Specification of nonmonotonic reasoning*

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ABSTRACT. Two levels of description of nonmonotonic reasoning are distinguished. For these levels semantical formalizations are given. The first level is defined semantically by the notion of belief state frame, the second level by the notion of reasoning frame. We introduce two specification languages to describe nonmonotonic reasoning at each of the levels: (1) a specification language for level 1, with formal semantics based on belief state frames, (2) a fragment of infinitary temporal logic as a general specification language for level 2, with formal semantics based on reasoning frames. In our framework every level 2 description can be abstracted to level 1, and for every level 1 description there are level 2 descriptions which are a specialization of it.

KEYWORDS: Nonmonotonic reasoning, temporal logic, specification.

1. Introduction

Nonmonotonic reasoning systems address applications where an agent reasoning about the world wants to draw conclusions that are not logically entailed by its (incomplete) knowledge about the world. Under such circumstances it is only possible to build a set of (additional) beliefs of hypothetical nature. Such a set of beliefs represents a hypothetical view on the world. In general it is not unique: multiple views are possible; an agent may (temporarily) commit itself to one view and switch its commitment to another one later. Such a view does not necessarily give a complete world description either. Each view leaves open a number of possible complete world descriptions. However, the additional knowledge defining the view an agent is

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committing to, may be sufficient for the agent to draw the required (defeasible) conclusions (within the context of that view).

One may focus on the intersection of the different possible sets of beliefs for the agent; this could be described by a nonmonotonic inference operator, e.g., as in [KLM 90]. A disadvantage of this (sceptical) approach may be that hypothetical conclusions that are possible within one of the belief sets may be lost due to the restriction to the common beliefs. So, the agent may not be able to draw the required conclusions, only taking into account the beliefs common in all views. Therefore we will concentrate on multiple belief sets rather than on their intersection. The non-multiple view will be incorporated as the special case of a single belief set.

A set of beliefs is usually not available to an agent immediately: usually for a given set X of world knowledge a set of beliefs is constructed by applying some type of reasoning (e.g., inference steps defined by default rules). As this type of reasoning should allow different alternative belief sets as conclusion sets (that may be mutually inconsistent), the type of reasoning used is essentially context-dependent: depending on a chosen context one of the alternative belief sets is generated.

In this paper we distinguish two levels of abstraction at which nonmonotonic reasoning can be specified:

1. *Specification of a set of intended multiple belief sets*
Specification of the possible belief sets for the agent abstracting from the specific reasoning patterns that lead to them.
2. *Specification of a set of intended reasoning patterns*
Specification of the reasoning patterns that lead to the intended possible belief states.

We will first semantically define the two levels. The first level is defined semantically by the notion of belief state frame (Section 2), the second level by the notion of reasoning frame (Section 3). In Section 4 the semantic connections between the two levels are described. Moreover, we introduce two specification languages to describe nonmonotonic reasoning at each of the levels. In Section 5 a specification language is defined for level 1, with formal semantics based on belief state frames. In Section 6 we introduce a fragment of infinitary (linear time) temporal logic as a general specification language for level 2, with formal semantics based on reasoning frames (interpreted as linear time temporal models).

Thus a specification framework is obtained by which any nonmonotonic system aiming at defining multiple belief sets (or reasoning traces leading to them) can be specified. Such a specification can be written either at the level of the possible alternative sets of beliefs that are the intended outcomes, or at the level of reasoning patterns to construct these outcomes (or both). The first type of specifications at least covers approaches such as preferential semantics (cf. [SHO 87]) and the notion of S-expansions for modal nonmonotonic logics (cf. [MT 93]). The second type at least covers approaches like default logic (cf. [REI 80]). The general specification framework (and its semantics) introduced here provides a unifying perspective on these and other well-known approaches. For example, the general connections between the two levels developed in the current paper provide connections between preferential semantics and default logic (cf. [ETH 87], [VOO 93]) as a special case.

Part of the material in this paper was previously published in [ET 96]. Apart from the two levels of abstraction considered here, [EHT 95] distinguishes three more levels of abstraction. Sections 2, 3 and 4 review and extend some of the material presented there.

2. Belief state frames

Classical (propositional) logic lies at the basis of most nonmonotonic formalisms, so we will assume we have a propositional language, $\mathbf{L}$, its corresponding set of models, $\mathbf{Mod}$, and the (semantic) consequence relation $\models\ \subseteq \mathbf{Mod} \times \mathbf{L}$. Furthermore, for a set $\mathbf{X} \subseteq \mathbf{L}$, we define the models of $\mathbf{X}$: $\mathbf{Mod(X) = \{ m \in Mod \mid m \models \varphi \text{ for all } \varphi \in X\}}$; the consequences of a set $\mathbf{X} \subseteq \mathbf{L}$ are defined by $\mathbf{Cn(X) = \{ \varphi \in L \mid Mod(X) \subseteq Mod(\varphi)\}}$. For a subset $\mathbf{K}$ of $\mathbf{Mod}$, the *theory* of $\mathbf{K}$ is defined by $\mathbf{Th(K) = \{ \varphi \mid m \models \varphi \text{ for all } m \in K \}}$. A set of models $\mathbf{K}$ is called *closed* if $\mathbf{K = Mod(Th(K))}$, or equivalently, if $\mathbf{K}$ is the set of models of a theory.

Semantically, nonmonotonic reasoning could be described by model operators which assign to a set of formulae $\mathbf{X}$, the initial facts, a set of models, the *intended models*, which should be a subset of the set of classical models of $\mathbf{X}$. These models are intended in the sense that these are the worlds the agent considers plausible on the basis of $\mathbf{X}$. Often, however, a reasoning agent does not have just one set of beliefs, but there are (many) alternative belief sets depending on which further assumptions the agent wants to make. For example, in the case of default reasoning, given initial facts and a set of default rules, there may be more than one Reiter extension. To formalize this semantically we introduce the notion of an information state and a belief state frame.

Definition 2.1 (Information state)

a) An *information state* $\mathbf{M}$ is a non-empty closed set of propositional models, that is, there is a consistent theory of which it is the model class. The truth of a propositional formula α in such a state is defined by:

$$\mathbf{M} \models \alpha \iff \mathbf{m} \models \alpha \text{ for each } \mathbf{m \in M}$$

b) The *refinement ordering* $\leq$ on information states is defined by:

$$\mathbf{M_1 \leq M_2} \iff \mathbf{M_2 \subseteq M_1}$$

c) The *set of all information states* is denoted by $\mathbf{IS}$.

An information state represents a possible view on the world. The information the agent has in a state $\mathbf{M}$ is $\mathbf{Th(M)}$, which is closed under propositional consequence. Although in practice this may not always be the case, we assume the agent is in principle capable of deriving any propositional consequence of its information. The condition that an information state is closed is merely technical, it improves readability of definitions and proofs, but is not essential. The refinement ordering on information states expresses degree of information: if $\mathbf{M_1 \leq M_2}$ then $\mathbf{Th(M_1) \subseteq Th(M_2)}$, so the agent has more information in $\mathbf{M_2}$ than it has in $\mathbf{M_1}$.

Definition 2.2 (Belief state operator and belief state frame)
a) A *belief state operator* $\boldsymbol{\Gamma}$ is a function $\boldsymbol{\Gamma} : \mathfrak{P}(\mathbf{L}) \to \mathfrak{P}(\mathbf{IS})$ satisfying the following conditions for every $\mathbf{X} \subseteq \mathbf{L}$:
(i) $\mathbf{K} \subseteq \mathbf{Mod(X)}$ for every $\mathbf{K} \in \boldsymbol{\Gamma}(\mathbf{X})$;
(ii) $\forall \mathbf{J}, \mathbf{K} \in \boldsymbol{\Gamma}(\mathbf{X}) \quad \mathbf{J} \subseteq \mathbf{K} \Rightarrow \mathbf{J} = \mathbf{K}$ (*noninclusiveness)*.
The tuple $\mathfrak{SB} = (\mathbf{L}, \mathbf{Mod}, \models, \boldsymbol{\Gamma})$ is said to be a *belief state frame*.
b) A belief state operator $\boldsymbol{\Gamma}$ is called *invariant* if $\boldsymbol{\Gamma}(\mathbf{X}) = \boldsymbol{\Gamma}(\mathbf{Cn(X)})$ for all $\mathbf{X} \subseteq \mathbf{L}$.

The first condition expresses conservativity: it means that a possible view on the world at least satisfies the initial facts. In our opinion this is an essential feature of nonmonotonic reasoning (about a single world situation; if the world changes, the agent may have to adapt its initial facts): it gives a method of extending partial information. The second condition is probably most subject to discussion. The idea is that if an agent has two possible views, one of which is included in the other, the agent will want to retain only the most informative one. Most nonmonotonic formalisms (though not all) satisfy this requirement; we may want to drop this condition in further work. The property of invariance is related to the fact that an information state is closed under propositional consequence: the syntactical form in which the initial facts are given, is not important.

Belief state operator for preferential semantics

Level 1 semantically consists of descriptions of nonmonotonic reasoning using belief state frames. As an example we consider preferential semantics (cf. [SHO 87]). Let a preference relation $<$ on $\mathbf{Mod}$ be given. A belief state operator $\boldsymbol{\Gamma}_<$ (with single belief states) can be defined in the following manner: for each $\mathbf{X} \subseteq \mathbf{L}$

$$\boldsymbol{\Gamma}_<(\mathbf{X}) = \{ \{ \mathbf{m} \in \mathbf{Mod} \mid \mathbf{m} \text{ is } <\text{-minimal in } \mathbf{Mod(X)} \} \}$$

Preferential semantics essentially provides a level 1 description, abstracting from lower levels. Also approaches with non-singleton belief states specified by preference relations exist (cf. [KLM 90], [VOO 93]).

