

Branching Time Semantics for the Dynamics of Reasoning by Default

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Abstract. In this paper we formalize default reasoning using branching time temporal models, in which an information state at a certain point in time describes what has been derived up until that moment. The branching character of the models reflects the fact that at a certain point in the reasoning process there might be a number of (conflicting) default rules which can be selected to be applied. We show how one can construct a branching time model, in which all possible reasoning patterns are brought together, making explicit the time points at which choices have to be made. The semantics of default reasoning is defined using this model.

1 Introduction

An important characteristic of default reasoning is that usually there are different lines of reasoning possible, each leading to a set of conclusions. In default logic these conclusion sets are described by (Reiter) extensions. In common examples this leads to a variety of extensions. In logic one is used to express semantics in terms of models that represent consistent descriptions of the world and semantic entailment relations based on a specific class of this type of models. These notions are not really adequate to describe alternative conclusion sets for default reasoning. Sometimes one introduces sceptical entailment (what is true in all conclusion sets) or credulous entailment (what is true in some conclusion set). From a semantic point of view both notions only give a

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limited description: they only indicate global upper and lower bounds for the conclusion set of particular lines of reasoning. In this paper we formalize default reasoning processes by temporal models. This enables us to integrate process aspects of the reasoning in the semantics in an explicit manner. Our approach extends the one introduced in [ET93, ET98], where it was shown how one line of default reasoning corresponds to one linear time model. In the current paper the branching character of the reasoning processes is described by a branching time temporal model. Each line of reasoning corresponds to a branch in the temporal model. We show how (under a particular topological condition, called extension completeness) one branching time model can be constructed in which precisely all possible lines of reasoning (and the resulting conclusion sets) can be represented (even though they might be mutually contradictory). The semantics of the default theory can be defined on the basis of this single model. In particular, we show how sceptical and credulous entailment relations can be defined as well on the basis of this model.

In Section 2 we will define the temporal logic we will use later. Section 3 begins with a brief introduction of Reiter's Default Logic and gives an interpretation of a default theory in temporal logic. Section 4 describes the construction of branching time models reflecting extensions. In Section 5 we describe entailment relations based on these models. The special case of normal default theories is treated in Section 6, after which conclusions follow in Section 7. A preliminary version of this work appeared as [ET96].

2 Branching Time Temporal Logic

In this section we introduce the temporal logic that we have defined to satisfy our requirements. The base language will consist of all classical propositional formulae of a certain signature Σ , an ordered sequence of atom names. The formulae in propositional logic based on Σ will be called propositional formulae.

Definition 2.1 (Information State)

a) An *information state*, or shortly state, is a non-empty closed set of propositional models, that is, there is a consistent set of formulae of which it is the model class.

The truth of a propositional formula α in an information state \mathbf{M} , denoted $\mathbf{M} \models \alpha$, is defined by:

$$M \models \alpha \Leftrightarrow m \models \alpha \text{ for all } m \in M$$

- b) The *theory of a state* M , denoted $\text{Th}(M)$ is defined by $\text{Th}(M) = \{ \alpha \mid M \models \alpha \}$.
- c) We call the state N a *refinement* of the state M , denoted by $M \leq N$, if $M \supseteq N$.
- d) For a set of formulae S , the *set of models of S* is denoted by $\text{Mod}(S)$.
- e) For a set of formulae S , the *deductive closure of S* is denoted by $\text{Cn}(S)$.
- f) The set of information states is denoted IS .

Note that for a consistent set S , $\text{Mod}(S)$ is an information state, and if $S \subseteq T$ then $\text{Mod}(S) \leq \text{Mod}(T)$.

We will now temporalize (see [FG92]) these states to temporal models, based on some flow of time.

Definition 2.2 (Flow of time)

A flow of time is a pair $(T, <)$ where T is a non-empty set of time points and $<$ is a binary relation over T , called the immediate successor relation. Here for s, t in T the expression $s < t$ denotes that t is an (immediate) successor of s , and that s is an (immediate) predecessor of t . In this paper we only consider forward branching structures: $(T, <)$ viewed as a graph has to be a forest, that is a disjoint union of trees, satisfying successor existence: each time point must have at least one successor. Furthermore the transitive (but not reflexive) closure \ll of $<$ is introduced. A flow of time is called linear if \ll is a total ordering. A time point without predecessor is called a root. A branch in a forest is a branch of any of its trees, that is an infinite path starting at a root.

Definition 2.3 (Temporal model)

Let Σ be a signature and $(T, <)$ a flow of time.

- a) A (propositional) temporal model of signature Σ and flow of time $(T, <)$ is a triple $(M, T, <)$ where $(T, <)$ is a flow of time and M is a mapping $M : T \rightarrow \text{IS}$, called the state assignment.

If no confusion is expected we will often denote a temporal model by M . Moreover, instead of t is a time point in $(M, T, <)$ we sometimes say t is a time point in M (or simply t in M), with meaning $t \in T$.

- b) We sometimes will use the notation $(M_t)_{t \in T}$ where each M_t is a state as an equivalent description of a temporal model M .

- c) A temporal model $(\mathbf{B}, \mathbf{T}', <')$ is called a branch of $(\mathbf{M}, \mathbf{T}, <)$ if $(\mathbf{T}', <')$ is a branch of $(\mathbf{T}, <)$ and $\mathbf{B}_t = \mathbf{M}_t$ for all $t \in \mathbf{T}'$.
- d) If \mathbf{K} is a set of propositional formulae for the signature Σ , a temporal model \mathbf{M} of signature Σ is called a model of \mathbf{K} if all formulae of \mathbf{K} are true in \mathbf{M}_t for all $t \in \mathbf{T}$. This is denoted by $\mathbf{M} \models \mathbf{K}$.
- e) The refinement relation \leq between temporal models is defined by: $\mathbf{M} \leq \mathbf{N}$ if they have the same flow of time and $\mathbf{M}(t) \leq \mathbf{N}(t)$ for all time points t .

The basic building blocks of our temporal language will be temporal operators applied to propositional formulae. Using these temporal "atoms" we can build complex formulae using the usual connectives and the temporal operators. Because our branching time models have a more differentiated structure towards the future than in the past, there are more operators for the future.

Definition 2.4 (Temporal language)

The temporal language L_T is the smallest set closed under:

- i) If α is a propositional formula, then $\mathbf{O}\alpha \in L_T$ for $\mathbf{O} \in \{ \exists F, \forall F, \exists G, \forall G, \mathbf{P}, \mathbf{C} \}$;
- ii) If $\phi, \psi \in L_T$ then $\neg\phi, \phi \wedge \psi, \phi \vee \psi, \phi \rightarrow \psi, \mathbf{O}\phi \in L_T$ for $\mathbf{O} \in \{ \exists F, \forall F, \exists G, \forall G, \mathbf{P}, \mathbf{C} \}$.

We will now give the semantics of our temporal logic. In these definitions, $(\mathbf{M}, t) \models \phi$ means that in the model \mathbf{M} at time point t the formula ϕ is true and $(\mathbf{M}, t) \not\models \phi$ means that this is not the case. For uniformity in notation we will also define this for propositional formulae.

