Problem 5.

(Problem 2 in Evans, page 345)

A function $u \in H_0^2(U)$ is a weak solution of the boundary value problem

$$\Delta^2 u = f \quad \text{in } \; U, \quad \text{and} \;\; u|_{\partial U} = \frac{\partial u}{\partial n}\Big|_{\partial U} = 0,$$

if

$$\int_U \Delta u \Delta v dx = \int_U f v dx, \quad \forall v \in H_0^2(U).$$

(a) Show that Δ^2 is an elliptic partial differential operator.

(b) Prove that for each $f \in L^2(U)$ the above boundary value problem has a unique weak solution (Hint: for $H_0^2(U)$ the following Poincaré inequality holds $\|u\|_{L^2(U)} \leq C\|\Delta u\|_{L^2(U)}$ for all $u \in H_0^2(U)$).