Belief state operator formalizing default logic

Default logic (cf. [REI 80]) can also be formalized using a belief state operator. Let $\mathbf{D}$ be a set of defaults. For $\mathbf{X} \subseteq \mathbf{L}$ let $\mathbb{E}(\langle \mathbf{X}, \mathbf{D} \rangle)$ denote the set of (Reiter) extensions of the default theory $\langle \mathbf{X}, \mathbf{D} \rangle$. The following belief state operator can be defined for $\mathbf{X} \subseteq \mathbf{L}$:

$$\Gamma_{\mathbf{D}}(\mathbf{X}) = \{ \mathbf{Mod(E)} \mid \mathbf{E} \in \mathbb{E}(\langle \mathbf{X}, \mathbf{D} \rangle) \}$$

In a similar manner other variants of default logic, but also for instance autoepistemic logic, can be formalized.

3. Reasoning frames

We have seen in Section 2 how (nonmonotonic) reasoning can be described by assigning to each set $\mathbf{X}$ of initial facts a set $\boldsymbol{\Gamma}(\mathbf{X})$ of belief states, abstracting from the way in which the conclusions of these states have been reached. On a less abstract level, one would also like to be able to describe types of reasoning by specifying not only the conclusions of the reasoning process, but also the reasoning path leading from the initial facts to the conclusions. To do this we will first formalize the notion of such a path, which we call a reasoning trace. After giving a formal (algebraic) description of these traces, in Section 6 we will introduce a specification language for these traces and investigate the power of this language. We will look at the links between the previous level and the current one in Section 4.

Intuitively, the path from initial set of formulae to final conclusions can be seen as the behaviour of a reasoning process which starts with the initial formulae, then makes some inferences to arrive at a new state, again makes some inferences, et cetera, possibly ad infinitum. The union of all conclusions drawn at all stages of such a process can be seen as the set of final conclusions of the process. A formalization of such reasoning behaviour would have to describe which formulae have been derived at each stage. We will formalize this semantically by the notion of a reasoning trace:

Definition 3.1 (Reasoning trace and limit model)
a) A *reasoning trace* $\mathcal{M}$ is a function from the set of natural numbers ($\mathbb{N}$) to $\mathbf{IS}$ such that for all $\mathbf{s} \in \mathbb{N}$:

(i) $\mathcal{M}_{\mathbf{s}} \leq \mathcal{M}_{\mathbf{s+1}}$

(ii) $\mathcal{M}_{\mathbf{s}} = \mathcal{M}_{\mathbf{s+1}} \quad \Rightarrow \quad \mathcal{M}_{\mathbf{s}} = \mathcal{M}_{\mathbf{t}}$ for all $\mathbf{t} \geq \mathbf{s}$.

b) The *refinement ordering* $\leq$ on reasoning traces is defined by:

$\mathcal{M} \leq \mathcal{N} \quad \Leftrightarrow \quad \mathcal{M}_{\mathbf{s}} \leq \mathcal{N}_{\mathbf{s}}$ for all $\mathbf{s} \in \mathbb{N}$

c) The *limit model*, $\mathbf{lim}\,\mathcal{M}$ of a reasoning trace $\mathcal{M}$ is defined by

$$\mathbf{lim}\ \mathcal{M} = \bigcap_{\mathbf{s=0}}^{\infty} \mathcal{M}_{\mathbf{s}}$$

d) A reasoning trace $\mathcal{M}$ is sometimes denoted by $(\mathcal{M}_{\mathbf{s}})_{\mathbf{s} \in \mathbf{N}}$.

e) A reasoning trace is called *finitely generated* if each $\mathbf{Th}(\mathcal{M}_{\mathbf{s}})$ is finitely generated over $\mathbf{Th}(\mathcal{M}_{\mathbf{0}})$, i.e., if $\mathbf{Th}(\mathcal{M}_{\mathbf{s}}) = \mathbf{Cn}(\mathbf{Th}(\mathcal{M}_{\mathbf{0}}) \cup \{\alpha_{\mathbf{s}}\})$ for some formula $\alpha_{\mathbf{s}}$.

We will motivate the two conditions. The first condition expresses conservativity: the information states become more informative over time. This means that the conclusions the agent draws at a certain point in time, remain valid in the future. Of course we do not wish to claim that backtracking never occurs in an *implementation* of a reasoning process; it will. However, we would like to abstract away from particular implementations, and look only at the increase of information of the agent over time: the resulting valid reasoning patterns which may be generated by a backtracking implementation. The second condition states that once the agent does not reach any new conclusions in a reasoning step, it has finished reasoning. This relates on the one hand to the assumption that the agent is reasoning about a particular fixed world situation (it cannot perform any observations during the reasoning), and on the other hand to the above mentioned motivation: although in an implementation the agent may be idle for some time, we are interested in the essentials of a reasoning process. The reason that a trace is always infinite, is again technical and not essential. We want to be able to model infinite reasoning processes (in the case of for instance default logic with an infinite set of defaults), and finite ones. In order to simplify definitions and proofs we model these finite processes as infinite traces which stabilize (become constant) at a certain point in time, instead of as finite traces.

Since we assume a countable language **L**, each theory can be approximated by a chain of finitely generated theories. Therefore:

Proposition 3.2

a) For any reasoning trace its limit is an information state.

b) Any information state is the limit model of a finitely generated reasoning trace.

Proof

a) Let $\mathcal{M}$ be a reasoning trace. As each $\mathcal{M}_i$ is closed we have $\mathcal{M}_i = \mathbf{Mod(Th(}\mathcal{M}_i\mathbf{)}$ so $\bigcap_{s=0}^{\infty} \mathcal{M}_s = \bigcap_{s=0}^{\infty} \mathbf{Mod(Th(}\mathcal{M}_s\mathbf{)} = \mathbf{Mod(}\bigcup_{s=0}^{\infty} \mathbf{Th(}\mathcal{M}_s\mathbf{)\ )}$ and $\bigcup_{s=0}^{\infty} \mathbf{Th(}\mathcal{M}_s\mathbf{)}$ is consistent as each $\mathbf{Th(}\mathcal{M}_s\mathbf{)}$ is consistent and propositional logic is compact. (Note that $\mathbf{Th(lim}\ \mathcal{M}\mathbf{)} = \bigcup_{s=0}^{\infty} \mathbf{Th(}\mathcal{M}_s\mathbf{)}$).

b) Let **M** be an information state. Let $\{\varphi_1, \varphi_2, \ldots\}$ be an enumeration of **Th(M)**. Now define $\mathcal{M}_i = \mathbf{Mod(}\{\varphi_1, \varphi_2, \ldots \varphi_i\}\mathbf{)}$, then $\mathcal{M}$ is finitely generated and $\mathbf{lim}\ \mathcal{M} = \mathbf{M}$. It need not be a reasoning trace, as condition (ii) of Definition 3.1 need not be satisfied. This can be easily overcome by deleting duplicate states in $\mathcal{M}$ (if $\mathcal{M}$ has a constant tail, then this tail should be left as is).

A (nonmonotonic) type of reasoning can now be described by giving its intended reasoning traces. Given a set of initial formulae, there may of course be several traces leading to different conclusion sets. We do, however, assume that the reasoning is deterministic in the sense that given the set of initial formulae *and* the final conclusion set, the trace between them is uniquely determined. This can be explained in the sense that at each stage of the reasoning process all conclusions that possibly can be drawn, actually are drawn in the next step. (Although most nonmonotonic formalisms satisfy this requirement, we may want to drop this assumption in further work.) Moreover, we do not allow two distinct traces with the same initial facts leading to limit models of which one is a refinement of the other (non-inclusiveness of traces), in analogy with the requirement of non-inclusiveness for belief state operators.

Definition 3.3 (Reasoning frame)
a) A *reasoning frame* is a tuple $(\mathbf{L}, \mathbf{Mod}, \models, \mathfrak{T})$ with $\mathfrak{T}$ a set of reasoning traces such that for all $\mathcal{M}$ and $\mathcal{N}$ in $\mathfrak{T}$: if $\mathcal{M}_0 = \mathcal{N}_0$ and $\lim \mathcal{M} \leq \lim \mathcal{N}$ then $\mathcal{M} = \mathcal{N}$.
For shortness, sometimes we also call $\mathfrak{T}$ by itself a reasoning frame.
b) If for all sets of formulae $\mathbf{X}$ there exists a trace $\mathcal{M}$ in $\mathfrak{T}$ such that $\mathbf{Th}(\mathcal{M}_0) = \mathbf{Cn}(\mathbf{X})$ then $\mathfrak{T}$ is called a *complete* reasoning frame. Otherwise it is called *partial*.

This provides a semantical formalization of level 2.

Reasoning frame formalizing default logic

As an example of a formalization using reasoning frames we again consider default logic. Let $\mathbf{D}$ be a set of defaults, and remember that, for $\mathbf{X} \subseteq \mathbf{L}$, we let $\mathbb{E}(\langle \mathbf{X}, \mathbf{D} \rangle)$ denote the set of (Reiter) extensions of the default theory $\langle \mathbf{X}, \mathbf{D} \rangle$. For a given $\mathbf{X}$ and $\mathbf{E} \in \mathbb{E}(\langle \mathbf{X}, \mathbf{D} \rangle)$ the following reasoning trace $\mathcal{M}$ can be associated in a canonical manner:

$$\mathcal{M}_i = \mathbf{Mod}(\mathbf{E}_i)$$

with $\mathbf{E}_0 = \mathbf{Cn}(\mathbf{X})$, and for all $i \geq 0$

$$\mathbf{E}_{i+1} = \mathbf{Cn}(\mathbf{E}_i \cup \{ \omega \mid (\alpha : \beta_1, \ldots, \beta_n) / \omega \in \mathbf{D}, \ \alpha \in \mathbf{E}_i \text{ and } \neg \beta_1 \notin \mathbf{E}, \ldots, \neg \beta_n \notin \mathbf{E} \})$$

Note that this is a trace definition based on the given set of defaults $\mathbf{D}$. We can form the reasoning frame $\mathfrak{T}_\mathbf{D}$ consisting of all the traces associated to all $\mathbf{X} \subseteq \mathbf{L}$ and $\mathbf{E} \in \mathbb{E}(\langle \mathbf{X}, \mathbf{D} \rangle)$.