Definition 2.5 (Semantics)

Let a temporal model \mathbf{M} and a time point $t \in \mathbf{T}$ be given, then:

- a) For a propositional formula α :

$$(\mathbf{M}, t) \models \alpha \Leftrightarrow \mathbf{M}_t \models \alpha$$

- b) For ϕ either propositional or in L_T :

$$(\mathbf{M}, t) \models \exists F\phi \Leftrightarrow \exists s \in \mathbf{T} [t \ll s \ \& \ (\mathbf{M}, s) \models \phi]$$

$$(\mathbf{M}, t) \models \forall F\phi \Leftrightarrow \text{for all branches through } t \text{ there exists an } s \text{ in that branch such that } [t \ll s \ \& \ (\mathbf{M}, s) \models \phi]$$

$$(\mathbf{M}, t) \models \forall G\phi \Leftrightarrow \forall s \in \mathbf{T} [t \ll s \Rightarrow (\mathbf{M}, s) \models \phi]$$

$$(\mathbf{M}, t) \models \exists G\phi \Leftrightarrow \text{there exists a branch through } t \text{ such that for all } s \text{ in that branch } [t \ll s \Rightarrow (\mathbf{M}, s) \models \phi]$$

$$(M, t) \models P\phi \Leftrightarrow \exists s \in T [s \ll t \ \& \ (M, s) \models \phi]$$

$$(M, t) \models C\phi \Leftrightarrow (M, t) \models \phi$$

c) For $\phi, \psi \in L_T$:

$$(M, t) \models \phi \wedge \psi \Leftrightarrow (M, t) \models \phi \ \text{and} \ (M, t) \models \psi$$

$$(M, t) \models \neg\phi \Leftrightarrow (M, t) \not\models \phi$$

d) For a temporal model M , by $M \models \phi$ we mean $(M, t) \models \phi$ for all $t \in T$ and by $M \models K$ we mean $M \models \phi$ for all $\phi \in K$, where K is a set of temporal formulae. We say that M is a model of K .

e) For a set K of temporal formulae, we say that a temporal model M is a minimal model of K if $M \models K$ and whenever a model $N \leq M$ is a model of K then $N = M$.

f) If T is a temporal theory, then by $\mathbf{LT}(T)$ we denote the set of all minimal linear time models of T .

Note that the case of negation in this definition does not hold for propositional formulae.

Suppose we want to express the fact that a propositional formula α should never be true in a model (meaning that it should be true in none of the information states at any point in time). Using the formula $\neg\alpha$ will ensure that α is never true, but also that $\neg\alpha$ is always true (in each information state). However, if we use the formula $\neg C\alpha$ then in no information state will α be true. This does not enforce $\neg\alpha$ to be true: the information state may contain models in which α is true (as long as it contains at least one model in which α is false). This explains the use of the C operator.

If an element t lies in a tree with root r , the length of the unique path of r to t is called the *depth* of the element t . This defines a mapping from the flow of time to the natural numbers; in the case of a branch this mapping is a successor relation isomorphism. We can identify a branch with a model based on the natural numbers as flow of time.

What is still interesting about a reasoning process, is of course its set of final conclusions. To be able to talk about *final* conclusions, we have to assume that the reasoning is *conservative*, which means that once a fact is established, it will remain true in the future of the reasoning process. In that case a fact is a final conclusion of a process if it is established at the branch representing the process at any point in time. So, besides reasoning paths also the conclusions they result in are defined in a branching time model in the following manner:

Definition 2.6 (Limit models of a conservative model)

Let M be a temporal model.

- a) M is conservative if $M_t \leq M_s$ whenever $t \ll s$.
- b) The set of branches of the model M is denoted by $\mathbb{B}(M)$.
- c) Let B be a branch of M , and identify its flow of time with the natural numbers.

The limit model of B , denoted by $\lim_B M$, is the information state defined by:

$$\lim_B M = \bigcap_{i=0}^{\infty} B_i$$

If $B = M$, we will simply write $\lim M$.

Note that the intersection of a decreasing sequence of information states is indeed an information state, and that $\text{Th}(\lim_B M) = \bigcup_{i=0}^{\infty} \text{Th}(B_i)$.

3 Interpreting Default Logic in Temporal Logic

We will first give a brief overview of Reiter's default logic, restricted to a propositional language. A *default rule* is an expression of the form $(\alpha : \beta) / \gamma$, where α , β and γ are propositional formulae. A *default theory* Δ is then a pair $\langle W, D \rangle$ where W is a set of sentences (the *axioms* of Δ), and D a set of default rules. We will not give Reiter's original definition of an extension (see [Be89], [Re80]), but a slight variation of it, which in [ET93, ET98] has been shown to be equivalent.

Definition 3.1 (Reiter Extension)

Let $\Delta = \langle W, D \rangle$ be a default theory of signature Σ , and let E be a set of sentences for Σ . Then E is a Reiter extension of Δ if $E = \bigcup_{i=0}^{\infty} E_i$ where $E_0 = \text{Cn}(W)$, and for all $i \geq 0$: $E_{i+1} = \text{Cn}(E_i \cup \{ \gamma \mid (\alpha : \beta) / \gamma \in D, \alpha \in E_i \text{ and } \neg\beta \notin E_i \})$

If E is a Reiter extension, then throughout the paper by E_i we will denote the subsets of E as defined in this lemma.

We will establish an interpretation mapping from default theories to temporal theories. Under this interpretation the Reiter extensions of a default theory and temporal models which obey a number of rules correspond to each other. The correspondence we are aiming at will be such that the propositional formulae true in a branch B of the temporal model at depth i will be exactly those which are element of E_i and the

formulae in E will be those true in the limit model of a branch B in M : $\text{Th}(B_i) = E_i$ and $\text{Th}(\lim_{\mathbf{B}} M) = E$, or $B_i = \text{Mod}(E_i)$ and $\lim_{\mathbf{B}} M = \text{Mod}(E)$

We will investigate what requirements should be imposed on the temporal model M . Firstly, since the E_i are non-decreasing, our model should be conservative. If we define

$$C' = \{ P(\alpha) \rightarrow C(\alpha) \mid \alpha \text{ propositional formula} \}$$

it can easily be shown that $M \models C'$ if and only if M is conservative. Next we will try to see which rules will ensure in the model the effect of application of the default rules. To this end we have to look at how a default rule is used in our definition of an extension. The meaning of a default rule $(\alpha : \beta) / \gamma$ is that if $\alpha \in E_i$, and $\neg \beta \notin E$, then γ has to be in E_{i+1} , and consequently in E_j for all $j > i$. The requirement $\neg \beta \notin E$ is equivalent to $\neg \beta \notin E_i$ for all $i \in \mathbb{N}$, and as the sets E_i are non-decreasing, it is equivalent to $\neg \beta \notin E_j$ for all $j > i$. If we want to enforce a corresponding effect of the use of defaults in our temporal model, we have to make sure that at all times, if α has become true in the past, and there is at least a reasoning path where $\neg \beta$ is not true at any point in the future, then γ has to be true in the current state. This leads us to the rule: $P\alpha \wedge \neg \forall F \neg \beta \rightarrow C\gamma$, which has to be true in the model at all time points. As the semi-construction of an extension starts with W , all formulae of W should be true in all roots of M . As the theory C' ensures conservativity, this is equivalent to saying that the formulae of W should be true in M_s for all $s \in T$.

Now we can define a temporal interpretation of Δ as a temporal theory associated to Δ . As we do not want any extra conclusions in the corresponding model than those which have to be drawn, we will take the minimal models with respect to \leq .

