**Problem 6.**
(Evans problems 11 and 12, page 427)

1. Let $\{S(t)\}_{t\geq 0}$ be a contraction semi-group on a real Banach space $X$, with infinitesimal generator $A : D(A) \subset X \to X$. Show that the following identity holds:

$$A \int_0^\infty e^{-\lambda t} S(t) u dt = \int_0^\infty e^{-\lambda t} A S(t) u dt,$$

for all $u \in D(A)$.

2. Show that for $\lambda, \mu \in \rho(A)$ it holds that

$$R_\lambda - R_\mu = (\mu - \lambda) R_\lambda R_\mu,$$

and

$$R_\lambda R_\mu = R_\mu R_\lambda,$$

where $R_\lambda = (\lambda I - A)^{-1} : X \to X$ is the resolvent operator at $\lambda$.

3. Consider the parabolic equation

$$u_t + \Delta^2 u = 0,$$

on $D_T = D \times (0, T)$, where $D \subset \mathbf{R}^m$ is a bounded open set with smooth boundary. Consider the boundary conditions: $u(x, 0) = g \in L^2(D)$ and $u = \frac{\partial u}{\partial \mathbf{n}} = 0$ on $\partial D \times [0, T]$. Show the existence of a weak solution (Hint: show that $-\Delta^2$ generates a contraction semi-group on $L^2(U)$).