4. Connections between belief state frames and reasoning frames

Belief state frames and reasoning frames both provide a means of defining (nonmonotonic) types of reasoning. A specification of a belief state frame (level 1) is more abstract whereas a specification of a reasoning frame (level 2) provides more details of the reasoning process. But can they describe the same types of reasoning, or put differently: are there clear connections between the two levels? We would like every level 1 specification to be "implementable" (by specialisation) by a level 2 specification. On the other hand, for every specification on level 2 it should be possible to find an abstraction of it on level 1 of which it is an "implementation". These issues will be addressed in this section.

4.1. Abstraction

Level 1 descriptions give the final conclusion sets of a type of reasoning given the initial formulae, abstracting from the reasoning process. So if we want to abstract from a level 2 specification, given a trace we should look at the initial formulae and the final outcome, that is the limit model. If we have a reasoning frame of level 2, we can define an invariant belief state operator in a straightforward way.

Definition 4.1 (Belief state operator of a reasoning frame)
a) Given a complete reasoning frame $\mathfrak{F}$ the *associated belief state operator* $\Gamma_{\mathfrak{F}}$ is defined as follows: for any set $\mathbf{X} \subseteq \mathbf{L}$,

$$\Gamma_{\mathfrak{F}}(\mathbf{X}) = \{ \lim \mathcal{M} \mid \mathcal{M} \in \mathfrak{F},\ \mathrm{Th}(\mathcal{M}_0) = \mathrm{Cn}(\mathbf{X}) \}$$

b) For a given invariant belief state operator Γ we define the set $\mathfrak{F}(\Gamma)$ of reasoning frames with Γ as their associated belief state operator:

$$\mathfrak{F}(\Gamma) = \{ \mathfrak{F} \mid \mathfrak{F} \text{ is a reasoning frame with } \Gamma_{\mathfrak{F}} = \Gamma \}$$

It is easy to see that for a given reasoning frame $\mathfrak{F}$ the defined $\Gamma_{\mathfrak{F}}$ is an invariant belief state operator.

4.2. Specialisation

Many reasoning frames can yield the same associated belief state operator, so we want to analyse the set $\mathfrak{F}(\Gamma)$ of possible reasoning frames connected with a belief state operator Γ by defining a parametrization of $\mathfrak{F}(\Gamma)$.

If we want to specify, for an invariant belief state operator Γ, an associated reasoning frame, what we have to do is, for all $\mathbf{X}$ and $\mathbf{M} \in \Gamma(\mathbf{X})$, specify a trace from the information state related to the set of initial formulae $\mathbf{X}$ to the belief state $\mathbf{M} \in \Gamma(\mathbf{X})$. Therefore a parameter should specify when the formulae of $\mathbf{Th(M)\backslash Cn(X)}$ can and have to be added during the reasoning. We can do this by assuming that each formula may depend on some other formulae and can only be added if the formulae it depends on have already been added in earlier stages. So, for each $\mathbf{X}$ and $\mathbf{M} \in \Gamma(\mathbf{X})$ such a dependency ordering (or information ordering) $\boxed{\prec}$ between propositional formulae has to be specified, where $\psi \boxed{\prec} \varphi$ means that φ depends on ψ. Therefore we have chosen a parametrization of $\mathfrak{F}(\Gamma)$ by means of functions $\boldsymbol{p}$ which assign to each pair of theories $\mathbf{X}$, $\mathbf{Y}$, where $\mathbf{X}$ consists of the initial facts and $\mathbf{Y}$ is the theory of a belief state $\mathbf{M} \in \Gamma(\mathbf{X})$ an ordering on the propositional formulae. In the reasoning trace one has to make sure the formulae are added to $\mathbf{X}$ respecting this order. Doing this, for each $\mathbf{X}$ and $\mathbf{M} \in \Gamma(\mathbf{X})$, the traces are specified unambiguously.

Definition 4.2 (Reasoning trace parameters)
a) A *reasoning trace parameter* is a partial order $(\mathbf{L}, \boxed{\prec})$ on propositional formulae such that for each $\varphi \in \mathbf{L}$ there is an $\mathbf{n} \in \mathbb{N}$ such that $\{\psi \mid \psi \boxed{\prec} \varphi\}$ does not contain a chain of length more than $\mathbf{n}$. A reasoning trace parameter is called *finitely grounded* if for each $\varphi \in \mathbf{L}$ the set $\{\psi \mid \psi \boxed{\prec} \varphi\}$ is finite.
b) Let $\mathbb{TH} := \{\mathbf{X} \subseteq \mathbf{L} \mid \mathrm{Cn}(\mathbf{X}) = \mathbf{X}\}$ be the set of theories, and let $\mathbb{BE} = \{ (\mathbf{X}, \mathbf{Y}) \mid \mathbf{X}, \mathbf{Y} \in \mathbb{TH}, \mathbf{X} \subseteq \mathbf{Y}\}$ be the set of pairs of possible starting points

and endpoints. Given a trace parameter $\boxed{\prec}$ and $(X, Y) \in \mathbb{BE}$ we define a chain of sets of formulae $(S_t)_{t \in N}$ as follows:

$$S_0 = X$$

$$S_{t+1} = Cn(S_t \cup \{ \varphi \in Y \mid \{\psi \in Y \mid \psi \boxed{\prec} \varphi\} \subseteq S_t\})$$

Now we define a reasoning trace $(\mathcal{M}_t)_{t \in N}$ by $\mathcal{M}_t = Mod(S_t)$. This trace will be denoted by $\mathcal{M}(\boxed{\prec}, X, Y)$.

It is easy to see that this indeed defines a reasoning trace with $Th(\mathcal{M}_0) = X$ and $Th(\lim \mathcal{M}(\boxed{\prec}, X, Y)) = Y$.

Definition 4.3 (Parametrized reasoning frame of a belief state operator)
Let $\mathbb{TP}$ be the set of all reasoning trace parameters. A function $p : \mathbb{B} \to \mathbb{TP}$ with $\mathbb{B} \subseteq \mathbb{BE}$ such that if $(X, Y), (X, Y') \in \mathbb{B}$ and $Y \subseteq Y'$ then $Y = Y'$ is called a *reasoning frame parameter*. For a reasoning frame parameter p let $\mathfrak{T}_p$ be the following reasoning frame:

$$\mathfrak{T}_p = \{ \mathcal{M}(p(X,Y), X, Y) \mid (X,Y) \in \mathbb{B}\}$$

A reasoning frame parameter p is *suited for* an invariant belief state operator Γ if $\mathbb{B} = \{(X, Y) \mid X \in \mathbb{TH}, Y = Th(M)$ for some $M \in \Gamma(X)\}$.

It is easy to verify that in case all $X \in \mathbb{TH}$ occur as first coordinate in a pair in $\mathbb{B}$, this defines a complete reasoning frame and that all reasoning frames are parametrized by these parameters:

Theorem 4.4
Let an invariant belief state operator Γ and a complete reasoning frame $\mathfrak{T}$ be given. The following conditions are equivalent:

(i) $\Gamma_{\mathfrak{T}} = \Gamma$

(ii) There exists a reasoning frame parameter p suited for Γ such that $\mathfrak{T}_p = \mathfrak{T}$.

The reasoning frame parameter can be taken finitely grounded.

Proof
(i) $\Rightarrow$ (ii) Let $\mathbb{B} = \{(X, Y) \mid X \in \mathbb{TH}, Y = Th(M)$ for some $M \in \Gamma(X)\}$. We will define a finitely grounded reasoning frame parameter $p : \mathbb{B} \to \mathbb{TP}$. Fix a set $X \in \mathbb{TH}$ and $M \in \Gamma(X)$. Since $\Gamma_{\mathfrak{T}} = \Gamma$ there exists $\mathcal{M} \in \mathfrak{T}$ such that $Th(\mathcal{M}_0) = Cn(X) = X$ and $\lim \mathcal{M} = M$. Let $n = \sup \{ i > 0 \mid \mathcal{M}_{i-1} < \mathcal{M}_i \}$, where $\mathcal{M}_{i-1} < \mathcal{M}_i$ if $\mathcal{M}_{i-1} \leq \mathcal{M}_i$ and $\mathcal{M}_{i-1} \neq \mathcal{M}_i$ (we allow $n = \infty$ or $n = -\infty$ if $\mathcal{M}$ is constant). Then since each $\mathcal{M}_i$ is closed we have that $Th(\mathcal{M}_i)$ is a proper subset of $Th(\mathcal{M}_{i+1})$ for $0 \leq i < n$. Choose for each $0 < i < n$ a formula $\varphi_i \in Th(\mathcal{M}_i) \setminus Th(\mathcal{M}_{i-1})$.