Definition 3.2 (Temporal interpretation of a default theory)

Let $\Delta = \langle W, D \rangle$ be a default theory of signature Σ . Define

$$\begin{aligned} C' &= \{ P\alpha \rightarrow C\alpha \mid \alpha \text{ propositional formula} \} \\ D' &= \{ P\alpha \wedge \neg \forall F \neg \beta \rightarrow C\gamma \mid (\alpha : \beta) / \gamma \in D \} \\ W' &= \{ C\alpha \mid \alpha \in W \} \end{aligned}$$

The temporal interpretation of Δ is the temporal theory $T_\Delta = C' \cup D' \cup W'$. The set of minimal linear time models of T_Δ is denoted by $\mathbf{LT}(\Delta)$.

This temporal interpretation enables us to attribute semantics to default reasoning based on temporal models. In a previous publication [ET93] it has been shown how a linear time partial temporal model of (a temporal interpretation of) a default theory can describe one line of reasoning (i.e., can play in a sense the role of a Reiter extension of

the default theory). In [ET93] we gave a treatment restricted to the linear time case, using three-valued partial models (in which an atom may have truth-value true, false or unknown) instead of information states, and a slightly different translation, but the result is easily transferred (see the proof in [ET98]):

Theorem 3.3

Let $\Delta = \langle W, D \rangle$ be a default theory.

- a) If M is a minimal linear time temporal model of T_Δ , then $\text{Th}(\lim M)$ is a Reiter extension E of Δ . Moreover, $E_i = \text{Th}(M_i)$ for all $i \in \mathbf{N}$.
- b) If W is consistent and E a Reiter extension of Δ , then the temporal model M defined by $M = (\text{Mod}(E_i))_{i \in \mathbf{N}}$ is a minimal linear time temporal model of T_Δ with $\text{Th}(\lim M) = E$.

Proof

We will only make some remarks about the different translation. In [ET93] and [ET98], only linear temporal models are considered, and a default rule $(\alpha : \beta) / \gamma$ is translated into the rule $C\alpha \wedge \neg F \neg \beta \rightarrow G\gamma$, where $F\phi$ means "sometimes in the future ϕ ", and $G\phi$ means "always in the future ϕ ". It is easy to see that on linear models, $\neg F \neg \beta$ is equivalent to $\neg \forall F \neg \beta$. We will show that any conservative linear model satisfies $C\alpha \wedge \neg F \neg \beta \rightarrow G\gamma$ if and only if it satisfies $P\alpha \wedge \neg \forall F \neg \beta \rightarrow C\gamma$. Suppose $M \models C\alpha \wedge \neg F \neg \beta \rightarrow G\gamma$ and that for some $t \in \mathbf{N}$, $(M, t) \models P\alpha \wedge \neg \forall F \neg \beta$. Remark that $(M, t) \models P\alpha$ implies that $t > 0$. Then it easily follows (given conservativity) that $(M, t-1) \models C\alpha \wedge \neg F \neg \beta$, so $(M, t-1) \models G\gamma$ whence $(M, t) \models C\gamma$. Now suppose that $M \models P\alpha \wedge \neg \forall F \neg \beta \rightarrow C\gamma$ and for some $t \in \mathbf{N}$, $(M, t) \models C\alpha \wedge \neg F \neg \beta$. Then $(M, t+1) \models P\alpha \wedge \neg \forall F \neg \beta$, so $(M, t+1) \models C\gamma$. With conservativity it follows that $(M, t) \models G\gamma$.

For the case of lines of default reasoning that do not stabilize after a finite number of steps, topological properties of the space of reasoning patterns become relevant. Before defining a metric on the space of linear time models, we recall the following definitions. A sequence $(a_i)_{i \in \mathbf{N}}$ in a metric space X with metric d is called *convergent* with limit $a \in X$ if for each $\varepsilon > 0$ there exists an $N \in \mathbf{N}$ such that for all $i \geq N$ it holds $d(a_i, a) < \varepsilon$. A subset Y of X is called *closed* if for every convergent sequence in X with all a_i in Y , its limit is included in Y .

Definition 3.4 (Metric)

Define the following metric d on the set of linear time models LT : for M, N linear models:

$$d(M, N) = \begin{cases} 0 & \text{if } M = N \\ 2^{-i} & \text{where } i = \sup\{j \in \mathbf{N} \mid \forall k \leq j : M_k = N_k\}, \text{ otherwise} \end{cases}$$

It is easy to see that the metric space (LT, d) is complete, i.e., that every Cauchy-sequence has a limit. The following definition will play an important role in the next section:

Definition 3.5 (Extension complete)

A default theory Δ is called *extension complete* if $LT(\Delta)$ is a closed subset of the metric space (LT, d) .

Proposition 3.6

Every default theory with a finite set of defaults is extension complete.

Proof

A default theory Δ with a finite set of defaults has finitely many extensions (this follows easily from the fact that every extension is the propositional closure of W and the set of generating defaults, see [Re80]), so by Theorem 3.3 $LT(\Delta)$ is finite. In a metric space, all finite sets are closed.

As an example of a default theory which is *not* extension complete, let

$$W = \{a_0\} \cup \{b \rightarrow a_i \mid i \in \mathbf{N}\} \text{ and} \\ D = \{ :b/b \} \cup \{ a_i : a_{i+1} / a_{i+1} \mid i \in \mathbf{N} \} \cup \{ a_i : \neg a_{i+1} / \neg a_{i+1} \mid i \in \mathbf{N} \}.$$

This (normal) default theory has infinitely many extensions: $F = Cn(W \cup \{b\})$ and for each $n \in \mathbf{N}$, $E^{(n)} = Cn(W \cup \{a_i \mid i \leq n\} \cup \{\neg a_{n+1}\})$. In this example the linear time models corresponding to these extensions form a convergent sequence in (LT, d) , but its limit is the model $(M_t)_{t \in \mathbf{N}}$, with $M_t = Mod(W \cup \{a_i \mid i \in \mathbf{N}\})$, which is not in $LT(\Delta)$ (See Figure 1). In Figure 1, we have indicated the (non-trivial) formulas that are true in the various time points, where a formula is not repeated if it was true earlier.

