norm. Since this is simply a formal calculation, we will proceed as if we already know that \( u \) is as smooth as we like.

\[
\int_{\Omega} |\Delta u|^2 \, dx = \sum_{i,j=1}^n \int_{\Omega} u_{x_i x_j} u_{x_j x_i} \, dx
\]

\[
= - \sum_{i,j=1}^n \int_{\Omega} u_{x_i} u_{x_j x_i x_j} \, dx
\]

\[
+ \sum_{i,j=1}^n \int_{\partial \Omega} u_{x_i} u_{x_j} \eta_i \, dS
\]

\[
= \sum_{i,j=1}^n \int_{\Omega} u_{x_i x_j} u_{x_j x_i} \, dx
\]

\[
+ \sum_{i,j=1}^n \int_{\partial \Omega} u_{x_i} u_{x_j} \eta_i - u_{x_i} u_{x_i} \eta_j \, dS.
\]

We also have

\[
\int_{\Omega} |\nabla u|^2 \, dx = \int_{\Omega} |f|^2 \, dx.
\]

(9.156)

Combining these two results gives us

\[
\int_{\Omega} |\nabla^2 u|^2 \, dx \leq \int_{\Omega} |f|^2 \, dx + |\text{boundary terms}|.
\]

(9.157)

Thus, if we had some additional information on the boundary terms, we could derive an a priori estimate on the \( H^2(\Omega) \) norm of a solution \( u \) in terms of the data \( f \).

Unfortunately, estimates on boundary terms are rather delicate, so we will put off this subject until the next section. In the meantime, we will concentrate on interior estimates of higher-order derivatives. For example, let \( \Omega' \) be any domain such that \( \Omega' \subset\subset \Omega \). (The notation \( \Omega' \subset\subset \Omega \) means
that \( \Omega' \) is \textit{compactly contained} in the open set \( \Omega \); i.e., \( \overline{\Omega'} \) is compact and \( \overline{\Omega'} \subset \Omega \). We now choose a cutoff function \( \zeta \in \mathcal{D}(\Omega) \) such that \( 0 \leq \zeta \leq 1 \) and \( \zeta \equiv 1 \) on \( \Omega' \). We can now make some calculations very similar to those above, but without any boundary terms getting in the way.

\[
\int_\Omega \zeta^2 |\Delta u|^2 \, dx = \sum_{i,j=1}^{n} \int_\Omega u_{x_i} u_{x_j} u_{x_i} u_{x_j} \zeta^2 \, dx \\
= -\sum_{i,j=1}^{n} \int_\Omega u_{x_i} u_{x_j,x_i} \zeta^2 \, dx \\
- \sum_{i,j=1}^{n} \int_\Omega u_{x_i} u_{x_j,x_j} 2\zeta \zeta_{x_i} \, dx \\
= \sum_{i,j=1}^{n} \int_\Omega u_{x_i} u_{x_j} \zeta^2 \, dx \\
+ \sum_{i,j=1}^{n} \int_\Omega u_{x_i} u_{x_j} 2\zeta \zeta_{x_j} - u_{x_i} u_{x_j,x_j} 2\zeta \zeta_{x_i} \, dx.
\]

We now use this with inequalities of the form

\[
|u_{x_i} u_{x_j,x_j} 2\zeta \zeta_{x_j}| \leq \epsilon u_{x,j}^2 \zeta^2 + \frac{1}{\epsilon} u_{x,j}^2 \zeta_{x_j}^2
\]

and

\[
\int_\Omega |\Delta u|^2 \zeta^2 \, dx = \int_\Omega |f|^2 \zeta^2 \, dx \leq \int_\Omega |f|^2 \, dx
\]

to get

\[
\int_\Omega |\nabla^2 u|^2 \zeta^2 \, dx \leq \int_\Omega [||f||^2 + \epsilon |\nabla^2 u|^2 \zeta^2 + C(\epsilon) |\nabla u|^2 |\nabla \zeta|^2] \, dx.
\]

We now let \( \epsilon = 1/2 \) and use the fact that \( \zeta \equiv 1 \) on \( \Omega' \) to get

\[
\int_{\Omega'} |\nabla^2 u|^2 \, dx \leq \int_\Omega \zeta^2 |\nabla^2 u|^2 \, dx \leq C \left(||f||_2^2 + ||\nabla u||_2^2\right).
\]

Thus, we have an estimate on the \( H^2(\Omega') \) norm of a solution \( u \) for any \( \Omega' \subset \subset \Omega \) in terms of the \( L^2(\Omega) \) of the data \( f \) and the \( H^1(\Omega) \) norm of \( u \).

Of course, one of the major objections to the calculations performed above is that we needed to make unwarranted assumptions about the smoothness of the solution \( u \) in order to perform the integrations by parts involved. In the rigorous versions of these calculations below, these operations are replaced by analogous techniques involving difference quotients. Because the technique of using difference quotients is so important in this section, we present the following short digression on this topic.
9.5.1 Difference Quotients

Let $\Omega \subset \mathbb{R}^n$ and let $\{e_1, \ldots, e_n\}$ be the standard orthonormal basis for $\mathbb{R}^n$. For any function $u \in L^p(\Omega)$ we can formally define the difference quotient in the direction $e_i$ to be

$$D_i^h u(x) := \frac{u(x + he_i) - u(x)}{h}.$$  \hfill (9.162)

Of course, since $x + he_i$ might extend beyond $\Omega$ for $x$ near the boundary, this function might not be well defined for all $x \in \Omega$. However, we can get the following result.

