MATHEMATICS

THE CARTAN-BRAUER-HUA THEOREM

BY

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(Communicated by Prof. H. Freudenthal at the meeting of January 29, 1977)

In this note we present an elementary proof which hardly uses any computation; cf. [4] p. 2. First a lemma.

LEMMA: Suppose K is a skew field and E is a left K-linear space. Assume $M \subset E$ is closed under addition and contains at least two independent elements over K. Let $\phi: M \to E$ be a map satifying $\phi(a+b) = \phi(a)+\phi(b)$ and $\phi(a) = \lambda_a a$ for all $a, b \in M$ and certain $\lambda_a \in K$. Then there exists one element $\lambda \in K$ such that $\phi(a) = \lambda a$ for all $a \in M$.

PROOF: Let $a, b \in M$ be nonzero; choose $c \in M$ such that a and c are independent. From $\lambda_a a + \lambda_c c = \phi(a) + \phi(c) = \phi(a+c) = \lambda_{a+c}(a+c)$ it follows that $\lambda_a = \lambda_{a+c} = \lambda_c$. Since b is independent of a or c, also $\lambda_b = \lambda_a = \lambda_c$.

THEOREM (Cartan-Brauer-Hua): Suppose L/K is an extension of skew fields and $M \subset L$ is closed under addition and $1 \in M$. If $aK \subset Ka$ for all $a \in M$ then either $M \subset K$ or $M \subset Z_L(K)$ (here $Z_L(K)$ denotes the centralizer of K in L). In particular, if $aKa^{-1} = K$ for all nonzero $a \in L$ then either $K = L$ or K is contained in the center of L.

PROOF: Suppose M is not contained in K, then M contains at least two elements left independent over K. Let $b \in K$ be given; for each $a \in M$, $ab \in aK \subset Ka$. Hence $ab = \lambda_a a$ for some $\lambda_a \in K$. Apply the lemma to the right multiplication $r_b: a \mapsto ab: M \to L$. We have an element $\lambda \in K$ such that $ab = \lambda a$ for all $a \in M$. In particular this holds for $a = 1$, hence $\lambda = b$. Therefore $ab = ba$ for all $a \in M$ and $b \in K$. For the last statement, take $M = L$.

REMARK: The above proof remains valid if L is replaced by any ring extension of the skew field K.

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