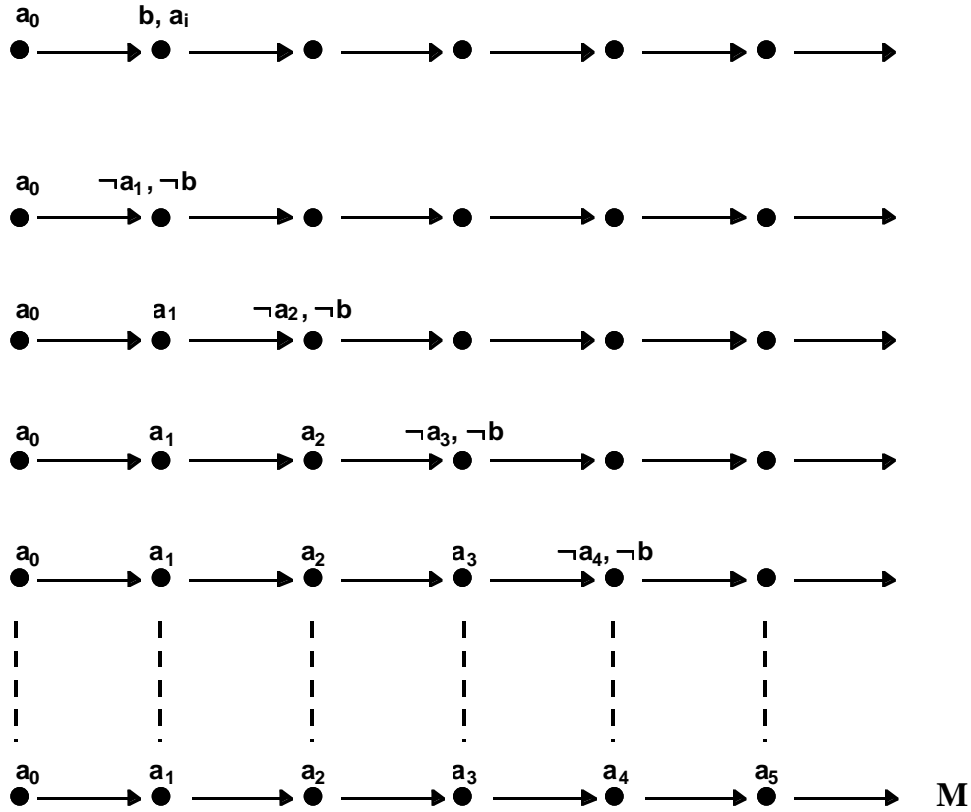


Figure 1. Not extension complete theory

4 Joint Embeddings of Linear Time Models of Default Theories

In the previous section we summarized results showing a correspondence between the set of Reiter extensions of a default theory Δ and the set of minimal linear time models of its temporal interpretation. These results provide semantics for default reasoning in the form of a set of linear time models that represent the possible default reasoning patterns. An alternative manner of representing these reasoning patterns is by means of one branching time model, where each branch represents one alternative reasoning pattern (with a Reiter extension as its limit). This would provide semantics for default reasoning in the form of one "standard" model. The aim of this section is for any given default theory to indeed construct such a branching time model, under certain conditions (extension completeness). To this end we apply some algebraic (category-theoretic) techniques developed in [ET02] to the model theory of the temporal translation of default theories.

Subsequently \mathbf{M} and \mathbf{M}' will be temporal models based on the flows of time $(\mathbf{T}, <)$, $(\mathbf{T}', <')$ respectively.

Definition 4.1 (Homomorphism)

- a) A mapping $f: \mathbf{T} \rightarrow \mathbf{T}'$ is called a *homomorphism* of \mathbf{M} to \mathbf{M}' (also denoted $f: \mathbf{M} \rightarrow \mathbf{M}'$) if
- (i) $s < t \Rightarrow f(s) <' f(t)$ for all $s, t \in \mathbf{T}$
 - (ii) $\mathbf{M}(s) = \mathbf{M}'(f(s))$ for all $s \in \mathbf{T}$
 - (iii) If s is a root of \mathbf{T} then $f(s)$ is a root of \mathbf{T}' .
- b) A homomorphism $f: \mathbf{T} \rightarrow \mathbf{T}'$ is called branch-surjective if for every branch \mathbf{B}' of \mathbf{M}' there exists a branch \mathbf{B} of \mathbf{M} such that $f[\mathbf{B}] = \mathbf{B}'$.

Note that a branch-surjective homomorphism is always surjective.

The *coproduct* $*\mathbf{C} = (\mathbf{M}, \mathbf{T}, <)$ of a set \mathbf{C} of temporal models is a temporal model which can be constructed by taking $(\mathbf{T}, <)$ as the disjoint union of the flows of time of the models in \mathbf{C} , and the union of the respective state assignments as \mathbf{M} . (See [ET02] for motivation and more properties of the coproduct.) This construction preserves minimal models:

Proposition 4.2

Let \mathbf{K} be a temporal theory and let \mathbb{B} be a set of models. Then all models in \mathbb{B} are minimal models of \mathbf{K} if and only if $*\mathbb{B}$ is a minimal model of \mathbf{K} .

Proof

Suppose \mathbb{B} is a set of minimal models of a temporal theory \mathbf{K} , and let $*\mathbb{B}$ be its coproduct. Since the evaluation of a formula in a point depends only on the connected component in which it lies, it is easy to see that $*\mathbb{B}$ is a model of \mathbf{K} . Now suppose that there is a smaller model \mathbf{M} of \mathbf{K} . Then there is a point s in \mathbf{M} such that $\mathbf{M}_s < (*\mathbb{B})_s$. This point s is an element of one of the models \mathbf{N} in \mathbb{B} . Now let's look at the model \mathbf{M}' which is the restriction of \mathbf{M} to the flow of time of \mathbf{N} . It is easy to verify that $\mathbf{M}' < \mathbf{N}$ and that \mathbf{M}' is a model of \mathbf{K} , contradicting the assumption that \mathbb{B} contains only minimal models of \mathbf{K} . Thus, $*\mathbb{B}$ is a minimal model of \mathbf{K} . For the other direction, suppose \mathbb{B} contains a model \mathbf{M} which is not a minimal model of \mathbf{K} . If it is not a model of \mathbf{K} , then it is easy to see that $*\mathbb{B}$ can not be a model of \mathbf{K} . Otherwise there is a model $\mathbf{N} < \mathbf{M}$ which is a model of \mathbf{K} .

Consider $\mathbb{B}' = (\mathbb{B} \setminus \{M\}) \cup \{N\}$. Then $*\mathbb{B}' < *\mathbb{B}$ and $*\mathbb{B}'$ is a model of \mathbf{K} , so $*\mathbb{B}$ is not a minimal model of \mathbf{K} .

As we want to study minimal models of \mathbf{T}_Δ and connections between them, the following proposition is useful:

Proposition 4.3

Let Δ be a default theory. If \mathbf{M} is a minimal model of \mathbf{T}_Δ and $f: \mathbf{M} \rightarrow \mathbf{M}'$ is branch-surjective then \mathbf{M}' is also a minimal model of \mathbf{T}_Δ .

Proof

a) Suppose \mathbf{M} has flow of time $(\mathbf{T}, <)$ and \mathbf{M}' has flow of time $(\mathbf{T}', <')$. First we will show that \mathbf{M}' is a model of \mathbf{T}_Δ . Take a point $s' \in \mathbf{T}'$. Then s' lies on at least one branch, say \mathbf{B}' . As f is branch-surjective, there must be a branch \mathbf{B} in \mathbf{M} such that $f[\mathbf{B}] = \mathbf{B}'$. Note that \mathbf{B}' is an isomorphic copy of \mathbf{B} . It follows that $(\mathbf{M}', s') \models \mathbf{W}$ and $(\mathbf{M}', s') \models \mathbf{C}$.

Now take a rule $\mathbf{P}\alpha \wedge \neg \forall \mathbf{F} \neg \beta \rightarrow \mathbf{C}\gamma$ in \mathbf{D}' , and suppose $(\mathbf{M}', s') \models \mathbf{P}\alpha \wedge \neg \forall \mathbf{F} \neg \beta$. This means that there must exist a branch \mathbf{B}' in \mathbf{M}' such that s' lies on \mathbf{B}' , there is a $t' \in \mathbf{B}'$ with $t' \ll s'$ and $\mathbf{M}'_{t'} \models \alpha$, and for all $u' \in \mathbf{B}'$: if $s' \ll u'$ then $\mathbf{M}'_{u'} \not\models \neg \beta$. Since f is branch-surjective, there is a branch in \mathbf{M} with $f[\mathbf{B}] = \mathbf{B}'$. Thus, there is a (unique) $s \in \mathbf{B}$ with $f(s) = s'$, and it is easy to verify that $(\mathbf{M}, s) \models \mathbf{P}\alpha \wedge \neg \forall \mathbf{F} \neg \beta$. But then $(\mathbf{M}, s) \models \mathbf{C}\gamma$, as \mathbf{M} is a model of \mathbf{T}_Δ , and therefore $(\mathbf{M}, s) \models \mathbf{C}\gamma$. We have proved that \mathbf{M}' is a model of \mathbf{T}_Δ .