**Lemma 9.48.** Let $u \in W^{1,p}(\Omega)$, $1 \leq p \leq \infty$. Then for any $\Omega' \subset \subset \Omega$ and any $h < \text{dist}(\Omega', \partial \Omega)$, we have $D_i^h u \in L^p(\Omega')$ and

$$\|D_i^h u\|_{L^p(\Omega')} \leq \|u_{x_i}\|_{L^p(\Omega)}. \hfill (9.163)$$

**Proof.** Let $\Omega'$ and $h$ satisfy the hypotheses of the lemma. For any $\xi \in [0, h]$ we define

$$\Omega'_{\xi,i} := \{x \in \Omega \mid x = \bar{x} + \xi e_i; \quad \bar{x} \in \Omega'\}. \hfill (9.164)$$

For $p$ in $[1, \infty)$ we first consider the case where $u \in C^1_b(\Omega) \cap W^{1,p}(\Omega)$. Then for any $x \in \Omega'$, we can use the fundamental theorem of calculus to write

$$D_i^h u(x) = \frac{u(x + he_i) - u(x)}{h} = \frac{1}{h} \int_0^h u_{x_i}(x + \xi e_i) d\xi.$$  \hfill (9.165)

Thus, using Hölder’s inequality, and switching orders of integration, we get

$$\int_{\Omega'} |D_i^h u(x)|^p \, dx \leq \int_{\Omega'} \frac{1}{h} \int_0^h |u_{x_i}(x + \xi e_i)|^p \, d\xi \, dx \leq \frac{1}{h} \int_0^h \int_{\Omega'_{\xi,i}} |u_{x_i}(x)|^p \, dx \, d\xi \leq \int_{\Omega} |u_{x_i}|^p \, dx.$$  

By Theorem 7.48, this inequality extends to the whole space by taking limits.

For $p = \infty$ we note that (9.165) holds in the sense of distributions. Since the test functions are dense in $L^1$, the inequality follows from manipulating and estimating $D_i^h u$ and $u_{x_i}$ as linear functionals on sets of test functions. We leave this to the reader. \qed

Of course, it is not at all surprising that if a function is in $W^{1,p}(\Omega)$, its difference quotients obey some bound in terms of its partial derivatives. The following result is more substantial; it says that if we start out knowing
that \( u \) is in \( L^p(\Omega) \) and can obtain a bound on its difference quotients that is independent of \( h \), then we can deduce that \( u \) is in the space \( W^{1,p}(\Omega) \).

**Lemma 9.49.** Let \( u \in L^p(\Omega), 1 < p \leq \infty \), and suppose there exists a constant \( \bar{C} \) such that for any \( \Omega' \subset \subset \Omega \) and \( h < \text{dist}(\Omega', \partial \Omega) \) we have \( D^h_i u \in L^p(\Omega') \) and

\[
\|D^h_i u\|_{L^p(\Omega')} \leq \bar{C}. \tag{9.166}
\]

Then \( u \) (which is a priori well defined as a distribution) is in fact in \( L^p(\Omega) \) and satisfies

\[
\|u\|_{L^p(\Omega)} \leq \bar{C}. \tag{9.167}
\]

**Proof.** Recall that at the end of Section 5.5.1 we showed the distributional derivative of a function could be obtained as the limit of difference quotients. In terms of the present problem we have

\[
(u_{x_i}, \phi) = \lim_{h \to 0} (D^h_i u, \phi) \tag{9.168}
\]

for each \( \phi \in \mathcal{D}(\Omega) \).

We now note that there exists a function \( v \in L^p(\Omega) \) with \( \|v\|_{L^p(\Omega)} \leq \bar{C} \) and a sequence \( h_m \to 0 \) such that for every \( \phi \in \mathcal{D}(\Omega) \)

\[
\lim_{h_m \to 0} \int_{\Omega} \phi D^{h_m}_i u \, dx = \int_{\Omega} \phi v \, dx. \tag{9.169}
\]

This follows from the weak compactness of the bounded set \( D^h_i u \) in \( L^p(\Omega') \) for every \( \Omega' \subset \subset \Omega \) (cf. Theorem 6.64) and the fact that \( \Omega \) can be covered with a countable collection of closed subsets \( \Omega_j \subset \subset \Omega \) such that each \( \Omega' \subset \subset \Omega \) intersects at most a finite number of the \( \Omega_j \).

Thus,

\[
(u_{x_i}, \phi) = \int_{\Omega} \phi v \, dx; \tag{9.170}
\]

i.e., the distributional derivative of \( u \) is given (uniquely) by the function \( v \in L^p(\Omega) \). \( \square \)

We also need to develop a few important tools using difference operators; namely, the analogues of the product rule and integration by parts in differential calculus.

