
Assignment III
Introduction Partial Differential Equations
for students Mathematics and Physics

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Due: Friday December 16, 2005.

Instructions: include input and output; *motivate all answers*.

Individual assignment, written in English.

Given the eigenvalue problem

$$\begin{aligned}\varphi'' &= -\lambda\varphi, & x \in (0, \pi), \\ \varphi(0) &= \varphi(\pi) = 0.\end{aligned}$$

The eigenfunctions are given by $\varphi_n(x) = \sin(nx)$, and the associated eigenvalues are $\lambda_n = n^2$, $n = 1, 2, \dots$. Given a function $v \in L^2(0, 1)$, then $v(x) = \sum_{n=1}^{\infty} v_n \sin(nx)$, with $v_n = \frac{2}{\pi} \int_0^{\pi} v(x) \sin(nx) dx$, and $\int_0^{\pi} v^2(x) dx = \sum_{n=1}^{\infty} v_n^2 = \frac{\pi}{2} < \infty$.

Consider the equation

$$\begin{aligned}-u'' + u &= f(x), & x \in (0, \pi), \\ u(0) &= u(\pi) = 0.\end{aligned}$$

From general theory we know that there exists a Green's function $G(x, y) = G(y, x)$ on $[0, \pi] \times [0, \pi]$, such that

$$u(x) = \int_0^{\pi} G(x, y) f(y) dy.$$

1. Use the above described eigenfunction expansion and use the eigenfunction expansion method to derive a Fourier series for the Green's function G .
2. Use MAPLE or MATHEMATICA with $N = 100$ modes to plot the Green's function G in a 3d graph.
3. Use the above to compute and plot the solution of $-u'' + u = 1$. Compare the graphs of the approximate solution with the exact solution

$$u(x) = \frac{\cosh(\pi) - 1}{\sinh(\pi)} \sinh(x) - \cosh(x) + 1.$$

Good luck