Now let

$p(X,Th(M)) = \{ (\varphi_i, \psi) \mid 0 < i < n, \psi \in Th(\mathcal{M}_{i+1}) \setminus Th(\mathcal{M}_i) \}$. Let p be defined in this way for each $X \in \mathbb{TH}$ and $Y = Th(M)$ for some $M \in \Gamma(X)$. It is easy to see that p is a finitary reasoning frame parameter suited for Γ. Since $\Gamma_{\mathfrak{F}} = \Gamma$ and p is suited for Γ, both $\mathfrak{F}$ and $\mathfrak{F}_p$ have exactly one trace for each pair (X, Y) with $X \in \mathbb{TH}$, $Y = Th(M)$ for some $M \in \Gamma(X)$ and nothing more, so all we have to prove is that the corresponding traces are equal. Take such a pair and let $\mathcal{M}$ be the trace in $\mathfrak{F}$ with $Th(\mathcal{M}_0) = X$ and $Th(\lim \mathcal{M}) = Y$ and let $\mathcal{N}$ be the trace in $\mathfrak{F}_p$ with $Th(\mathcal{N}_0) = X$ and $Th(\lim \mathcal{N}) = Y$. Let $\boxed{\prec} = p(X,Y)$. We will prove by induction that $\mathcal{M}$ and $\mathcal{N}$ are equal:

- $i = 0$: $Th(\mathcal{M}_0) = X = Th(\mathcal{N}_0)$ and since both are closed we have $\mathcal{M}_0 = \mathcal{N}_0$.
- Induction step: suppose $\mathcal{M}_i = \mathcal{N}_i$, then $Th(\mathcal{N}_{i+1}) =$

$Cn(Th(\mathcal{N}_i) \cup \{\varphi \in Y \mid \{\psi \in Y \mid \psi \boxed{\prec} \varphi\} \subseteq Th(\mathcal{N}_i)\}) =$

$Cn(Th(\mathcal{M}_i) \cup \{\varphi \in Y \mid \{\varphi_j \in Y \mid \varphi_j \boxed{\prec} \varphi\} \subseteq Th(\mathcal{M}_i)\}) =$

$Cn(Th(\mathcal{M}_i) \cup \{\varphi \in Y \mid \varphi_j \boxed{\prec} \varphi$ for some $j \leq i\}) =$

$Cn(Th(\mathcal{M}_i) \cup \{\varphi \in Y \mid \varphi_i \boxed{\prec} \varphi\}) = Cn(Th(\mathcal{M}_i) \cup Th(\mathcal{M}_{i+1}) \setminus Th(\mathcal{M}_i))$

$= Th(\mathcal{M}_{i+1})$. As $\mathcal{M}_{i+1}$ and $\mathcal{N}_{i+1}$ are closed, we have $\mathcal{M}_{i+1} = \mathcal{N}_{i+1}$.

Thus $\mathcal{M}$ and $\mathcal{N}$ are equal so $\mathfrak{F}_p = \mathfrak{F}$.

(ii) $\Rightarrow$ (i) Let $p : \mathbb{B} \rightarrow \mathbb{TP}$ with $\mathbb{B} = \{(X, Y) \mid X \in \mathbb{TH}, Y = Th(M)$ for some $M \in \Gamma(X)\}$ be a reasoning frame parameter with $\mathfrak{F}_p = \mathfrak{F}$. Take $X \subseteq L$,

then $\Gamma_{\mathfrak{F}}(X) = \Gamma_{\mathfrak{F}_p}(X) = \{\lim \mathcal{M} \mid \mathcal{M} \in \mathfrak{F}_p, Th(\mathcal{M}_0) = Cn(X)\} = \{Mod(Y) \mid (X,Y) \in \mathbb{B}\} = \{M \mid M \in \Gamma(X)\} = \Gamma(X)$.

Thus $\Gamma_{\mathfrak{F}} = \Gamma$.

5. A specification language for belief state frames

Having introduced a semantical foundation of the level of multiple belief sets based on the notion of a belief state frame, a natural next question is how to specify such a belief state operator. In other words, can a standard language be defined in which a specific belief state frame can be described? To this end we introduce the following specification language.

Definition 5.1 (Belief state frame specification)

a) A *belief state frame expression* is a tuple $< A, B, C, \gamma >$ where A, B and C are sets of formulae and γ is a formula in L. This expression is called *finitary* if B is finite; otherwise it is called *infinitary*.

b) A *belief state frame theory* or *specification* is a set S of belief state frame expressions.

c) A belief state frame specification S *specifies* the belief state frame $\mathfrak{SB}$ if for any set of formula X and any closed set of models K the following are equivalent:

(i) $K \in \Gamma(X)$

(ii) $K = Mod(X \cup \{\gamma \mid \exists < A, B, C, \gamma > \in S$ such that $A \subseteq X$, $B \subseteq L \setminus X$ and for all $\beta \in C$: not $K \models \neg \beta\})$

In a belief state frame expression $< \mathbf{A}, \mathbf{B}, \mathbf{C}, \boldsymbol{\gamma} >$ the set $\mathbf{A}$ can be seen as the preconditions, the set $\mathbf{B}$ contains formulae which can be considered as "anti-preconditions" (the formulae in $\mathbf{B}$ should *not* be known initially in order for the rule to be applicable), and the set $\mathbf{C}$ is a kind of consistency check (as is also used in default logic). Examples as in [MTT 97] (see also [EMTT 96]) show that sometimes we really need an infinite $\mathbf{C}$.

Definition 5.2
Let $\mathfrak{SB} = (\mathbf{L}, \mathbf{Mod}, \models, \boldsymbol{\Gamma})$ be a belief state frame. Then the belief state frame specification $\mathbf{S}^{\mathfrak{SB}}$ is defined by the following set of belief state frame expressions for each set of formulae $\mathbf{X} : < \mathbf{X}, \mathbf{L} \backslash \mathbf{X}, \mathbf{C}, \boldsymbol{\gamma} >$ where $\boldsymbol{\gamma}$ ranges over a set of generators of $\mathbf{Th(K)}$ over $\mathbf{X}$, for each $\mathbf{K} \in \boldsymbol{\Gamma}(\mathbf{X})$ and $\mathbf{B} = \{ \neg \boldsymbol{\alpha}_{\mathbf{J,K}} \mid \mathbf{J} \in \boldsymbol{\Gamma}(\mathbf{X}), \mathbf{J} \neq \mathbf{K} \}$ with $\boldsymbol{\alpha}_{\mathbf{J,K}}$ defined as follows: as $\mathbf{Th(J)} \backslash \mathbf{Th(K)} \neq \varnothing$ (non-inclusiveness), we choose $\boldsymbol{\alpha}_{\mathbf{J,K}}$ in $\mathbf{Th(J)} \backslash \mathbf{Th(K)}$.

Notice that $\mathbf{S}^{\mathfrak{SB}}$ can be taken finitary if there is only a finite number of atomic proposition symbols. The following theorem shows that any belief state frame $\mathfrak{SB}$ is specified by a belief state frame specification and, conversely, that for every belief state frame specification an associated belief state frame can be defined that is specified by it.

Theorem 5.3
a) Let $\mathbf{S}$ be a belief state frame specification. The operator $\boldsymbol{\Gamma}^{\mathbf{S}}$ defined by

$$\boldsymbol{\Gamma}^{\mathbf{S}}(\mathbf{X}) = \{ \mathbf{K} \mid \mathbf{K} = \mathbf{Mod}(\mathbf{X} \cup \{ \boldsymbol{\gamma} \mid \exists < \mathbf{A}, \mathbf{B}, \mathbf{C}, \boldsymbol{\gamma} > \in \mathbf{S} \text{ such that } \mathbf{A} \subseteq \mathbf{X}, \mathbf{B} \subseteq \mathbf{L} \backslash \mathbf{X} \text{ and for all } \boldsymbol{\beta} \in \mathbf{C}: \text{not } \mathbf{K} \models \neg \boldsymbol{\beta} \})\}$$

for every set of formulae $\mathbf{X}$, is a belief state operator and $\mathfrak{SB}^{\mathbf{S}} = (\mathbf{L}, \mathbf{Mod}, \models, \boldsymbol{\Gamma}^{\mathbf{S}})$ is the belief state frame specified by $\mathbf{S}$.
b) Let $\mathfrak{SB} = (\mathbf{L}, \mathbf{Mod}, \models, \boldsymbol{\Gamma})$ be a belief state frame. Then the belief state frame specification $\mathbf{S}^{\mathfrak{SB}}$ specifies $\mathfrak{SB}$. If $\boldsymbol{\Gamma}$ is invariant and there is only a finite number of atomic proposition symbols, then we can restrict $\mathbf{S}^{\mathfrak{SB}}$ to a finite set of finitary expressions.

Proof
a) It is easy to see that each $\mathbf{K} \in \boldsymbol{\Gamma}^{\mathbf{S}}(\mathbf{X})$ is a closed subset of $\mathbf{Mod(X)}$. Suppose $\mathbf{J}, \mathbf{K} \in \boldsymbol{\Gamma}^{\mathbf{S}}(\mathbf{X})$ and $\mathbf{J} \subseteq \mathbf{K}$. Take a rule $< \mathbf{A}, \mathbf{B}, \mathbf{C}, \boldsymbol{\gamma} > \in \mathbf{S}$ with $\mathbf{A} \subseteq \mathbf{X}$, $\mathbf{B} \subseteq \mathbf{L} \backslash \mathbf{X}$ and for all $\boldsymbol{\beta} \in \mathbf{C}$: $\mathbf{J} \not\models \neg \boldsymbol{\beta}$, then also for all $\boldsymbol{\beta} \in \mathbf{C}$: $\mathbf{K} \not\models \neg \boldsymbol{\beta}$. Thus $\{ \boldsymbol{\gamma} \mid \exists < \mathbf{A}, \mathbf{B}, \mathbf{C}, \boldsymbol{\gamma} > \in \mathbf{S}$ **such that** $\mathbf{A} \subseteq \mathbf{X}, \mathbf{B} \subseteq \mathbf{L} \backslash \mathbf{X}$ **and for all** $\boldsymbol{\beta} \in \mathbf{C}$**: not** $\mathbf{J} \models \neg \boldsymbol{\beta}]\} \subseteq \{ \boldsymbol{\gamma} \mid \exists < \mathbf{A}, \mathbf{B}, \mathbf{C}, \boldsymbol{\gamma} > \in \mathbf{S}$ **such that** $\mathbf{A} \subseteq \mathbf{X}, \mathbf{B} \subseteq \mathbf{L} \backslash \mathbf{X}$ **and for all** $\boldsymbol{\beta} \in \mathbf{C}$**: not** $\mathbf{K} \models \neg \boldsymbol{\beta}]\}$, but then $\mathbf{K} \subseteq \mathbf{J}$ which implies $\mathbf{J} = \mathbf{K}$. So $\boldsymbol{\Gamma}^{\mathbf{S}}$ is a belief state operator. The second statement is trivial.
b) Abbreviate $\mathbf{S}^{\mathfrak{SB}}$ as $\mathbf{S}$. Take a set $\mathbf{X} \subseteq \mathbf{L}$.
- Suppose $\mathbf{K} \in \boldsymbol{\Gamma}(\mathbf{X})$. We call a rule $< \mathbf{A}, \mathbf{B}, \mathbf{C}, \boldsymbol{\gamma} > \in \mathbf{S}$ *applicable* in $\mathbf{K}$ if $\mathbf{A} \subseteq \mathbf{X}$, $\mathbf{B} \subseteq \mathbf{L} \backslash \mathbf{X}$ and for all $\boldsymbol{\beta} \in \mathbf{C}$: $\mathbf{K} \not\models \neg \boldsymbol{\beta}$. It is easy to see that the set of rules applicable in $\mathbf{K}$ is the set of rules $< \mathbf{X}, \mathbf{L} \backslash \mathbf{X}, \mathbf{C}, \boldsymbol{\gamma} >$ with