**Lemma 9.50.** Suppose \( \Omega' \subset \subset \Omega \) and \( h < \text{dist}(\Omega', \partial \Omega) \). Then for any \( u \in L^p(\Omega), 1 < p < \infty \), and any \( v \in C_b(\Omega) \) with \( \text{supp} \, v \subset \Omega' \) we have

\[
D^h_i (uv)(x) = u(x)D^h_i v(x) + D^h_i u(x)v(x + h e_i) \tag{9.171}
\]

and

\[
\int_{\Omega} D^h_i u(x)v(x) \, dx = -\int_{\Omega} u(x)D^{-h}_i v(x) \, dx. \tag{9.172}
\]

The proof is left to the reader.
9.5.2 Second-Order Scalar Equations

In order to eliminate many technical details, we will give a proof of an interior regularity result only in the case of a second-order scalar equation. We already gave statements of results for higher-order equations and systems in Section 9.4 above.

**Theorem 9.51 (Interior regularity).** Let $L$ be a uniformly elliptic second-order operator of the form

$$L(x, D)u := \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} a_{ij}(x) \frac{\partial u}{\partial x_j} + \sum_{i=1}^{n} b_i(x) \frac{\partial u}{\partial x_i} + c(x)u,$$

with corresponding bilinear form

$$B[v, u] := -\sum_{i,j=1}^{n} \int_{\Omega} a_{ij}(x) u_{x_j} v_{x_i} \, dx + \sum_{i=1}^{n} \int_{\Omega} b_i(x) u_{x_i} v \, dx + \int_{\Omega} c(x) uv \, dx.$$

Suppose the coefficients satisfy $a_{ij} \in W^{1,\infty}(\Omega)$, $b_i, c \in L^\infty(\Omega)$ and that $f \in L^2(\Omega)$. Let $u \in H^1_0(\Omega)$ be a weak solution of the Dirichlet problem for $L(D, x)u = f$. Then $u \in H^2(\Omega')$ for every $\Omega' \subset \subset \Omega$, and

$$\|u\|_{H^2(\Omega')} \leq C(\|u\|_{H^1(\Omega)} + \|f\|_{L^2(\Omega)}). \quad (9.173)$$

**Proof.** We begin with the identity

$$\int_{\Omega} a_{ij} u_{x_j} v_{x_i} \, dx = \int_{\Omega} g v \, dx \quad (9.174)$$

for all $v \in H^1_0(\Omega)$, where $g \in L^2(\Omega)$ is given by

$$g := b_i u_{x_i} + cu - f. \quad (9.175)$$

Now suppose $v \in H^1(\Omega)$ and $\text{supp } v \subset \subset \Omega$ and let

$$|2h| < \text{dist}(\text{supp } v, \partial \Omega).$$

Then we have (for any $k = 1, \ldots, n$) $D_k^{-h}v \in H^1_0(\Omega)$, and thus we can use (9.174) and the “differentiating by parts formula” (9.172) to get

$$\int_{\Omega} D_k^{-h}(a_{ij} u_{x_j}) v_{x_i} \, dx = -\int_{\Omega} a_{ij} u_{x_j} D_k^{-h}v_{x_i} \, dx$$

$$= -\int_{\Omega} g D_k^{-h}v \, dx.$$ 

We can now use this and the product rule for difference quotients (9.171) to get

$$\int_{\Omega} a_{ij}(x + h e_k)(D_k^{-h}u_{x_j})v_{x_i} \, dx = -\int_{\Omega} (D_k^{-h}a_{ij}) u_{x_j} v_{x_i} + g D_k^{-h}v \, dx. \quad (9.176)$$

We can now use Lemma 9.48 to estimate this by

$$\int_{\Omega} a_{ij}(x + h e_k)(D_k^{-h}u_{x_j})v_{x_i} \, dx \leq C(\|u\|_{1,2} + \|f\|_2)\|\nabla v\|_2. \quad (9.177)$$
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Let $\Omega' \subset \subset \Omega$. We now choose a cutoff function $\zeta \in D(\Omega)$ with the following properties.

1. $\zeta \equiv 1$ on $\Omega'$.
2. $|\nabla \zeta| < 2/d$ where $d = \text{dist}(\Omega', \partial \Omega)$.

We now use the ellipticity condition (9.6), elementary inequalities, and the previous estimate (with $v = \zeta^2 D_h^k u$) to obtain

$$\theta \int_{\Omega} |\zeta \nabla D_h^k u|^2 \, dx \leq - \int_{\Omega} \zeta^2 a_{ij}(x + h e_k) D_h^k u_{x_i} D_h^k u_{x_j} \, dx$$

$$= - \int_{\Omega} a_{ij}(x + h e_k) D_h^k u_{x_i} [((\zeta^2 D_h^k u)_{x_j} - 2 D_h^k u \zeta \zeta_{x_j}) \, dx$$

$$\leq C(\|u\|_{1,2} + \|f\|_2)(\|\zeta \nabla D_h^k u\|_2 + 2\|D_h^k u \zeta \zeta\|_2)$$

$$+ \sup_{i,j} \|a_{ij}\|_{\infty} \|\zeta \nabla D_h^k u\|_2 \|D_h^k u \zeta \zeta\|_2$$

$$\leq C(\epsilon)(\|u\|_{1,2} + \|f\|_2 + \|D_h^k u \zeta \zeta\|_2^2 + \epsilon \|\zeta \nabla D_h^k u\|_2^2)$$

for any $\epsilon > 0$.