Suppose that \mathbf{M}' is not minimal, then there exists a model $\mathbf{N}' < \mathbf{M}'$, such that $\mathbf{N}' \models \mathbf{T}_\Delta$. We will define a model \mathbf{N} of \mathbf{T}_Δ which is smaller than \mathbf{M} , contradicting the hypothesis that \mathbf{M} is minimal. Let \mathbf{N} be based on the flow of time $(\mathbf{T}, <)$, and define $\mathbf{N}_s = \mathbf{N}'_{f(s)}$. Then $\mathbf{N}_s = \mathbf{N}'_{f(s)} \leq \mathbf{M}'_{f(s)} = \mathbf{M}_s$, and there is at least one point $u' \in \mathbf{T}'$ such that $\mathbf{N}'_{u'} \neq \mathbf{M}'_{u'}$. But as f is surjective, there is a $u \in \mathbf{T}$ with $f(u) = u'$, so we have that $\mathbf{N}_u \neq \mathbf{M}_u$. Take a point $s \in \mathbf{T}$, then the path from the root of the tree in which s lies is mapped isomorphically to the path from a root to $f(s)$, so since \mathbf{N}' is a model of \mathbf{C}' and \mathbf{W}' , it is easy to see that $(\mathbf{N}, s) \models \mathbf{C}' \cup \mathbf{W}'$. Now take a rule $\mathbf{P}\alpha \wedge \neg \forall \mathbf{F} \neg \beta \rightarrow \mathbf{C}\gamma$ in \mathbf{D}' and suppose $(\mathbf{N}, s) \models \mathbf{P}\alpha \wedge \neg \forall \mathbf{F} \neg \beta$. This means that there is a branch \mathbf{B} in \mathbf{N} on which s lies, such that there is a $t \ll s$ with $\mathbf{N}_t \models \alpha$ and for all $u \in \mathbf{B}$ with $u \gg s$, $\mathbf{N}_u \not\models \neg \beta$. But then $f[\mathbf{B}]$ is a branch in \mathbf{N}' with $f(t) \ll f(s)$ and $\mathbf{N}'_{f(t)} \models \alpha$, and for all $u' \in \mathbf{B}'$ with $u' \gg f(s)$ it must be the case that $u' = f(u)$ for some $u \in \mathbf{B}$ with $s \ll u$, so $\mathbf{N}'_{u'} = \mathbf{N}'_{f(u)} \not\models \neg \beta$. As \mathbf{N}' is a model of \mathbf{D}' , we have $\mathbf{N}'_{f(s)} \models \gamma$, so

that $N_s \models \gamma$. Thus N is a model of D' , so it is a model of T_Δ in contradiction with the hypothesis that M was a minimal model of T_Δ . We have proved that M' is a minimal model of T_Δ .

Sometimes properties of branching time temporal models can be related to properties of the linear time models that are their branches. In our case we have the following results for the property of being a minimal model of T_Δ .

Theorem 4.4

Let Δ be a default theory.

- a) If M is a (branching time) temporal model such that $\mathbb{B}(M) \subseteq \mathbf{LT}(\Delta)$, then M is a minimal model of T_Δ .
- b) Suppose Δ is extension complete and $\mathbb{B} \subseteq \mathbf{LT}(\Delta)$. If $f : *B \rightarrow M$ is a surjective homomorphism, then M is a minimal model of T_Δ .

Proof

- a) Suppose $\mathbb{B}(M) \subseteq \mathbf{LT}(\Delta)$. By definition, $\mathbf{LT}(\Delta)$ are (linear) minimal models of T_Δ , so the same holds for $\mathbb{B}(M)$. By Proposition 4.2, the coproduct $*B(M)$ is a minimal model of T_Δ . Now define the function $f : *B(M) \rightarrow M$ mapping every branch in $*B(M)$ into M . It is easy to see that f is a branch-surjective homomorphism, so by Proposition 4.3, M is a minimal model of T_Δ .
- b) We will show that $\mathbb{B}(M) \subseteq \mathbf{LT}(\Delta)$, from which the desired result follows by part a).
 - a). Take any branch D of M , and assume (without loss of generality) that it has the natural numbers as flow of time. Now take an arbitrary $n \in \mathbf{N}$. Since f is surjective, there must be a point s in $*B$ such that $f(s) = n$. This point s must lie on a branch D' of $*B$, and this D' is a linear time model in B . From the definition of homomorphism, it follows that f maps this branch up to point s isomorphically onto D (up to point n). This means that $d(D, D') \leq 2^{-n}$. As n was chosen arbitrarily, we can find a sequence of linear time models in B that have D as their limit. The models of B are in $\mathbf{LT}(\Delta)$, which is closed as Δ is extension complete. This means that $D \in \mathbf{LT}(\Delta)$. By a) we have that M is a minimal model of T_Δ .

It can easily be shown that in general minimal models of T_Δ can have branches that are *not* in $\mathbf{LT}(\Delta)$. Consider the default theory $\Delta = \langle W, D \rangle$ with $W = \emptyset$ and $D = \{ :a / a, :c / b, a : c / c, a : \neg c / \neg c \}$, and the following model M :

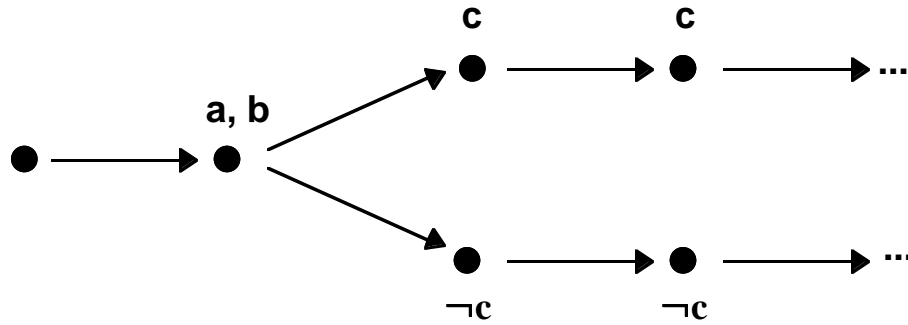


Figure 2. Minimal model with non-minimal branches

In Figure 2, we have again indicated the (non-trivial) formulas that are true in the various time points, where a formula is not repeated if it was true earlier (so in the points labelled c , also a and b are true). It can easily be checked that \mathbf{M} is a model of \mathbf{T}_Δ , and that it is minimal (if any true formulae are deleted anywhere, the result is not a model of \mathbf{T}_Δ). However, the lower branch is not a minimal linear time model of \mathbf{T}_Δ . This can be seen either by considering the smaller model where b is deleted from the second point onwards (which would still be a model of \mathbf{T}_Δ), or by verifying that its limit model (in which a, b and $\neg c$ are known), does not correspond to an extension of Δ . The equivalence of these two methods follows from Theorem 3.3.

