1. Let $J$ be the set of real $2 \times 2$ matrices $J$, with the defining property

$$J^2 = -I.$$  

Such matrices $J$ define complex multiplication on $\mathbb{R}^2$.

(a) Show that $J$ is a smooth manifold of dimension 2.

(b) Construct a smooth embedding $\iota : J \to \mathbb{R}^3$.

(c) Prove that $J$ is diffeomorphic to $\mathbb{R}^2 \sqcup \mathbb{R}^2$.

2. Given the 2-torus

$$\mathbb{T}^2 = \{(x, y, z, w) : x^2 + y^2 = 1, \quad z^2 + w^2 = 1\} \subset \mathbb{R}^4$$

and consider the 1-form

$$\sigma = -x^3 dy + y^3 dx + z^3 dw - w^3 dz$$

on $\mathbb{T}^2$.

(a) Show that $\mathbb{T}^2$ is a smooth manifold of dimension 2.

(b) Prove that $\sigma$ is a smooth 1-form on $\mathbb{T}^2$.

(c) Let $S^1 = \{(p, q) \in \mathbb{R}^2 : p^2 + q^2 = 1\}$ and consider the mapping $g : S^1 \to \mathbb{T}^2$ given by

$$g(p, q) = (p, q, (q - p)/\sqrt{2}, (p + q)/\sqrt{2}).$$

Compute the pullback form $g^* \sigma$ on $S^1$.

Good luck