Thus, after making an appropriate choice of $\epsilon$ and rearranging, we have

$$\|D_h^k \nabla u\|_{L^2(\Omega')} \leq \|\zeta D_h^k \nabla u\|_{L^2(\Omega)}$$

$$\leq C(\|u\|_{H^1(\Omega)} + \|f\|_{L^2(\Omega)} + \|D_h^k u \zeta \zeta\|)$$

$$\leq C(\|u\|_{H^1(\Omega)} + \|f\|_{L^2(\Omega)}).$$

Here we have used Lemma 9.48 and our pointwise bound on $|\nabla \zeta|$.

Finally, we can use this estimate on the difference quotients of $\nabla u$ and Lemma 9.49 to deduce that $\nabla u \in H^1(\Omega')$. The estimate (9.173) follows immediately.

Problems

9.12. Show that if $\Omega$ is bounded, $f : \mathbb{R} \to \mathbb{R}$ is uniformly Lipschitz, and $u \in W^{1,p}(\Omega)$ with $1 < p \leq \infty$, then the composite $f \circ u$ belongs to $W^{1,p}(\Omega)$.


9.6 Boundary Regularity

In the previous section we showed that if the data and coefficients are sufficiently smooth, then weak solutions of elliptic problems are “as smooth as one could expect” in the interior of the domain on which the problem is posed. In the following example we see that a solution is not necessarily smooth up to the boundary of the domain on which the problem is posed if that boundary is not sufficiently smooth. (The reader should also note that
Example 9.52. Let \( i \) and \( j \) give an orthonormal basis for \( \mathbb{R}^2 \), and let \((r, \theta)\) be the standard polar coordinates for \( \mathbb{R}^2 \) defined by the map

\[
\hat{x}(r, \theta) := re_1(\theta),
\]

where

\[
e_1(\theta) := \cos \theta i + \sin \theta j, \quad (9.179)
\]

\[
e_2(\theta) := -\sin \theta i + \cos \theta j. \quad (9.180)
\]

Recall that for a real-valued function \( f(r, \theta) \) we can calculate the gradient and Laplacian as follows:

\[
\nabla f = f_r e_1 + \frac{1}{r} f_\theta e_2, \quad (9.181)
\]

\[
\Delta f = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial f}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 f}{\partial \theta^2}. \quad (9.182)
\]

In addition, the Hessian or second gradient matrix is given by

\[
\begin{pmatrix}
    f_{rr} & -\frac{1}{r} f_{r\theta} - \frac{1}{r^2} f_{\theta}
    \\
    -\frac{1}{r} f_{r\theta} - \frac{1}{r^2} f_{\theta} & \frac{1}{r} f_r + \frac{1}{r^2} f_{\theta\theta}
\end{pmatrix}. \quad (9.183)
\]

We now consider the following problem for Laplace’s equation with nonhomogeneous boundary conditions. Let \( 0 < \beta < 2\pi \) and define

\[
\Omega_\beta := \{ \mathbf{x} \in \mathbb{R}^2 \mid \mathbf{x} = \hat{x}(r, \theta), \ 0 < r < 1, \ 0 < \theta < \beta \}. \quad (9.184)
\]

We seek \( u : \Omega_\beta \to \mathbb{R} \) satisfying

\[
\Delta u = 0 \quad \text{in} \ \Omega_\beta, \quad (9.185)
\]

and the boundary conditions

\[
u(r, 0) = 0, \quad 0 < r < 1
\]

\[
u(r, \beta) = 0, \quad 0 < r < 1
\]

\[
u(1, \theta) = \sin \frac{\pi \theta}{\beta}, \quad 0 < \theta < \beta.
\]

Using separation of variables, we can find the solution of the problem to be

\[
u(r, \theta) = r^{\pi/\beta} \sin \frac{\pi \theta}{\beta}. \quad (9.186)
\]

This function (as the results of the previous section assure us) is in \( C^\infty(\Omega') \) for any \( \Omega' \) compactly contained in \( \Omega_\beta \). However, note that if \( \beta > \pi \), then
our solution decays at the origin like \( r^\alpha \) with \( \alpha < 1 \). To see the impact of this we calculate the gradient of \( u \)

\[
\nabla u = \frac{\pi}{\beta} r^{\pi/\beta - 1} \left( \sin \frac{\pi \theta}{\beta} e_1(\theta) + \cos \frac{\pi \theta}{\beta} e_2(\theta) \right)
\]

and its \( L^2(\Omega_\beta) \) norm

\[
\int_{\Omega_\beta} |\nabla u|^2 \, dx = C \int_{0}^{\beta} \int_{0}^{1} r^{2(\pi/\beta - 1)} r \, dr \, d\theta \leq \infty.
\]

We see that (as our basic existence theorem for weak solutions implies) our solution \( u \) is in \( H^1(\Omega_\beta) \). However, by calculating the second gradient and computing its norm, we see that if \( \beta > \pi \), then \( u \) is not in \( H^2(\Omega_\beta) \).

Thus, despite the fact that we have all of the interior regularity guaranteed by the results of the previous section, \textit{we do not have regularity up to the boundary}. The culprit here is the lack of smoothness of the boundary.

As the example above indicates, we will need to assume that the boundary has some smoothness properties in order to get a boundary regularity result (also called a global regularity result). In order to emphasize the most important techniques in the proof (breaking up the domain using a partition of unity and mapping the pieces containing portions of the boundary to a half-space) we will give the proof only for second-order scalar equations and in the proof we will ignore lower-order terms.