Given the set of linear time minimal models $\mathbf{LT}(\Delta)$ of a temporal interpretation \mathbf{T}_Δ of a default theory, these models can be jointly embedded in their coproduct $*\mathbf{LT}(\Delta)$, which also is a minimal model of \mathbf{T}_Δ . This provides one model to describe the complete semantics of the default theory. However, this model may contain a lot of redundant information: all branches at least have the same starting point, but in the coproduct a copy is included of this (actually identical information state) starting point for every branch. Moreover, branches can contain longer initial subsequences that are identical. In a coproduct these are not shared but present in a copy for each of the branches. A more compact form of a joint embedding of the minimal linear time models can be obtained if all these copies are identified with each other in order to share these (sub)structures. For this construction based on identification of equal substructures homomorphisms can be used. This can be stated in other words as follows: we want to find a model which "contains" all minimal linear time models as a submodel, and which is as small as possible in the sense that its image under any surjective homomorphism is isomorphic to the model itself. This last property is called closedness (see also [ET02]):

Definition 4.5 (Closed model)

A model \mathbf{M} is called closed if one of the following equivalent conditions is satisfied:

(i) For all \mathbf{t}, \mathbf{u} roots or with a common immediate predecessor :

$$\mathbf{M}_{\mathbf{t}} = \mathbf{M}_{\mathbf{u}} \text{ implies } \mathbf{t} = \mathbf{u}.$$

(ii) Every homomorphism $f : \mathbf{M} \rightarrow \mathbf{M}'$ is injective.

In [ET02] we established the following result.

Proposition 4.6

Every branching time model \mathbf{M} can be mapped by a surjective homomorphism onto a closed branching time model, which is unique up to isomorphism.

The required homomorphism identifies all common initial subbranches.

Definition 4.7 (Closure and joint closure)

a) Let \mathbf{M} be a branching time model. The unique closed model on which \mathbf{M} can be mapped by a surjective homomorphism is called the closure of \mathbf{M} , denoted by $\mathbf{cl}(\mathbf{M})$.

b) Let \mathbf{M} be a set of models. The closure of the coproduct $*\mathbf{M}$ is called the joint closure of \mathbf{M} , denoted by $\mathbf{jcl}(\mathbf{M})$.

In [ET02] a more direct, but equivalent definition of the notion of joint closure in general category-theoretic terms is given. The following was proved there.

Proposition 4.8

A closed model is the joint closure of its branches.

Definition 4.9

The joint closure $\mathbf{jcl}(\mathbf{LT}(\Delta))$ of $\mathbf{LT}(\Delta)$ is shortly denoted by $\mathbf{LT}^*\Delta$.

Theorem 4.10

a) Let Δ be a default theory and $\mathbf{S} \subseteq \mathbf{LT}(\Delta)$ a set of minimal linear time models of \mathbf{T}_Δ . Then the joint closure $\mathbf{jcl}(\mathbf{S})$ of \mathbf{S} is a closed minimal temporal model of \mathbf{T}_Δ . If \mathbf{S} is closed (in $(\mathbf{LT}, \mathbf{d})$), then $\mathbf{B}(\mathbf{jcl}(\mathbf{S})) = \mathbf{S}$.

b) This holds in particular for the set $\mathbf{LT}(\Delta)$ of all minimal linear time models of \mathbf{T}_Δ : the model \mathbf{LT}^*_Δ is a minimal model of \mathbf{T}_Δ and if Δ is extension complete, then $\mathbb{B}(\mathbf{LT}^*_\Delta) = \mathbf{LT}(\Delta)$.

Proof

The joint closure $\mathbf{jcl}(\mathbf{S})$ is closed by definition, and the (unique) homomorphism \mathbf{f} mapping $*\mathbf{S}$ into $\mathbf{jcl}(\mathbf{S})$ is surjective, so $\mathbf{jcl}(\mathbf{S})$ is a minimal model of \mathbf{T}_Δ by Theorem 4.4b). Now suppose \mathbf{S} is closed. Using a similar argument as in the proof of Theorem 4.4b), one can show that every branch of $\mathbf{jcl}(\mathbf{S})$ can be approximated by elements of \mathbf{S} , which is closed and therefore contains such a branch. If Δ is extension complete, then by definition $\mathbf{LT}(\Delta)$ is closed.

The aim of this section was to find a branching time model containing just the Reiter extensions of Δ as limits of its branches. The following theorem shows that for an extension complete default theory Δ the model \mathbf{LT}^*_Δ indeed fulfils this requirement.

Theorem 4.11

Let $\Delta = \langle \mathbf{W}, \mathbf{D}, \rangle$ be an extension complete default theory with \mathbf{W} consistent.

- a) For every minimal linear time model of \mathbf{T}_Δ there is a unique homomorphism into \mathbf{LT}^*_Δ ; this homomorphism is injective.
- b) There is a bijection from the set $\mathbb{E}(\Delta)$ of all Reiter extensions of Δ onto the set $\mathbb{B}(\mathbf{LT}^*_\Delta)$ of branches of \mathbf{LT}^*_Δ .

More precisely, the mapping $\Psi : \mathbb{E}(\Delta) \rightarrow \mathbb{B}(\mathbf{LT}^*_\Delta)$ defined by

$$\Psi(\mathbf{E}) = (\text{Mod}(\mathbf{E}_i))_{i \in \mathbf{N}}, \text{ has the inverse } \Phi : \mathbb{B}(\mathbf{LT}^*_\Delta) \rightarrow \mathbb{E}(\Delta) \text{ defined by } \Phi(\mathbf{B}) = \text{Th}(\lim_{\mathbf{B}} \mathbf{LT}^*_\Delta)$$

Furthermore, for every $i \in \mathbf{N}$ it holds $\Phi(\mathbf{B})_i = \text{Th}(\mathbf{B}_i)$

Proof

a) Every model \mathbf{M} of $\mathbf{LT}(\Delta)$ is mapped by inclusion (which is a homomorphism) into $*\mathbf{LT}(\Delta)$ which is mapped into \mathbf{LT}^*_Δ . The composition of these two homomorphisms is again a homomorphism. If there are two homomorphisms \mathbf{f}, \mathbf{g} mapping \mathbf{M} into \mathbf{LT}^*_Δ , then $\mathbf{f}[\mathbf{M}]$ and $\mathbf{g}[\mathbf{M}]$ are two isomorphic branches in a closed model, so these images must coincide by (the first condition) in Definition 4.5. But this means that \mathbf{f} and \mathbf{g} are equal. It can easily be checked that a homomorphism from a linear model is always injective. Note that extension completeness is not needed here.

b) From Theorem 4.10 we have that $\mathbb{B}(\mathbf{LT}^*\Delta) = \mathbf{LT}(\Delta)$, and Theorem 3.3 established a bijection between $\mathbf{LT}(\Delta)$ and $\mathbb{E}(\Delta)$.

For the existence of a closed temporal model containing as branches just the minimal linear time models of a given default theory, the condition of extension completeness is not only sufficient, but also necessary, as is shown in the following proposition.

Proposition 4.12

For any default theory Δ the following are equivalent:

- (i) Δ is extension complete.
- (ii) There exists a closed model \mathbf{M} with $\mathbb{B}(\mathbf{M}) = \mathbf{LT}(\Delta)$.

Proof

From (i) to (ii) is easy: the required model is $\mathbf{LT}^*\Delta$. For the other direction, suppose we have a closed model \mathbf{M} with $\mathbb{B}(\mathbf{M}) = \mathbf{LT}(\Delta)$. Take a converging sequence $\{\mathbf{B}_1, \mathbf{B}_2, \dots\}$ of models in $\mathbf{LT}(\Delta)$, with limit \mathbf{B} . The models in the sequence are all present as branches in \mathbf{M} , and as \mathbf{M} is closed, if two models in the sequence have an initial common subbranch, then these are mapped onto the same subbranch in \mathbf{M} . Take any initial subbranch of \mathbf{B} , then we can find a model \mathbf{B}_i with the same initial subbranch, the image of which is in \mathbf{M} . If we extend this initial subbranch by one point, then we can again find a model \mathbf{B}_j with this initial subbranch. Its image in \mathbf{M} then extends the image of the subbranch of \mathbf{B}_i , as \mathbf{M} is closed. In this fashion we find that \mathbf{B} is a branch of \mathbf{M} , and therefore is in $\mathbf{LT}(\Delta)$.

