

THE CARTAN-BRAUER-HUA THEOREM

BY

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In this note we present an elementary proof which hardly uses any computation; cf. [4] p. 2. First a lemma.

LEMMA: Suppose  $K$  is a skew field and  $E$  is a left  $K$ -linear space. Assume  $M \subset E$  is closed under addition and contains at least two independent elements over  $K$ . Let  $\phi: M \rightarrow E$  be a map satisfying  $\phi(a+b) = \phi(a) + \phi(b)$  and  $\phi(a) = \lambda_a a$  for all  $a, b \in M$  and certain  $\lambda_a \in K$ . Then there exists one element  $\lambda \in K$  such that  $\phi(a) = \lambda a$  for all  $a \in M$ .

PROOF: Let  $a, b \in M$  be nonzero; choose  $c \in M$  such that  $a$  and  $c$  are independent. From  $\lambda_a a + \lambda_c c = \phi(a) + \phi(c) = \phi(a+c) = \lambda_{a+c}(a+c)$  it follows that  $\lambda_a = \lambda_{a+c} = \lambda_c$ . Since  $b$  is independent of  $a$  or  $c$ , also  $\lambda_b = \lambda_a = \lambda_c$ .

THEOREM (Cartan-Brauer-Hua): Suppose  $L/K$  is an extension of skew fields and  $M \subset L$  is closed under addition and  $1 \in M$ . If  $aK \subset Ka$  for all  $a \in M$  then either  $M \subset K$  or  $M \subset Z_L(K)$  (here  $Z_L(K)$  denotes the centralizer of  $K$  in  $L$ ). In particular, if  $aKa^{-1} = K$  for all nonzero  $a \in L$  then either  $K=L$  or  $K$  is contained in the center of  $L$ .

PROOF: Suppose  $M$  is not contained in  $K$ , then  $M$  contains at least two elements left independent over  $K$ . Let  $b \in K$  be given; for each  $a \in M$ ,  $ab \in aK \subset Ka$ . Hence  $ab = \lambda_a a$  for some  $\lambda_a \in K$ . Apply the lemma to the right multiplication  $r_b: a \mapsto ab: M \rightarrow L$ . We have an element  $\lambda \in K$  such that  $ab = \lambda a$  for all  $a \in M$ . In particular this holds for  $a=1$ , hence  $\lambda=b$ . Therefore  $ab = ba$  for all  $a \in M$  and  $b \in K$ . For the last statement, take  $M=L$ .

REMARK: The above proof remains valid if  $L$  is replaced by any ring extension of the skew field  $K$ .

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## REFERENCES

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