
Make up exam
Introduction Partial Differential Equations
for students Mathematics and Physics

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Date: Tuesday February 8, 18:30 - 21:30 (3 hours)

Instructions: 5 problems; *motivate all answers*.

No calculators, no books, no formula sheets.

1. (a) Calculate Fourier transform of the following functions

$$f(x) = 2e^{-x^2+4x}, \quad g(x) = e^{-2x}, \quad 1 < x < 4, \quad \text{and } g(x) = 0 \text{ elsewhere.}$$

- (b) Compute, using the definition, the Laplace transform of

$$g(x) = x \sin(2x).$$

- (c) Find the inverse Fourier transform

$$\mathcal{F}^{-1}\left(\frac{2}{\alpha^2 - 2\alpha + 2}\right).$$

2. Consider the fourth order equation

$$u_t = u_{xx} + u, \quad x \in [0, \pi], \quad t > 0$$

with initial condition $u(0, x) = f(x) \in C^0([0, \pi])$ and boundary conditions $u(t, 0) = 0$, and $u(t, \pi) = 0$.

- (a) Give the general solution using separation of variables.
(b) Derive the formulas for the Fourier coefficients in terms of the initial function f .
(c) Let $f(x) = 1$ on $[0, \pi]$. Compute the solution $u(t, x)$.
(d) Prove that $\lim_{t \rightarrow \infty} u(t, x) = \frac{4}{\pi} \sin(x)$ uniformly in $x \in [0, \pi]$.

3. Consider the elliptic equation on a domain $\Omega \subset \mathbb{R}^2$

$$\begin{aligned} Lu &= \Delta u + 2u = f, & \mathbf{x} \in \Omega, \\ Bu &= u + \frac{\partial u}{\partial n} = g, & \mathbf{x} \in \partial\Omega, \end{aligned}$$

where \mathbf{n} the outward unit normal on $\partial\Omega$, and $\frac{\partial u}{\partial n} = \nabla u \cdot \mathbf{n}$.

- (a) Show that the adjoint differential operator L^* is given by

$$L^* \phi = \Delta \phi + 2\phi$$

by establishing the identity

$$\int_{\Omega} [\phi L\psi - \psi L^*\phi] d_{\Omega} \mathbf{x} = 0,$$

for all $\phi, \psi \in C_0^\infty(\Omega)$. Recall that $C_0^\infty(\Omega)$ are infinitely smooth functions whose support is strictly contained in Ω ; no boundary contributions.

(b) Derive the Green's identity

$$\int_{\Omega} [vLu - uL^*v] d_{\Omega}\mathbf{x} = \dots$$

(c) Find the adjoint boundary operator B^* .

(d) Formulate the differential equation for the Green's function $G(\mathbf{x}, \mathbf{y})$.

(e) Assuming that the Green's function G exists, write down a representation formula for the solution $u(\mathbf{x})$.

4. (a) Determine the general solution of the following first order equation

$$u_x + yu_y = 0.$$

(b) Find the solution that matches the the initial condition $u(0, y) = \sin(y)$.

5. (a) Calculate the Fourier series of x^2 on $[-\pi, \pi]$.

(b) Compute the sine Fourier series of the function $f(x) = 1$ on $[0, \pi]$.

The cosine Fourier series for f are $\sum_{n=0}^{\infty} a_n \cos(nx)$, $a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos(nx) dx$, $n \geq 1$, and $a_0 = \frac{1}{\pi} \int_0^{\pi} f(x) dx$.

The sine Fourier series for f are $\sum_{n=1}^{\infty} a_n \sin(nx)$, $a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin(nx) dx$, $n \geq 1$.

The Fourier series for f are $\frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos(nx) + b_n \sin(nx)]$, $a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) dx$, and $b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx) dx$.

The Fourier transform of f is given by $\mathcal{F}(\alpha) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{-i\alpha x} dx$, and the Laplace transform by $\hat{f}(s) = \int_0^{\infty} f(x) e^{-sx} dx$.

Good luck