So if a default theory Δ is not extension complete, then $\mathbf{LT}^*\Delta$ contains a branch which is not a member of $\mathbf{LT}(\Delta)$. Such a branch does not correspond to an extension. This means that the use of a temporal model construction as introduced here heavily depends on the topological properties of the given default theory: constructions fulfilling the requirements we imposed are not possible for non-extension complete default theories. However, recall Proposition 3.6, stating that this can only occur in the case of an infinite set of defaults. For almost all applications of default logic, the condition of extension completeness is fulfilled due to finiteness of the set of defaults.

5 Semantic Entailment Relations

We can define the following minimal semantic entailment relations (where Δ is any extension complete default theory):

$$\Delta \vDash_{LT} \varphi \iff \forall M \text{ [} M \text{ is a minimal linear time model of } T_{\Delta} \Rightarrow M \vDash \varphi \text{]}$$

$$\Delta \vDash_{LT^*} \varphi \iff LT^*_{\Delta} \vDash \varphi$$

For a certain class of formulae we can give logical relations between these entailment relations.

Definition 5.1 (Backward persistency)

Let $f : M \rightarrow M'$ be a homomorphism. The backward persistency property for a formula φ (under f) is defined by

$$(M, t) \vDash \varphi \iff (M', f(t)) \vDash \varphi \quad \text{for all time points } t \text{ in } T.$$

In [ET02] an overview of results on persistency is given. Here we confine ourselves to the following:

Proposition 5.2

For any default theory Δ its temporal interpretation T_{Δ} is backward persistent under any homomorphism.

The following theorem gives more precise connections between the two semantic consequence relations.

Theorem 5.3

Let T_{Δ} be the temporal interpretation of an extension complete default theory Δ and φ any formula.

a) If φ is backward persistent under injections, then

$$\Delta \vDash_{LT^*} \varphi \implies \Delta \vDash_{LT} \varphi$$

b) If φ is propositional, then

$$\Delta \vDash_{LT^*} \forall F \varphi \iff \Delta \vDash_{LT} \forall F \varphi$$

Proof

- a) Suppose $\Delta \models_{\text{LT}^*} \varphi$ and let \mathbf{M} be a minimal linear time model of \mathbf{T}_Δ . By Theorem 4.11a) there is an injective homomorphism f mapping \mathbf{M} into LT^*_Δ . Take any time point s of \mathbf{M} , then since $\text{LT}^*_\Delta \models \varphi$, in particular $(\text{LT}^*_\Delta, f(s)) \models \varphi$. As φ is backward persistent under injections, we have $(\mathbf{M}, s) \models \varphi$. This proves that $\mathbf{M} \models \varphi$, and therefore $\Delta \models_{\text{LT}} \varphi$.
- b) If φ is propositional, it is easy to see that $\forall F \varphi$ is backward persistent under any homomorphism. So the left to right direction follows with part a). For the other direction, by Theorem 4.10 we have that $\mathbf{B}(\text{LT}^*_\Delta) = \mathbf{LT}(\Delta)$. Take a point s in LT^*_Δ , and a branch \mathbf{B} through s . Then $\mathbf{B} \in \mathbf{LT}(\Delta)$, so $(\mathbf{B}, s) \models \forall F \varphi$, which means there must be a point $t \gg s$ with $\mathbf{B}_t \models \varphi$. But then also $(\text{LT}^*_\Delta)_t \models \varphi$. As the branch was arbitrary, we have $(\text{LT}^*_\Delta, s) \models \forall F \varphi$. This proves that $\text{LT}^*_\Delta \models \forall F \varphi$, so $\Delta \models_{\text{LT}^*} \forall F \varphi$.

We will show in Theorem 5.5 how these formulae $\forall F \varphi$ are related to sceptical entailment.

The model LT^*_Δ of an extension complete default theory Δ gives an overview of both all possible reasoning paths from a default theory (the branches) and the resulting conclusion sets (the limit models). Therefore in principle it contains all information that is relevant for an intended semantics. As a special case also sceptical and credulous entailment relations can be based on this model.

Definition 5.4 (sceptical and credulous entailment)

Let Δ be an extension complete default theory and let φ be a propositional formula.

- a) We define the sceptical entailment relation by:

$$\Delta \models_{\text{scep}} \varphi \quad \text{if } \varphi \text{ is in all extensions of } \Delta.$$

- b) We define the credulous entailment relation by:

$$\Delta \models_{\text{cred}} \varphi \quad \text{if } \varphi \text{ is in some extension of } \Delta.$$

- c) We define $\text{LT}^*_\Delta^\omega$ as the set of the limit models of all branches of LT^*_Δ , i.e.,

$$\text{LT}^*_\Delta^\omega = \{ \lim_{\mathbf{B}} \text{LT}^*_\Delta \mid \mathbf{B} \text{ branch of } \text{LT}^*_\Delta \}$$

Theorem 5.5

Let Δ be an extension complete default theory, r the root of LT^*_Δ and let φ be a propositional formula.

- a) The following are equivalent:

- (i) $\Delta \models_{\text{scep}} \varphi$
 - (ii) $\text{LT}^*_{\Delta} \models \varphi$
 - (iii) $(\text{LT}^*_{\Delta}, r) \models \forall F \varphi$
 - (iv) $(L, s) \models \forall F \varphi$ for every minimal linear time model L of Δ with root s
- b) The following are equivalent:
- (i) $\Delta \models_{\text{cred}} \varphi$
 - (ii) $\lim_{\mathbf{B}} \text{LT}^*_{\Delta} \models \varphi$ for some branch \mathbf{B}
 - (iii) $(\text{LT}^*_{\Delta}, r) \models \exists F \varphi$
 - (iv) $(L, s) \models \exists F \varphi$ for some minimal linear time model L of Δ with root s
 - (v) $(L, s) \models \forall F \varphi$ for some minimal linear time model L of Δ with root s

Proof

- a) From Theorem 4.11, we know that the function $\Psi : \mathbb{E}(\Delta) \rightarrow \mathbb{B}(\text{LT}^*_{\Delta})$ defined by $\Psi(E) = (\text{Mod}(E_i))_{i \in \mathbb{N}}$, is a bijection. Now for any propositional φ , we have that $\varphi \in E \Leftrightarrow \varphi \in \cup E_i \Leftrightarrow \text{Mod}(E_i) \models \varphi$ for some $i \Leftrightarrow \lim((\text{Mod}(E_i))_{i \in \mathbb{N}}) \models \varphi$ (this uses the fact that information states are closed). From these facts, it is easy to see that (i) and (ii) are equivalent. The equivalence of (ii) and (iii) is immediate. From Theorem 4.10 we know that $\mathbb{B}(\text{LT}^*_{\Delta}) = \mathbb{LT}(\Delta)$, from which we get the equivalence of (iii) and (iv).
- b) These equivalences can be proved analogously to those in part a). The equivalence of (iv) and (v) is an easy consequence of the semantic definitions of $\exists F$ and $\forall F$.

Using the model LT^*_{Δ} we can define many more different consequence relations. Sceptical and credulous entailment used the formulae $\forall F \varphi$ and $\exists F \varphi$, but our temporal language is much more expressive. We can check for instance whether a certain propositional formula is true in every branch at a point with depth less than 5.

In the case of normal default theories, there are even stronger connections between linear minimal models, branching time minimal models and the joint closures of these classes. We will treat them in the next section.