\textbf{Theorem 9.53 (Global regularity).} Suppose that the hypotheses of Theorem 9.51 hold and that in addition \( \partial \Omega \) is of class \( C^2 \). Then \( u \in H^2(\Omega) \) and

\[
\|u\|_{H^2(\Omega)} \leq C(\|u\|_{L^2(\Omega)} + \|f\|_{L^2(\Omega)}).
\]

The proof of this result is rather long and involved, so we will break it up by proving a number of preliminary lemmas. One of our basic techniques is to decompose the domain into pieces using a partition of unity and “flattening out” any portion of the boundary. As we see in our first lemma (which is essentially a version of the main result in the case where the boundary is already flat) a flat boundary allows us to use difference quotients to our advantage.

\textbf{Lemma 9.54.} Let \( R > 0, \lambda \in (0,1) \), and define

\[
D^+ := B_R(0) \cap \{x \in \mathbb{R}^n \mid x_n > 0\},
\]

\[
Q^+ := B_{\lambda R}(0) \cap \{x \in \mathbb{R}^n \mid x_n > 0\}.
\]

Let \( L \) be a uniformly elliptic second-order operator of the form

\[
L(x, D)u := \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} a_{ij}(x) \frac{\partial u}{\partial x_j} + \sum_{i=1}^{n} b_i(x) \frac{\partial u}{\partial x_i} + c(x)u
\]
with corresponding bilinear form

\[
B[v, u] := - \sum_{i,j=1}^{n} \int_{D^+} a_{ij}(x) u_{x_j} v_{x_i} \, dx + \sum_{i=1}^{n} \int_{D^+} b_i(x) u_x v \, dx + \int_{D^+} c(x) uv \, dx.
\]

Suppose the coefficients satisfy \( a_{ij} \in W^{1,\infty}(D^+) \), \( b_i, c \in L^\infty(D^+) \) and that \( f \in L^2(D^+) \). Suppose \( u \in H^1(D^+) \) satisfies the variational equation

\[
B[v, u] = (v, f) \quad (9.193)
\]

for all \( v \in H^1_0(D^+) \) and that \( u \equiv 0 \) in the sense of trace on \( \{ x \in \mathbb{R}^n \mid x_n = 0 \} \). Then \( u \in H^2(Q^+) \) and there exists a constant \( C \) depending on \( R \) such that

\[
\|u\|_{H^2(Q^+)} \leq C(\|f\|_{L^2(D^+)} + \|u\|_{L^2(D^+)}). \quad (9.194)
\]

Proof. Let \( h \in (0, R(1-\lambda)/2) \) and fix an index \( k = 1, \ldots, n-1 \) (i.e., \( k \neq n \)). Now choose \( \zeta \in C_0^\infty(D^+) \) such that

1. \( 0 \leq \zeta \leq 1 \),
2. \( \zeta \equiv 1 \) on \( Q^+ \),
3. \( U := \text{supp} \zeta \subset B_{R(1+\lambda)/2}(0) \).

Note: The function \( \zeta \) in not in \( D(D^+) \) since \( \zeta \neq 0 \) on the flat part of the boundary of \( D^+ : \{ x_n \equiv 0 \} \).

Now define

\[
v := -D_k^{-h}(\zeta^2 D_k^h u). \quad (9.195)
\]

After some manipulations using the definition of the difference quotients, we get the following identity.

\[
v(x) = -\frac{1}{h^2} (\zeta^2(x)[u(x + he_k) - u(x)] + \zeta^2(x - he_k)[u(x - he_k) - u(x)]).
\]

Note that in constructing \( v \) we have used translations only in directions tangential to the plane \( x_n = 0 \). The key idea is that we can “slide the support of \( u \)” along the plane \( x_n = 0 \) without destroying the boundary conditions. Also note that none of the translations moves the support of \( \zeta \) outside of \( \overline{D^+} \). These facts ensure that

1. \( v \) is well defined on \( D^+ \),
2. \( v \in H^1(D^+) \) (since \( u \in H^1(D^+) \)),
3. \( v \in H^1_0(D^+) \) (since \( \zeta \) is zero on the curved part of the boundary of \( D^+ \) and \( u \) is zero on the flat part of the boundary (and the same goes for any of the translations of \( \zeta \) and \( u \))).
We define
\[ w := \zeta D^h_k u, \] (9.197)
so that \( v = -D^{-h}_k(\zeta w) \).

Now, we follow a procedure similar to the derivation of the estimate (9.177) of Theorem 9.51, but using \( v \) as defined above. We get
\[
- \int_{D^+} gD^{-h}_k(\zeta w) \, dx = \int_{D^+} a_{ij} u_{x_j} v_{x_i} \, dx
\]
\[
= \int_{D^+} D^h_k(a_{ij} u_{x_j})(\zeta w)_{x_i} \, dx
\]
\[
= \int_{D^+} a_{ij}(x + he_k)D^h_k u_{x_j}(\zeta w)_{x_i} \, dx
\]
\[
+ \int_{D^+} D^h_k a_{ij} u_{x_j} (\zeta w)_{x_i} \, dx
\]
\[
= \int_{D^+} a_{ij}(x + he_k) w_{x_j} w_{x_i} \, dx
\]
\[
- \int_{D^+} a_{ij}(x + he_k) \zeta_{x_j} D^h_k w_{x_i} \, dx
\]
\[
+ \int_{D^+} a_{ij}(x + he_k)D^h_k u_{x_j} \zeta_{x_i} w \, dx
\]
\[
+ \int_{D^+} D^h_k a_{ij} u_{x_j} (\zeta w)_{x_i} \, dx.
\]