$C = \{ \neg \alpha_{J,K} \mid J \in \Gamma(X),\ J \neq K \}$. The conclusions of these rules form a set of generators G for $Th(K)$ over X, so that $Th(K) = Cn(X \cup G)$, and as K is closed we have $K =$
$Mod(X \cup \{ \gamma \mid \exists < A, B, C, \gamma > \in S$ such that $A \subseteq X,\ B \subseteq L \backslash X$ and for all $\beta \in C$: not $K \models \neg \beta]\})$.
- Suppose $K = Mod(X \cup \{ \gamma \mid \exists < A, B, C, \gamma > \in S$ such that $A \subseteq X$, $B \subseteq L \backslash X$ and for all $\beta \in C$: not $K \models \neg \beta]\})$.
Take a rule $< X, L \backslash X, C, \gamma > \in S$ with $B = \{ \neg \alpha_{J,K'} \mid J \in \Gamma(X),$ $J \neq K' \}$ for some $K' \in \Gamma(X)$ and suppose it is applicable in K. Then all rules of the form $< X, L \backslash X, C, \varphi >$ are applicable in K, so $\{ \gamma \mid \exists < A, B, C, \gamma > \in S$ such that $A \subseteq X,\ B \subseteq L \backslash X$ and for all $\beta \in C$: not $K \models \neg \beta]\}$ contains a set of generators G for $Th(K')$ over X, so $Th(K) \supseteq Th(K')$ so $K \subseteq K'$. Now take any other rule (if possible) $< X, L \backslash X, D, \psi > \in S$
with $D = \{ \neg \alpha_{J,N} \mid J \in \Gamma(A),\ J \neq N \}$, then $K' \neq N$, so $\neg \alpha_{K',N} \in D$ with $\alpha_{K',N} \in Th(K') \backslash Th(N)$. Then $\alpha_{K',N} \in Th(K') \subseteq Th(K)$, so $K \models \alpha_{K',N}$, so $K \models \neg (\neg \alpha_{K',N})$ which means that the rule $< X, L \backslash X, D, \psi >$ is not applicable. This means that $K = Mod(A \cup G) = Mod(Th(K')) = K' \in \Gamma(A)$. Thus S specifies $\mathfrak{S}\mathfrak{B}$. If Γ is invariant we need only specify Γ^S for theories, and if there is only a finite number of atomic proposition symbols then there are only finitely many theories. Also there are only finitely many different information states, so for each theory X, $\Gamma(X)$ is finite, therefore in any rule $< A, B, C, \gamma > \in S$, the set C is finite. Furthermore we can take a set of generators of $Th(K)$ over X which is finite.

Belief state frame specifications for default logic

We can give the belief state frame specification for the belief state operator Γ_D of default logic, for a set of defaults D. Suppose first that all defaults in D are prerequisite-free. Every such rule $(: \beta_1, ..., \beta_n)/\gamma$ can be translated into the belief state frame expression $< \emptyset, \emptyset, \{ \beta_1, ..., \beta_n \}, \gamma >$. If there are rules in D *with* prerequisite, then the construction is slightly more involved. For every $X \subseteq L$, the default theory $\langle X, D \rangle$ can be transformed into an equivalent default theory $\langle X, D' \rangle$, where D' is prerequisite-free (see [MT 93]). Every rule $(: \beta_1, ..., \beta_n)/\gamma \in D'$ can be translated into the expression $<X, L \backslash X, \{ \beta_1, ..., \beta_n \}, \gamma >$. The reason that the first two parts of this expression can not be empty, is that the set D' depends in general on X.

6. A temporal specification language for reasoning frames

A simple observation allows us to find a natural description language for reasoning traces: the steps in a reasoning trace can be viewed as temporal steps. This means that the transition from an information state to the next one (as the result of a number of inference steps) can be seen as a temporal one. In this view a trace is a temporal model based on the set of natural numbers as the flow of time. An obvious candidate for describing these models is temporal logic. However, the full (tense) logic will turn out to be not completely appropriate: on the one hand it can describe models which are not traces but on the other hand it is not powerful enough. Therefore we will introduce a limited fragment of infinitary tense logic (based on our essentially three-valued information states).

Definition 6.1 (Temporal model)

i) A *temporal model* is a function $\mathcal{M} : \mathbb{N} \rightarrow \mathbf{IS}$.

ii) A temporal model $\mathcal{M}$ is *conservative* if $\mathcal{M}_s \leq \mathcal{M}_{s+1}$ for all $s \in \mathbb{N}$.

iii) The *refinement ordering* $\leq$ on temporal models is defined by:

$\mathcal{M} \leq \mathcal{N} \Leftrightarrow$ for all s : $\mathcal{M}_s \leq \mathcal{N}_s$ and $\mathcal{M}_0 = \mathcal{N}_0$.

iv) The *limit model*, $\mathbf{lim}\ \mathcal{M}$, of a temporal model $\mathcal{M}$ is the information state defined by

$$\mathbf{lim}\ \mathcal{M} = \bigcap_{s=0}^{\infty} \mathcal{M}_s$$

v) A temporal model $\mathcal{M}$ is sometimes denoted by $(\mathcal{M}_s)_{s \in \mathbf{N}}$.

Note that the notion of a temporal model as such is close to the notion of a reasoning trace: any reasoning trace can be considered a conservative temporal model with the property that if it stabilises one step, then it stabilises forever. However, in a temporal model temporal operators and temporal formulae are interpreted. The temporal language we will use is built from temporal atoms of the form $\mathbf{O}\varphi$ (with $\mathbf{O} \in \{ \mathbf{F}, \mathbf{G}, \mathbf{C}, \mathbf{H_0} \}$ and φ a propositional formula) using negation and infinite conjunctions (we do not need nesting of temporal operators here, although the definition can easily be extended to include them). The truth of a formula α in a temporal model $\mathcal{M}$ at time point $\mathbf{i}$, denoted as $(\mathcal{M}, \mathbf{i}) \models \alpha$, is defined inductively:

Definition 6.2 (Temporal interpretation)

a) For a propositional formulae φ:

$(\mathcal{M}, s) \models \mathbf{F}\varphi$	$\Leftrightarrow$	there exists $t \in \mathbb{N}$, $t > s$ such that $\mathcal{M}_t \models \varphi$
$(\mathcal{M}, s) \models \mathbf{G}\varphi$	$\Leftrightarrow$	for all $t \in \mathbb{N}$ with $t > s$: $\mathcal{M}_t \models \varphi$
$(\mathcal{M}, s) \models \mathbf{C}\varphi$	$\Leftrightarrow$	$\mathcal{M}_s \models \varphi$
$(\mathcal{M}, s) \models \mathbf{H_0}\varphi$	$\Leftrightarrow$	$\mathcal{M}_0 \models \varphi$

b) For a temporal formula α:

$(\mathcal{M}, s) \models \neg\alpha \Leftrightarrow$ it is not the case that $(\mathcal{M}, s) \models \alpha$

c) For a set $\mathbf{A}$ of temporal formula:

$(\mathcal{M}, s) \models \bigwedge\mathbf{A} \Leftrightarrow$ for all $\varphi \in \mathbf{A}$: $(\mathcal{M}, s) \models \varphi$

d) A formula φ is true in a model $\mathcal{M}$, denoted $\mathcal{M} \models \varphi$, if for all $s \in \mathbb{N}$: $(\mathcal{M}, s) \models \varphi$

e) A set of formulae $\mathbf{T}$ is true in a model $\mathcal{M}$, denoted $\mathcal{M} \models \mathbf{T}$, if for all $\varphi \in \mathbf{T}$, $\mathcal{M} \models \varphi$. We call $\mathcal{M}$ a model of $\mathbf{T}$.

Furthermore the connectives $\vee$ and $\rightarrow$ are introduced as the usual abbreviations.

The temporal language we have just introduced is still too powerful: we want to use only a fragment to describe models which can be seen as reasoning traces. So the question is: which fragment is appropriate for reasoning? As steps in a reasoning process are taken whenever a number of (nonmonotonic) inference steps is used, it seems that temporal rules should prescribe taking the equivalent of inference steps in the temporal model. So the next question is what the nature is of a generalized nonmonotonic inference step that a reasoning process can execute. A general format of (temporal) inference rules is $\alpha \rightarrow \mathbf{G}\beta$ where α is a condition for the inference, and β is its conclusion: if the condition α is fulfilled, the conclusion β can be drawn, and will be true henceforth. The condition α may of course include reference to the initial facts and the facts which have been derived earlier (and therefore are still true at the present moment). But in nonmonotonic reasoning there is often also a kind of global consistency check. In default logic for instance, a rule is applicable if certain formulae, called the justifications, are consistent with the final outcome of the reasoning process (called an extension), which means they should be consistent throughout the entire reasoning process. Consistency of a formula usually means that its negation should not be true. Therefore we also allow conditions which state that a certain formula should never (in the future of the reasoning process) be true.

