(c) If $Q(x,t) = Q(x)$, take the limit as $t \to \infty$ of part (b) in order to determine the Green's function for

$$\frac{d^2u}{dx^2} = f(x) \quad \text{with} \quad u(0) = 0 \quad \text{and} \quad \frac{du}{dx}(L) = 0.$$

(a) Derive (9.3.29) from (9.3.28) [*Hint:* Let $f(x) = 1$.]

(b) Show that (9.3.33) satisfies (9.3.31).

(c) Derive (9.3.30) [*Hint*: Show for any continuous $f(x)$ that

$$\int_{-\infty}^{\infty} f(x_0)\delta(x - x_0)\, dx_0 = \int_{-\infty}^{\infty} f(x_0)\delta(x_0 - x)\, dx_0$$

by letting $x_0 - x = s$ in the integral on the right.]

(d) Derive (9.3.34) [*Hint:* Evaluate $\int_{-\infty}^{\infty} f(x)\delta[c(x - x_0)]\, dx$ by making the change of variables $y = c(x - x_0)$.]

Consider

$$\frac{d^2u}{dx^2} = f(x) \quad \text{with} \quad u(0) = 0 \quad \text{and} \quad \frac{du}{dx}(L) = 0.$$

*(a) Solve by direct integration.

*(b) Solve by the method of variation of parameters. $\longrightarrow$ $u(x_0) = \int_0^L G(x, x_0) f(x) dx$

*(c) Determine $G(x, x_0)$ so that (9.3.15) is valid.

(d) Solve by the method of eigenfunction expansion. Show that $G(x, x_0)$ is given by (9.3.23).

Consider

$$\frac{d^2G}{dx^2} = \delta(x - x_0) \quad \text{with} \quad G(0, x_0) = 0 \quad \text{and} \quad \frac{dG}{dx}(L, x_0) = 0.$$

*(a) Solve directly.

*(b) Graphically illustrate $G(x, x_0) = G(x_0, x)$.

(c) Compare to Exercise 9.3.5.

Redo Exercise 9.3.5 with the following change: $\dfrac{du}{dx}(L) + hu(L) = 0,\ h > 0.$

Redo Exercise 9.3.6 with the following change: $\dfrac{dG}{dx}(L) + hG(L) = 0,\ h > 0.$

Consider

$$\frac{d^2u}{dx^2} + u = f(x) \quad \text{with} \quad u(0) = 0 \quad \text{and} \quad u(L) = 0.$$

Assume that $(n\pi/L)^2 \neq 1$ (i.e., $L \neq n\pi$ for any n).

(a) Solve by the method of variation of parameters.

*(b) Determine the Green's function so that $u(x)$ may be represented i terms of it [see (9.3.15)].

9.3.10. Solve the problem of Exercise 9.3.9 using the method of eigenfunction ex pansion.

9.3.11. Consider

$$\frac{d^2G}{dx^2} + G = \delta(x - x_0) \quad \text{with} \quad G(0, x_0) = 0 \quad \text{and} \quad G(L, x_0) = 0.$$

*(a) Solve for this Green's function directly. Why is it necessary to assum that $L \neq n\pi$?

(b) Show that $G(x, x_0) = G(x_0, x)$.

9.3.12. For the following problems, determine a representation of the solution i terms of the Green's function. Show that the nonhomogeneous boundar conditions can also be understood using homogeneous solutions of the dif ferential equation:

(a) $\dfrac{d^2u}{dx^2} = f(x),\ u(0) = A,\ \dfrac{du}{dx}(L) = B.$ (See Exercise 9.3.6.)

(b) $\dfrac{d^2u}{dx^2} + u = f(x),\ u(0) = A,\ u(L) = B.$ Assume $L \neq n\pi$. (See Exercis 9.3.11.)

(c) $\dfrac{d^2u}{dx^2} = f(x),\ u(0) = A,\ \dfrac{du}{dx}(L) + hu(L) = 0.$ (See Exercise 9.3.8.)

9.3.13. Consider the one-dimensional infinite space wave equation with a periodi source of frequency ω:

$$\frac{\partial^2\phi}{\partial t^2} = c^2\frac{\partial^2\phi}{\partial x^2} + g(x)e^{-i\omega t}. \tag{9.3.53}$$

(a) Show that a particular solution $\phi = u(x)e^{-i\omega t}$ of (9.3.53) is obtaine if u satisfies a nonhomogeneous Helmholtz equation

$$\frac{d^2u}{dx^2} + k^2u = f(x).$$

*(b) The Green's function $G(x, x_0)$ satisfies

$$\frac{d^2G}{dx^2} + k^2G = \delta(x - x_0).$$

Determine this infinite space Green's function so that the correspondin $\phi(x,t)$ is an outward-propagating wave.

(c) Determine a particular solution of (9.3.53).