Here \( g \) is defined as in (9.175). Rearranging, we get
\[
\int_{D^+} a_{ij}(x + he_k) w_{x_j} w_{x_i} \, dx = I_1 + I_2 + I_3 + I_4,
\] (9.198)
where
\[
I_1 := - \int_{D^+} gD^{-h}_k(\zeta w) \, dx,
\] (9.199)
\[
I_2 := \int_{D^+} a_{ij}(x + he_k) \zeta_{x_j} D^h_k w_{x_i} \, dx,
\] (9.200)
\[
I_3 := - \int_{D^+} a_{ij}(x + he_k)D^h_k u_{x_j} \zeta_{x_i} w \, dx,
\] (9.201)
\[
I_4 := - \int_{D^+} D^h_k a_{ij} u_{x_j} (\zeta w)_{x_i} \, dx.
\] (9.202)

We estimate these terms using techniques which should be familiar to the reader from the proof of the Theorem 9.51 (basically Hölder’s inequality and Lemma 9.48). We will make use of the estimate
\[
\|v\|_2 = \|D^{-h}_k(\zeta w)\|_2 \leq 2\|\zeta\|_{1,\infty} \|w\|_{1,2}.
\] (9.203)
(Here, $\| \cdot \|_2 = \| \cdot \|_{L^2(D^+)}$, etc.) We get the following:

$$
|I_1| \leq \sum_i \| b_i \|_\infty \| \nabla u \|_2 \| v \|_2 + \| c \|_\infty \| u \|_2 \| v \|_2 + \| f \|_2 \| v \|_2 \\
\leq C(\| u \|_{1,2} + \| f \|_2) \| w \|_{1,2}.
$$

$$
|I_2| \leq \sum_{i,j} \| a_{ij} \|_\infty \| D^k_u \|_2 \| w_x \|_2 \| \nabla \zeta \|_\infty \\
\leq C\| u \|_{1,2} \| w \|_{1,2}.
$$

$$
|I_3| \leq \sum_{i,j} \| a_{ij} \|_\infty \| \zeta_x \|_\infty \| D^k_u \|_2 \| w \|_2 \\
\leq C\| u \|_{1,2} \| w \|_{1,2}.
$$

$$
|I_4| \leq \sum_{i,j} \| a_{ij} \|_{1,\infty} \| \zeta \|_{1,\infty} \| u \|_{1,2} \| w \|_{1,2} \\
\leq C\| u \|_{1,2} \| w \|_{1,2}.
$$

Combining these estimates and the uniform ellipticity condition gives us

$$
\theta \int_{D^+} |\nabla w|^2 \, dx \leq - \int_{D^+} a_{ij}(x + h\mathbf{e}_k) w_{x_i} w_{x_j} \, dx \leq C(\| u \|_{1,2} + \| f \|_2) \| w \|_{1,2}.
$$

(9.204)

It follows that

$$
\int_{D^+} |\nabla w|^2 \leq C(\| f \|_2^2 + \| u \|_{1,2}^2).
$$

(9.205)

We can use this and the fact that $\zeta \equiv 1$ on $Q^+$ to show that

$$
\int_{Q^+} |D^k_h \nabla u|^2 \, dx = \int_{Q^+} |\nabla w|^2 \, dx \\
\leq \int_{D^+} |\nabla w|^2 \, dx \\
\leq C(\| f \|_2^2 + \| u \|_{1,2}^2).
$$

Thus, using Lemma 9.48, we can get an estimate for all second-order mixed partial derivatives except for $u_{x_n,x_n}$; i.e., we have

$$
\sum_{i,j=1}^n \int_{Q^+} |u_{x_i,x_j}|^2 \, dx \leq C(\| f \|_2^2 + \| u \|_{1,2}^2).
$$

(9.206)

To estimate $u_{x_n,x_n}$ we proceed as follows. From the interior regularity result we know that the differential equation

$$
L(D,x)u = f
$$
is satisfied in the strong or $L^2$ sense. Since $a_{nn} \in W^{1,\infty}(\Omega)$, the Sobolev embedding theorem ensures that it is continuous. Furthermore, the strong ellipticity condition ensures that $a_{nn} \geq \theta > 0$. Thus, we can divide the PDE by $a_{nn}$ and rearrange to get

$$u_{x_n x_n} = -\frac{1}{a_{nn}} \left[ \sum_{i,j=1 \atop i+j<2n} a_{ij} u_{x_i x_j} - g \right]. \quad (9.207)$$

Note that we have shown that the right-hand side of this equation is in $L^2(Q^+)$; and in addition, we have the estimate

$$\int_{Q^+} |u_{x_n x_n}|^2 \, dx \leq C(\|f\|_2^2 + \|u\|_{1,2}^2). \quad (9.208)$$

Combining this with our previous results we get our final estimate (9.194).

We now state without proof the “higher-order” version of Lemma 9.54.