Definition 6.3 (Reasoning theories)
a) A formula is called a *(nonmonotonic) reasoning formula* if it is of the form
$\alpha \wedge \beta \wedge \varphi \wedge \psi \rightarrow \mathbf{G}\gamma$, where

$\alpha = \bigwedge\{ \mathbf{H_0}\varepsilon \mid \varepsilon \in \mathbf{A} \}$ for a set of propositional formulae $\mathbf{A}$.

$\beta = \bigwedge\{ \neg \mathbf{H_0}\delta \mid \delta \in \mathbf{B} \}$ for a set of propositional formulae $\mathbf{B}$.

$\varphi = \bigwedge\{ \neg \mathbf{F}\theta \mid \theta \in \mathbf{C} \}$ for a set of propositional formulae $\mathbf{C}$.

$\psi = \bigwedge\{ \mathbf{C}\zeta \mid \zeta \in \mathbf{D} \}$ for a set of propositional formulae $\mathbf{D}$.

γ is a propositional formula.

A reasoning formula is called *finitary* if all sets of formulae involved are finite; otherwise it is called *infinitary*.
b) A set $\mathbf{Th}$ of reasoning formulae is called a *theory of reasoning*. It is called *finitary* if all its elements are; otherwise it is called *infinitary*.

So a reasoning formula prescribes the truth of a formula in the future based on knowledge of initial facts, truth of current facts and consistency of facts in the future (if $\neg \mathbf{F}\theta$ is true, then θ is never true in the future, so it is always either false or unknown).

Definition 6.4 (Conservativity)
The theory $\mathbf{Cons} = \{ \mathbf{C}\alpha \rightarrow \mathbf{G}\alpha \mid \alpha$ **a propositional formula** $\}$ is a theory of reasoning expressing conservativity of temporal models.

A theory of reasoning prescribes truth of facts in the future, analogously to inference steps. But what about facts which become true at a point in time spontaneously, that is without any inference rule prescribing their truth? We should have a way to make sure that this does not happen: we want the models to have minimal information in the sense that nothing becomes true if there are no rules saying so. This leads to the following notion of minimal models:

Definition 6.5 (Minimal temporal models)
A temporal model $\mathcal{M}$ is called a *minimal model* of a theory **Th** if it is a model of **Th** and for any model $\mathcal{N}$ of **Th**, if $\mathcal{N} \leq \mathcal{M}$ then $\mathcal{N} = \mathcal{M}$.

A minimal model of a theory is a model for which there are no smaller models of the theory, so they contain a minimum of information.

Given the fragment of temporal logic we have defined, a natural property to investigate is whether this fragment is suited for describing reasoning traces.
From now on we will assume that any theory includes the theory **Cons** described in Definition 6.4. The first result is that all minimal models of any theory are reasoning traces. On the other hand, any reasoning frame is the set of minimal models of a theory of reasoning:

Theorem 6.6
a) For any theory of reasoning **Th** its minimal models constitute a (partial) reasoning frame.
b) For any (partial) reasoning frame $\mathcal{T}$ there exists a theory of reasoning whose minimal models are exactly $\mathcal{T}$. If there is only a finite number of atomic proposition symbols, then such a theory of reasoning can be taken finite and finitary.

Proof
a) It is easy to see that a temporal model $\mathcal{M}$ is conservative if and only if $\mathcal{M} \models \mathbf{Cons}$.
Let **Th** be a theory of reasoning. First we will show that any minimal model of **Th** is a reasoning trace. Let $\mathcal{M}$ be a minimal model of **Th**. Since **Th** contains **Cons**, $\mathcal{M}$ is conservative. We call a rule $\alpha \wedge \beta \wedge \varphi \wedge \psi \rightarrow \mathbf{G}\gamma$ with α, β, φ, ψ and γ as in Definition 6.3 *applicable* in $\mathcal{M}$ at time point $\mathbf{t}$ if $(\mathcal{M}, \mathbf{t}) \models \alpha \wedge \beta \wedge \varphi \wedge \psi$. We claim that for $\mathbf{t} \geq 0$: $\mathcal{M}_{t+1} = \mathbf{Mod}(\mathbf{Th}(\mathcal{M}_t) \cup \{\gamma \mid \alpha \wedge \beta \wedge \varphi \wedge \psi \rightarrow \mathbf{G}\gamma \in \mathbf{Th}$ is applicable at $\mathbf{t}\})$. If for some $\mathbf{s}$, $\mathbf{Th}(\mathcal{M}_{s+1})$ is not a superset of $\mathbf{Cn}(\mathbf{Th}(\mathcal{M}_s) \cup \{\gamma \mid \alpha \wedge \beta \wedge \varphi \wedge \psi \rightarrow \mathbf{G}\gamma \in \mathbf{Th}$ is applicable at $\mathbf{s}\})$, then either $\mathcal{M}$ is not conservative (but $\mathcal{M} \models \mathbf{Cons}$) or some rule is not satisfied, so $\mathcal{M}$ is not a model of **Th**. Therefore $\mathbf{Th}(\mathcal{M}_{s+1}) \supseteq \mathbf{Cn}(\mathbf{Th}(\mathcal{M}_s) \cup \{\gamma \mid \alpha \wedge \beta \wedge \varphi \wedge \psi \rightarrow \mathbf{G}\gamma \in \mathbf{Th}$ is applicable at $\mathbf{s}\})$. Suppose it is a proper superset, then define the model $\mathcal{N}$ by $\mathcal{N}_t = \mathcal{M}_t$ for $\mathbf{t} \neq \mathbf{s+1}$ and $\mathcal{N}_{s+1} = \mathbf{Mod}(\mathbf{Th}(\mathcal{M}_s) \cup \{\gamma \mid \alpha \wedge \beta \wedge \varphi \wedge \psi \rightarrow \mathbf{G}\gamma \in \mathbf{Th}$ is applicable in $\mathcal{N}$ at $\mathbf{s}\})$. Then it is easy to see that $\mathcal{N} < \mathcal{M}$ and that $\mathcal{N}$ is conservative. Take any rule $\alpha \wedge \beta \wedge \varphi \wedge \psi \rightarrow \mathbf{G}\gamma \in \mathbf{Th}$ and suppose it is applicable in $\mathcal{N}$ at time point $\mathbf{t}$. Then it can be easily verified that it is also applicable in $\mathcal{M}$ at time point $\mathbf{t}$. If $\mathbf{t} \neq \mathbf{s}$ then (since $\mathcal{M} \models \mathbf{Th}$) we have

$\gamma \in \mathbf{Th}(\mathcal{M}_{t+1}) = \mathbf{Th}(\mathcal{N}_{t+1}) \subseteq \mathbf{Th}(\mathcal{N}_u)$ for $u > t+1$,
so $(\mathcal{N}, t) \vDash \alpha \wedge \beta \wedge \varphi \wedge \psi \rightarrow G\gamma$. For $t = s$ we have by definition of $\mathcal{N}_{s+1}$ that $\gamma \in \mathbf{Th}(\mathcal{N}_{t+1}) \subseteq \mathbf{Th}(\mathcal{N}_u)$ for $u > t+1$, so again
$(\mathcal{N}, t) \vDash \alpha \wedge \beta \wedge \varphi \wedge \psi \rightarrow G\gamma$. Therefore $\mathcal{N}$ is a model of **Th** and $\mathcal{N} < \mathcal{M}$ in contradiction with the fact that $\mathcal{M}$ was a minimal model of **Th**. We have proven the claim.
Now suppose that $\mathcal{M}_s = \mathcal{M}_{s+1}$ for some s, then there is no rule in **Th** which is applicable in $\mathcal{M}$ at time point s, but not before. It is easy to see that the same rules are applicable at time point $t > s$ and at time point s, and with the claim it follows that $\mathcal{M}_t = \mathcal{M}_s$ for $t > s$. So $\mathcal{M}$ is a reasoning trace. Now we will prove that the set of minimal models is a reasoning frame. Take two minimal models $\mathcal{M}$ and $\mathcal{N}$ of **Th** and suppose that $\mathcal{M}_0 = \mathcal{N}_0$ and $\lim \mathcal{M} \leq \lim \mathcal{N}$. We will prove by induction that for all i, $\mathcal{M}_i \leq \mathcal{N}_i$:
- $i = 0$: by assumption $\mathcal{M}_0 = \mathcal{N}_0$ (so in particular $\mathcal{M}_0 \leq \mathcal{N}_0$).
- Induction step: Suppose a rule $\alpha \wedge \beta \wedge \varphi \wedge \psi \rightarrow G\gamma \in \mathbf{Th}$ is applicable in $\mathcal{M}$ at time point i. Since $(\mathcal{M}, i) \vDash \alpha \wedge \beta$ and $\mathcal{M}_0 = \mathcal{N}_0$ we have $(\mathcal{N}, i) \vDash \alpha \wedge \beta$. Note that $\varphi = \bigwedge\{\neg F\theta \mid \theta \in C\}$ for a set of propositional formulae C. Take a formula $\neg F\theta$ with $\theta \in C$, then $(\mathcal{M}, i) \vDash \neg F\theta$, so $\theta \notin \mathbf{Th}(\mathcal{M}_t)$ for $t > i$, but then $\theta \notin \mathbf{Th}(\lim \mathcal{M}) \supseteq \mathbf{Th}(\lim \mathcal{N})$, so $\theta \notin \mathbf{Th}(\mathcal{N}_t)$ for all t, in particular for $t > i$, so
$(\mathcal{N}, i) \vDash \neg F\theta$. So $(\mathcal{N}, i) \vDash \varphi$. Also, $\psi = \bigwedge\{C\zeta \mid \zeta \in D\}$ for a set of propositional formulae D. Take $\zeta \in D$, then as $(\mathcal{M}, i) \vDash C\zeta$ we have $\zeta \in \mathbf{Th}(\mathcal{M}_i) \subseteq \mathbf{Th}(\mathcal{N}_i)$ (by induction hypothesis), so $(\mathcal{N}, i) \vDash C\zeta$. It follows that $(\mathcal{N}, i) \vDash \alpha \wedge \beta \wedge \varphi \wedge \psi$, so $\alpha \wedge \beta \wedge \varphi \wedge \psi \rightarrow G\gamma \in \mathbf{Th}$ is applicable in $\mathcal{N}$ at time point i. Since both $\mathcal{M}$ and $\mathcal{N}$ are minimal models of **Th**, with our claim we have:
$\mathbf{Th}(\mathcal{M}_{t+1}) =$
$\mathbf{Cn}(\mathbf{Th}(\mathcal{M}_t) \cup \{\gamma \mid \alpha \wedge \beta \wedge \varphi \wedge \psi \rightarrow G\gamma \in \mathbf{Th}$ is applicable in $\mathcal{M}$ at $t\}) \subseteq$
$\mathbf{Cn}(\mathbf{Th}(\mathcal{N}_t) \cup \{\gamma \mid \alpha \wedge \beta \wedge \varphi \wedge \psi \rightarrow G\gamma \in \mathbf{Th}$ is applicable in $\mathcal{N}$ at $t\}) =$
$\mathbf{Th}(\mathcal{N}_{t+1})$ and as $\mathcal{M}_{t+1}$ and $\mathcal{N}_{t+1}$ are closed we have $\mathcal{M}_{t+1} \leq \mathcal{N}_{t+1}$.
So $\mathcal{M}_i \leq \mathcal{N}_i$ for all i and $\mathcal{M}_0 = \mathcal{N}_0$ so $\mathcal{M} \leq \mathcal{N}$ and as $\mathcal{N}$ was a minimal model of **Th** we have $\mathcal{M} = \mathcal{N}$. We have proven that the set of minimal models is a reasoning frame.
b) Instead of giving a direct construction of the theory of reasoning, we will use Theorem 6.8. With a construction similar to the one described in the proof of Theorem 4.4 (restricting the domain of the parameter since $\mathfrak{F}$ need not be complete) we can show that there exists a reasoning frame parameter p such that $\mathfrak{F}_p = \mathfrak{F}$. With Theorem 6.8 there exists a theory $\mathbf{Th}_p$ such that the set of minimal models of $\mathbf{Th}_p$ is exactly $\mathfrak{F}_p$.