6 The Case of Normal Default Theories

A *normal* default rule is a default rule of the form $(\alpha : \beta) / \beta$, and a default theory consisting of solely normal default rules is called a *normal* default theory. In [ET94] we pointed out a branching time temporal semantics for the normal case only. Most of the results there follow as a special case of the general case in this paper; we will restate and

prove the main results here. In case Δ is normal, the minimal temporal models of T_Δ can be characterised completely by their branches.

Theorem 6.1

Let Δ be a normal default theory. Then M is a minimal temporal model of T_Δ if and only if $\mathbb{B}(M) \subseteq \mathbb{LT}(\Delta)$.

Proof

The right to left direction is Theorem 4.4a). Note that the counterexample for the other direction following this theorem (see Figure 2) is based on a default theory with a non-normal default. So let us prove the other direction. Suppose M is a minimal model of T_Δ but has a branch B which is not a minimal model of T_Δ . Consider the homomorphism f mapping B seen as a linear time model into M . As $M \models T_\Delta$ and T_Δ is backward persistent under homomorphisms (Proposition 5.2), we have that $B \models T_\Delta$. As it is not a minimal model of T_Δ by assumption, there must exist a linear time model N of T_Δ such that $N < B$. Suppose B and N are based on the flow of time $s_0 < s_1 < s_2 < \dots$. Let us consider the first point of time s_n (from the roots) at which N and B are different. If $N(s_0) < B(s_0)$, then define a new model M' based on the same flow of time as M but with $M'(s_0) = N(s_0)$ and $M'(t) = M(t)$ for all $t \neq s_0$. It can easily be checked that M' is a model of T_Δ and $M' < M$, which is impossible since M was minimal.

Now suppose $n > 0$ so $N(s_i) = B(s_i)$ for $i < n$ and $N(s_n) < B(s_n)$. Construct a model M' based on the same flow of time as M but with $M'(s_n) = N(s_n)$ and $M'(t) = M(t)$ for $t \neq s_n$. We will show that $M' \models T_\Delta$. It is clear that $M' \models W$, as this is evaluated per time point, and both $M \models W$ and $N \models W$. To show that $M' \models C$, it is sufficient to show that M' is conservative. The only interesting case is for a point $t \ll s_n$. But as M' has a flow of time which is a forest, the path from s_n to the root is unique, and as s_n lies on B , it must be the case that $t = s_i$ for some $i < n$. But then we have $M'(s_i) = N(s_i) \leq N(s_n) = M'(s_n)$, as N is conservative ($N \models C$).

Now take a rule $P\alpha \wedge \neg \forall F \neg \beta \rightarrow C\beta$ (remember that Δ is normal). It is easy to see that if at a point in M the left hand side is false, it will also be false in the corresponding point of M' (this uses conservativity of M'). So the only possibility of this rule to be false in M' , is at time point s_n . We will show that this can not occur. So suppose we have $(M', s_n) \models P\alpha \wedge \neg \forall F \neg \beta$, then it easily follows that $(M, s_n) \models P\alpha \wedge \neg \forall F \neg \beta$, which implies $(M, s_n) \models C\beta$, so $(B, s_n) \models C\beta$. This means (by conservativity of B) that $(B, s_i) \not\models C \neg \beta$ for all $i \in \mathbb{N}$. As $N < B$, we also have that

$(N, s_i) \models C\neg\beta$, so $(N, s_n) \models \neg\forall F\neg\beta$. Since $(M, s_n) \models P\alpha$, we have $(B, s_n) \models P\alpha$, from which it follows that $(N, s_n) \models P\alpha$ ($N(s_i) = B(s_i)$ for $i < n$). So $(N, s_n) \models P\alpha \wedge \neg\forall F\neg\beta$. As $N \models D'$, we get $(N, s_n) \models C\beta$, and from $M'(s_n) = N(s_n)$ we conclude that $(M', s_n) \models C\beta$. We have shown that $M' < M$ and $M' \models T_\Delta$, which contradicts the assumption that M is a minimal model of T_Δ . This means that B must be a minimal model of T_Δ , so $\mathbb{B}(M) \subseteq \mathbf{LT}(\Delta)$, which concludes the proof.

For the case of closed models this implies the following.

Proposition 6.2

Suppose Δ is an extension complete normal default theory and M a temporal model. Then M is a closed minimal temporal model of T_Δ if and only if M is the joint closure of a set \mathbb{B} of minimal linear time models of T_Δ .

Proof

The joint closure of a set \mathbb{B} of minimal linear time models of T_Δ is a closed minimal temporal model of T_Δ by Theorem 4.10 (extension completeness and normality of Δ is not used). For the other direction, by Proposition 4.8, M is the joint closure of its branches. These branches are minimal linear time models of T_Δ by Theorem 6.1.

For extension complete normal default theories, the model \mathbf{LT}^*_Δ also has stronger properties:

Definition 6.3 (Final minimal model)

The model F is called a final minimal temporal model of T_Δ if it is a minimal temporal model of T_Δ and for each minimal temporal model M of T_Δ there is a unique homomorphism $f : M \rightarrow F$.

We have the following result:

Theorem 6.4

Let Δ be a normal extension complete default theory. Then \mathbf{LT}^*_Δ is a (unique) final minimal temporal model of T_Δ ; it is the joint closure of all minimal temporal models of T_Δ . For every minimal temporal model of T_Δ there is a unique

homomorphism into $\mathbf{LT}^*\Delta$; for closed minimal temporal models of $\mathbf{T}\Delta$ this homomorphism is injective.

Proof

By definition, $\mathbf{LT}^*\Delta$ is the joint closure of $\mathbf{LT}(\Delta)$, so using Proposition 6.2 we have that it is a minimal temporal model of $\mathbf{T}\Delta$. Now consider any minimal temporal model \mathbf{M} of $\mathbf{T}\Delta$. By Theorem 6.1 it follows that $\mathbf{B}(\mathbf{M}) \subseteq \mathbf{LT}(\Delta)$, and from Theorem 4.10 it follows that $\mathbf{B}(\mathbf{LT}^*\Delta) = \mathbf{LT}(\Delta)$. The required unique homomorphism maps every branch of \mathbf{M} into its (unique) place in $\mathbf{LT}^*\Delta$. This uniqueness follows from the closedness of $\mathbf{LT}^*\Delta$. By Definition 4.5 (ii), any homomorphism from a closed model is injective.

7 Conclusions

In this paper we have given a temporal interpretation of default rules. This led us to a translation of default theories into temporal theories. In earlier publications (i.e. [ET93, ET98]) we showed that using this translation we can define semantics for default logic using minimal linear time temporal models. In the current paper we described the construction of a branching time temporal model in which all minimal linear time models are incorporated, and proved properties about this model. Under a topological condition (extension completeness), which is always satisfied for finite default theories, this model contains only branches which are minimal linear time models. As another main result we established that for any normal default theory satisfying the same condition, this model contains not only all minimal linear models of the temporal interpretation, but also all minimal branching time models. In this case we have a linear, branching time and final model semantics for default logic. Other semantics for default logic can be found in e.g. [Vo93], [BS94] (see [ET98] for a comparison between our linear time semantics and various other approaches).

This work enables one to use concepts from temporal logic to integrate process aspects into the study of formal semantics for default reasoning. We share the view also put forward in [Ga82], [Et87] that integrating such dynamics in the semantics is more transparent and fruitful than trying to abstract from them. We think that our work as presented in the current paper (and [ET93, ET94b, ET98]) contributes to the operationalization of this view.

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