**Lemma 9.55.** Let the hypotheses of Lemma 9.54 hold. In addition, assume that the coefficients satisfy $a_{ij} \in W^{k+1,\infty}(D^+)$, $b_i, c \in W^k,\infty(D^+)$ and that $f \in H^k(D^+)$. Then $u \in H^{k+2}(Q^+)$ and there exists a constant $C$ depending on $R$ such that

$$\|u\|_{H^{k+2}(Q^+)} \leq C(\|f\|_{H^k(D^+)} + \|u\|_{H^1(D^+)}). \quad (9.209)$$

To prove Theorem 9.53 we must consider a general domain $\Omega$. Of course, since we did all the work of proving Lemma 9.54 on a half-space, our next task is to “straighten out” the boundary of $\Omega$. Since $\partial\Omega$ is a $C^2$ surface, we know that for each $\bar{x} \in \partial\Omega$ there is an $R > 0$ and a $C^2(\mathbb{R}^{n-1})$ function $\psi$ such that (after a possible renumbering and reorientation of coordinates)

$$\partial\Omega \cap B_R(\bar{x}) = \{ x \in B_R(\bar{x}) \mid x_n = \psi(x_1, x_2, \ldots, x_{n-1}) \}, \quad (9.210)$$

$$\Omega \cap B_R(\bar{x}) = \{ x \in B_R(\bar{x}) \mid x_n > \psi(x_1, x_2, \ldots, x_{n-1}) \}; \quad (9.211)$$

and moreover, the mapping

$$B_R(\bar{x}) \ni x \mapsto y = \Psi(x) \in \mathbb{R}^n \quad (9.212)$$

defined by

$$y_i := x_i - \bar{x}_i, \quad i = 1, \ldots, n-1,$$

$$y_n := x_n - \psi(x_1, \ldots, x_{n-1})$$

is one-to one. Define $\Phi := \Psi^{-1}$. Note that $\Phi$ is a $C^2$ function which transforms the set $\Omega':= \Omega \cap B_R(\bar{x})$ (in what we refer to as $x$ space) into a set $\Omega''$ in the half-space $y_n > 0$ (of $y$ space). Note also that the point $\bar{x}$ is mapped to the origin of $y$ space (cf. Figure 9.1).
Our task now is obvious (and obviously unpleasant). We must change the differential equation \( L(x, D)u = f \) into \( y \) coordinates. To facilitate this task we define the following notation: for any function \( v : \Omega' \rightarrow \mathbb{R} \), we define
\[
\tilde{v} : \Omega'' \rightarrow \mathbb{R}
\]
by
\[
\tilde{v}(y) := v(\Phi(y)).
\]
Note that for any function \( v \in L^2(\Omega) \) there are constants \( c_1 \) and \( c_2 \) such that
\[
c_1 \|v\|_{L^2(\Omega')} \leq \|\tilde{v}\|_{L^2(\Omega'')} \leq c_2 \|v\|_{L^2(\Omega')}.
\]

The action of the change of variables on our partial differential operator is described by the following lemma.

**Lemma 9.56.** Let \( u \in H^1(\Omega') \) satisfy \( u \equiv 0 \) (in the sense of trace) on \( \partial\Omega \cap \partial\Omega' \) and let \( u \) be a solution of the variational equation
\[
B[v, u] = (f, v),
\]
for all \( v \in H^1_0(\Omega') \). Then \( \tilde{u} \in H^1(\Omega'') \) satisfies \( \tilde{u} \equiv 0 \) on \( \partial\Omega'' \cap \{y \mid y_n = 0\} \) and \( \tilde{u} \) is a solution of the variational equation
\[
\tilde{B}[\tilde{v}, \tilde{u}] = (\tilde{f}, \tilde{v}),
\]
for every \( \tilde{v} \in H^1_0(\Omega'') \). Here
\[
\tilde{B}[\tilde{v}, \tilde{w}] := -\sum_{k,l=1}^n \int_{\Omega''} \bar{a}_{kl}(y) \tilde{w}_{y_l} \tilde{v}_{y_k} \, dy + \sum_{k=1}^n \int_{\Omega''} \bar{b}_k(y) \tilde{w}_{y_k} \tilde{v} \, dy
\]
\quad + \int_{\Omega''} \bar{c}(y) \tilde{v} \tilde{w} \, dy,
\]
\begin{align*}
\quad + \int_{\Omega''} \bar{c}(y) \tilde{v} \tilde{w} \, dy,
\end{align*}

\[
\quad + \int_{\Omega''} \bar{c}(y) \tilde{v} \tilde{w} \, dy,
\]

\[
\quad + \int_{\Omega''} \bar{c}(y) \tilde{v} \tilde{w} \, dy,
\]

\[
\quad + \int_{\Omega''} \bar{c}(y) \tilde{v} \tilde{w} \, dy,
\]

\[
\quad + \int_{\Omega''} \bar{c}(y) \tilde{v} \tilde{w} \, dy,
\]
with
\[
\bar{a}_{kl} := \sum_{i,j=1}^{n} \tilde{a}_{ij}(y) \frac{\partial \Psi_k}{\partial x_i}(\Phi(y)) \frac{\partial \Psi_l}{\partial x_j}(\Phi(y)),
\]
(9.219)
\[
\bar{b}_k := \sum_{i=1}^{n} \tilde{b}_i(y) \frac{\partial \Psi_k}{\partial x_i}(\Phi(y)),
\]
(9.220)
\[
\bar{c}(y) := \tilde{c}(y).
\]
(9.221)

The proof uses standard techniques and is left to the reader.