Given a belief state operator Γ and a reasoning frame parameter p suited for Γ, we have defined the corresponding reasoning frame $\mathfrak{F}_p$. As this is a reasoning frame, by

Theorem 6.6 we can find a theory of reasoning whose minimal models are exactly $\mathfrak{T}_p$. However, we can find a more intuitive and direct definition of a theory for reasoning for Γ and p yielding the corresponding reasoning frame.

Definition 6.7

Let $p : \mathbb{B} \to \mathbb{TP}$ be a reasoning frame parameter. Take a pair $(\mathbf{X}, \mathbf{Y}) \in \mathbb{B}$, and a formula $\gamma \in \mathbf{Y}\backslash\mathbf{X}$, and denote $p(\mathbf{X}, \mathbf{Y})$ as $\boxed{\prec}$. Then we define the rule for γ as:

$$\alpha \wedge \beta \wedge \varphi \wedge \psi \to \mathbf{G}\gamma, \text{ with}$$

$$\alpha = \bigwedge\{ \mathbf{H_0 g} \mid \mathbf{g} \in \mathbf{X} \},$$

$$\beta = \bigwedge\{ \neg \mathbf{H_0}\, \delta \mid \delta \in \mathbf{L}\backslash\mathbf{X} \},$$

$$\varphi = \bigwedge\{ \neg\, \mathbf{F}\ \alpha_{\mathbf{Y},\mathbf{Y'}}(\mathbf{X}) \mid \mathbf{Y'} \neq \mathbf{Y},\ (\mathbf{X}, \mathbf{Y'}) \in \mathbb{B}\},$$

$$\psi = \bigwedge\{ \mathbf{C}\varepsilon \mid \varepsilon \in \mathbf{Y},\ \varepsilon\ \boxed{\prec}\ \gamma \}.$$

In the formula φ, the $\alpha_{\mathbf{Y},\mathbf{Y'}}$ are as follows: since $\mathbf{Y'}\backslash\mathbf{Y} \neq \varnothing$ (noninclusiveness) we choose $\alpha_{\mathbf{Y},\mathbf{Y'}}$ in $\mathbf{Y'}\backslash\mathbf{Y}$. This ensures that γ is added only if we are "heading" for the right limit, $\mathbf{Y}$. Let $\mathbf{Th}_{(\mathbf{X},\mathbf{Y})}$ denote the set of these rules for all $\gamma \in \mathbf{Y}\backslash\mathbf{X}$.

For any theory $\mathbf{X}$ such that there is no pair $(\mathbf{X}, \mathbf{Y}) \in \mathbb{B}$ we define its rule:

$$\alpha \wedge \beta \to \mathbf{G}\bot$$

with α and β as above. Let $\mathbf{Th_N}$ be the set of these rules for all such $\mathbf{X}$.

Then a *theory of (nonmonotonic) reasoning for* p, $\mathbf{Th}_p$ is defined by:

$$\mathbf{Th}_p = \bigcup_{(\mathbf{X},\mathbf{Y}) \in \mathbb{B}} \mathbf{Th}_{(\mathbf{X},\mathbf{Y})} \cup \mathbf{Th_N} \cup \mathbf{Cons}$$

Note that the formulae $\alpha_{\mathbf{Y},\mathbf{Y'}}$ are defined here differently from Definition 5.2.

Theorem 6.8

Let $p : \mathbb{B} \to \mathbb{TP}$ be a reasoning parameter, then the set of minimal models of $\mathbf{Th}_p$ is exactly $\mathfrak{T}_p$. If there is only a finite number of atomic proposition symbols, then we can restrict $\mathbf{Th}_p$ to a finite and finitary theory.

Proof

- Take a trace $\mathcal{M}$ in $\mathfrak{T}_p$, and suppose it is the trace associated with $p(\mathbf{X}, \mathbf{Y})$ for some $(\mathbf{X}, \mathbf{Y}) \in \mathbb{B}$. Denote $p(\mathbf{X}, \mathbf{Y})$ as $\boxed{\prec}$. Since $(\mathbf{X}, \mathbf{Y}) \in \mathbb{B}$, there are no rules $\alpha \wedge \beta \to \mathbf{G}\bot$ applicable in $\mathcal{M}$ at any time point. Now take any other rule $\alpha \wedge \beta \wedge \varphi \wedge \psi \to \mathbf{G}\gamma$ with

$$\alpha = \bigwedge\{ \mathbf{H_0 g} \mid \mathbf{g} \in \mathbf{X'} \},$$

$$\beta = \bigwedge\{ \neg \mathbf{H_0}\, \delta \mid \delta \in \mathbf{L}\backslash\mathbf{X'} \},$$

$$\varphi = \bigwedge\{ \neg\, \mathbf{F}\ \alpha_{\mathbf{Y'},\mathbf{Z}}(\mathbf{X'}) \mid \mathbf{Z} \neq \mathbf{Y'},\ (\mathbf{X'}, \mathbf{Z}) \in \mathbb{B}\},$$

$$\psi = \bigwedge\{ \mathbf{C}\varepsilon \mid \varepsilon \in \mathbf{Y'},\ p(\mathbf{X'}, \mathbf{Y'})(\varepsilon, \gamma) \}$$

and suppose it is applicable in $\mathcal{M}$ at time point $\mathbf{t}$ (that is, $(\mathcal{M}, \mathbf{t}) \models \alpha \wedge \beta \wedge \varphi \wedge \psi$). As $(\mathcal{M}, \mathbf{t}) \models \alpha \wedge \beta$ it must be the case that $\mathbf{X'} = \mathbf{X}$. Also, since $(\mathcal{M}, \mathbf{t}) \models \varphi$, $\mathbf{Y'}$ must be equal to $\mathbf{Y}$. As $(\mathcal{M}, \mathbf{t}) \models \psi$, for all $\varepsilon \in \mathbf{Y}$ with $\varepsilon\ \boxed{\prec}\ \gamma$ we have $\varepsilon \in \mathbf{Th}(\mathcal{M}_\mathbf{t})$. By definition of the reasoning trace associated with $\boxed{\prec}$ we have $\gamma \in \mathbf{Th}(\mathcal{M}_{\mathbf{t+1}}) \subseteq \mathbf{Th}(\mathcal{M}_\mathbf{s})$ for $\mathbf{s} > \mathbf{t+1}$ so

$(\mathcal{M}, t) \models G\gamma$. Thus $\mathcal{M}$ is a model of Th_p. Now suppose there exists a model $\mathcal{N}$ of Th_p with $\mathcal{N} \leq \mathcal{M}$. We will prove by induction that $\mathcal{N} = \mathcal{M}$:

* $i = 0$: since $\mathcal{N} \leq \mathcal{M}$ we have $\mathcal{N}_0 = \mathcal{M}_0$.

* Induction: we have $\mathcal{M}_{i+1} = Mod(Th(\mathcal{M}_i) \cup \{ \varphi \in Y \mid \{\psi \in Y \mid \psi \boxed{\prec} \varphi\} \subseteq Th(\mathcal{M}_i)\})$ by definition of $\mathcal{M}$. Take a $\gamma \in \{ \varphi \in Y \mid \{\psi \in Y \mid \psi \boxed{\prec} \varphi\} \subseteq Th(\mathcal{M}_i)\}$, then we have a rule

$\alpha \wedge \beta \wedge \varphi \wedge \psi \to G\gamma \in Th_p$ with

$$\alpha = \bigwedge\{ H_0 g \mid g \in X \},$$
$$\beta = \bigwedge\{ \neg H_0 \delta \mid \delta \in L\backslash X \},$$
$$\varphi = \bigwedge\{ \neg F\ \alpha_{Y,Z}(X') \mid Z \neq Y, (X, Z) \in \mathbb{B}\},$$
$$\psi = \bigwedge\{ C\varepsilon \mid \varepsilon \in Y, \varepsilon \boxed{\prec} \gamma \}$$

As $\mathcal{N}_0 = \mathcal{M}_0$ and $\mathcal{N}_s \leq \mathcal{M}_s$ for $s > i$ we have $(\mathcal{N}, i) \models \alpha \wedge \beta \wedge \varphi$, and as for all $\varepsilon \in Y$ with $\varepsilon \boxed{\prec} \gamma$ we have $\varepsilon \in Th(\mathcal{M}_i) = Th(\mathcal{N}_i)$, we have $(\mathcal{N}, i) \models \alpha \wedge \beta \wedge \varphi \wedge \psi$. Since $\mathcal{N}$ is a model of Th_p we have $(\mathcal{N}, i) \models G\gamma$ so $\gamma \in Th(\mathcal{N}_{i+1})$. Furthermore, $\mathcal{N} \models Cons$ so $\mathcal{N}$ is conservative whence $Th(\mathcal{M}_i) \cup \{ \varphi \in Y \mid \{\psi \in Y \mid \psi \boxed{\prec} \varphi\} \subseteq Th(\mathcal{M}_i)\} \subseteq Th(\mathcal{N}_{i+1})$. So also $\mathcal{M}_{i+1} \leq \mathcal{N}_{i+1}$ and as $\mathcal{N} \leq \mathcal{M}$ we have $\mathcal{M}_{i+1} = \mathcal{N}_{i+1}$.

We have proved that $\mathcal{M}$ is a minimal model of Th_p.

- Take a minimal model $\mathcal{M}$ of Th_p. The rules of the form $\alpha \wedge \beta \to G\bot$ ensure that there exists a pair $(X, W) \in \mathbb{B}$ with $Th(\mathcal{M}_0) = X$. If no rules are applicable in $\mathcal{M}$ at any time point, then with the claim in the proof of Theorem 6.6, $\mathcal{M}$ must be constant, equal to $\mathcal{M}_0$. Now take any trace $\mathcal{N}$ in $\mathfrak{T}_p$ with $Th(\mathcal{N}_0) = X$ (which exists by the remark above), then we have that $\mathcal{M} \leq \mathcal{N}$, and $\mathcal{N}$ is a minimal model of Th_p (by the first part of the proof), so $\mathcal{M} = \mathcal{N}$ which means that $\mathcal{M} \in \mathfrak{T}_p$.

So suppose a rule $\alpha \wedge \beta \wedge \varphi \wedge \psi \to G\gamma \in Th_p$ with

$$\alpha = \bigwedge\{ H_0 g \mid g \in X \},$$
$$\beta = \bigwedge\{ \neg H_0 \delta \mid \delta \in L\backslash X \},$$
$$\varphi = \bigwedge\{ \neg F\ \alpha_{Y,Z}(X) \mid Z \neq Y, (X, Z) \in \mathbb{B}\},$$
$$\psi = \bigwedge\{ C\varepsilon \mid \varepsilon \in Y, \varepsilon \boxed{\prec} \gamma \}$$

is applicable in $\mathcal{M}$ at time point t. Let $\mathcal{N}$ be the trace in $\mathfrak{T}_p$ with $Th(\mathcal{N}_0) = X$ and $Th(\lim \mathcal{N}) = Y$. By induction we will prove that $\mathcal{M} \geq \mathcal{N}$:

* $i = 0$: $Th(\mathcal{M}_0) = X = Th(\mathcal{N}_0)$ so $\mathcal{M}_0 = \mathcal{N}_0$

* Induction: remember that $\mathcal{N}_{i+1} = Mod(Th(\mathcal{N}_i) \cup \{ \varphi \in Y \mid \{\psi \in Y \mid \psi \boxed{\prec} \varphi\} \subseteq Th(\mathcal{N}_i)\})$. Take a $\chi \in \{ \varphi \in Y \mid \{\psi \in Y \mid \psi \boxed{\prec} \varphi\} \subseteq Th(\mathcal{N}_i)\}$. By induction hypothesis we have $Th(\mathcal{N}_i) \subseteq Th(\mathcal{M}_i)$, so $\{\varepsilon \in Y \mid \varepsilon \boxed{\prec} \chi \} \subseteq Th(\mathcal{M}_i)$. There is a rule $\alpha \wedge \beta \wedge \varphi \wedge \psi' \to G\chi$ in Th_p with α, β and φ as above and $\psi' = \bigwedge\{ C\varepsilon \mid \varepsilon \in Y, \varepsilon \boxed{\prec} \chi \}$, and that rule is applicable in $\mathcal{M}$ at time point i. Therefore we have $\chi \in Th(\mathcal{M}_{i+1})$. As $Cons \subseteq Th_p$, $\mathcal{M}$ is conservative, so

$\mathbf{Th}(\mathcal{N}_{i+1}) = \mathbf{Th}(\mathcal{N}_i) \cup \{ \varphi \in \mathbf{Y} \mid \{\psi \in \mathbf{Y} \mid \psi \boxed{\prec} \varphi\} \subseteq \mathbf{Th}(\mathcal{N}_i)\} \subseteq \mathbf{Th}(\mathcal{M}_{i+1})$,
so $\mathcal{N}_{i+1} \leq \mathcal{M}_{i+1}$.
We have proven that $\mathcal{N} \leq \mathcal{M}$, but $\mathcal{M}$ was a minimal model of $\mathbf{Th}_p$ so $\mathcal{M} = \mathcal{N}$. This gives us that $\mathcal{M} \in \mathfrak{T}_p$.
If there is a finite number of atomic proposition symbols then there are finitely many non-equivalent formulae and finitely many theories, so the second claim follows easily.

The logic we described is infinitary. On the one hand, this is needed to get the result that any reasoning frame can be described by a theory of reasoning (see [MTT 97] and [EMTT 96]). On the other hand, it is possible to work with infinitary logic if the temporal models have a finite representation and if the truth of an infinite conjunction in such a model is decidable (this may depend on the theory of reasoning).

Theories of reasoning for default logic

For default logic a temporal translation was already introduced in [ET 93]. Here a default rule $(\alpha : \beta) / \gamma$ is translated into the temporal rule $\mathbf{C}\,\alpha \wedge \neg\, \mathbf{F}(\neg\, \beta) \rightarrow \mathbf{G}\,\gamma$, which indeed is a reasoning formula in the sense defined above (notice that translated default rules form a strict subset of the temporal language - Default Logic is strictly less expressive than the temporal language, even when only finite conjunctions are allowed).

7. Conclusions, related work and further perspectives

In this paper two levels of specification of nonmonotonic reasoning are distinguished. The notions of belief state frame and reasoning frame were introduced and used as a semantical basis for these levels. Moreover, the semantical connections between the levels were identified. The notion of a belief state operator was inspired by the work on abstract (nonmonotonic) consequence relations (such as the studies of Gabbay, [GAB 85], Shoham, [SHO 87], and Kraus, Lehmann, and Magidor [KLM 90]) and inference operations (see for example [MAK 89] and [MAK 94]). The latter paper (but also [VOO 93]) already suggests to look at intended belief sets abstractly.

The level 2 description, using reasoning traces, is new, to our knowledge, in the field of (nonmonotonic) reasoning (with the possible exception of step-logic (see [EP 90]). Trace semantics for processes in general (not necessarily *reasoning* processes) is of course known from process algebra, and the temporal semantics of programs can also be seen as such, if we view a (linear) temporal model as a trace. The situation calculus semantics for logic programs of [LR 96] are based on the same idea of describing the dynamic reasoning process explicitly.

A specification language for level 1 was introduced and a fragment of infinitary temporal logic was introduced and proposed as a general specification language for level 2. There exist temporal logics for describing the mental states of agents over time (such as the Temporal Belief Logic of [FW 97] or, for example, the multi-agent system logic of [SIN 94]), but these are not geared towards describing nonmonotonic reasoning. A multi-epistemic variant of the temporal logic described here, can be used to formally describe the behaviour of compositional multi-agent systems (see [EJT 98] which descibes the use of this logic for verification purposes).

The use of temporal logic augmented with an ordering on temporal models, and the selection mechanism of taking minimal models of a theory, is analogous to the use of these techniques in nonmonotonic temporal reasoning, which is concerned with commonsense reasoning about actions and change in the world (see [SS 95] for an overview). Their intended semantics capture the idea that in the physical world, nothing changes unless there is an explicit reason for change, whereas our intended semantics capture the intuition that in the mind of a reasoning agent nothing changes unless there is an explicit reason for change.

It was shown that the temporal logic is suited for describing reasoning frames and that for a finite number of atomic proposition symbols only finitary formulae can be used. On the other hand, for a countable language examples exist where finitary formulae are insufficient; see [MTT 97] for an example with countably many belief states that cannot be expressed by finitary means. Furthermore, a correct translation of a reasoning frame parameter to a theory in the logic was given.

Our framework allows a unified way of looking at nonmonotonic reasoning at two different levels of abstraction. Abstract properties of nonmonotonic reasoning at these levels can be studied in a way analogous to for instance in [KLM 90] (see [EHT 95]). Also specification of reasoning can be done in a unified way. Logical properties of the specification language for level 2 are studied in [ENG 96].

Although many nonmonotonic formalisms can be described and specified in our framework, we cannot claim that any reasoning process falls in its scope. We would like to relax some of the conditions for belief state operators and reasoning frames. Since the specification languages we proposed exactly capture these (restricted) notions of belief state frames and reasoning frames, it means we would also have to adapt the specification languages (for instance by allowing a more general form of formulae, or by dropping the conservativity axioms).

In [ET 93], [ET 94] it is shown for a number of examples how the language of temporal logic can be used to specify sets of intended reasoning patterns. The dualism between multiple outcomes and multiple reasoning traces of a nonmonotonic reasoning process is also studied in the context of default logic, leading to representation theory: for which set of outcomes can a default theory be found with these outcomes (for a number of results in this area, see [MTT 97] and [EMTT 96]).

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