Before applying Lemma 9.54 we need to show that the transformed differential operator is uniformly elliptic.

**Lemma 9.57.** The operator \( \tilde{L} \) defined by
\[
\tilde{L}(y,D)v := \sum_{k,l=1}^{n} (\bar{a}_{kl}(y)v_{y_l})_{y_k} + \sum_{k=1}^{n} \bar{b}_k(y)v_{y_k} + \bar{c}(y)v
\]
(9.222)
is uniformly elliptic in \( \Omega'' \).

**Proof.** We must show that there exists a constant \( \tilde{\theta} > 0 \) such that
\[
-\sum_{k,l=1}^{n} \bar{a}_{kl}(y)\xi_k\xi_l \geq \tilde{\theta}|\xi|^2
\]
(9.223)
for every \( \xi \in \mathbb{R}^n \) and every \( y \in \Omega'' \).

For any \( \xi \in \mathbb{R}^n \) let \( \eta := A\xi \) where
\[
A(y) := \nabla \Psi(\Phi(y))^T
= \begin{pmatrix}
1 & 0 & \cdots & 0 & -\psi_{x_1} \\
0 & 1 & \cdots & 0 & -\psi_{x_2} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 1 & -\psi_{x_{n-1}} \\
0 & 0 & \cdots & 0 & 1
\end{pmatrix}.
\]
Note that \( A(y) \) is invertible. Let
\[
\tilde{C} := \sup_{y \in \Omega''} |A^{-1}(y)|.
\]
(9.224)
Then
\[
|\xi| \leq \tilde{C}|\eta|.
\]
(9.225)
Now, using (9.225) and the uniform ellipticity of $L$ we get
\[ -\sum_{k,l=1}^{n} \bar{a}_{kl} \xi_k \xi_l = -\sum_{i,j=1}^{n} a_{ij}(\phi(y)) \left( \frac{\partial \Psi_k}{\partial x_i}(\Phi(y)) \xi_k \right) \left( \frac{\partial \Psi_l}{\partial x_j}(\Phi(y)) \xi_l \right) \]
\[ = -\sum_{i,j=1}^{n} a_{ij}(\phi(y)) \eta_i \eta_j \]
\[ \geq \theta |\eta|^2 \]
\[ \geq \frac{\theta}{C^2} |\xi|^2. \]
Thus, $\tilde{L}$ is uniformly elliptic with constant $\tilde{\theta} := \frac{\theta}{\tilde{C}^2}$. \qed

We can now put the previous lemmas together to get the following result.

**Lemma 9.58.** Let the hypotheses of Theorem 9.53 be satisfied. Then for each $\bar{x} \in \partial \Omega$ there exists an open set $\tilde{Q} \subset \mathbb{R}^n$ containing $\bar{x}$ such that $u \in H^2(\tilde{Q} \cap \Omega)$, and furthermore
\[ \|u\|_{H^2(\tilde{Q} \cap \Omega)} \leq C(\|f\|_{L^2(\Omega')} + \|u\|_{H^1(\Omega')}). \] (9.226)

**Proof.** For each $\bar{x} \in \partial \Omega$ we let the sets $\Omega'$ in $x$ space, $\Omega''$ in $y$ space and the maps $\Psi : \Omega' \to \Omega''$ and $\Phi : \Omega'' \to \Omega'$ be defined as above (cf. Figure 9.1). Let $R$ be such that $B_R(0) \cap \{y \mid y_n > 0\} \subset \Omega''$ and define
\[ Q^+ := B_R(0) \cap \{y \mid y_n > 0\}, \] (9.227)
\[ \tilde{Q} := \Phi(B_R(0)), \] (9.228)
\[ \tilde{Q}^+ := \Phi(Q^+). \] (9.229)
Now, we can use Lemmas 9.54 and 9.57 to get
\[ \|\tilde{u}\|_{H^2(Q^+)} \leq C(\|f\|_{L^2(\Omega'')} + \|\tilde{u}\|_{H^1(\Omega'')}). \] (9.230)
From inequalities such as (9.215) we get
\[ \|u\|_{H^2(\tilde{Q}^+)} \leq C(\|f\|_{L^2(\Omega')} + \|u\|_{H^1(\Omega')}), \] (9.231)
which leads immediately to (9.226). \qed

We now prove Theorem 9.53.

**Proof.** It is now a simple matter to put together the proof of the global regularity theorem. We simply provide an open cover for $\overline{\Omega}$ using the neighborhoods $\tilde{Q}$ constructed in Lemma 9.58 for each point $\bar{x} \in \partial \Omega$ and one additional set $\Omega_0 \subset \subset \Omega$ to cover the interior. Since $\overline{\Omega}$ is compact, there is a finite subcover (in which we assume $\Omega_0$ is included and which we label $\{\Omega_i\}_{i=0}^{N}$) such that
\[ \Omega \subset \bigcup_{i=0}^{N} \Omega_i. \] (9.232)
Now, using the interior regularity result (Theorem 9.51) for $\Omega_0$ and Lemma 9.58 for each of the other sets we get

$$\|u\|_{H^2(\Omega)} \leq \sum_{i=0}^{N} \|u\|_{H^2(\Omega_i)} \leq C(\|f\|_{L^2(\Omega)} + \|u\|_{H^1(\Omega)}).$$  \hfill (9.233)

A standard application of Ehrling’s lemma gives us the final result. \qed