An Analytical Model for Mathematical Analysis of Smart Daily Energy Management for Air to Water Heat Pumps

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Abstract

Having a continuous growth in the energy demand for heating, together with decreasing the availability of fossil fuels, renewable energy sources have become an important point of departure and specially air to water heat pumps have been suggested for domestic heating. Using them the main challenge is to be sensitive to both outdoor air temperature and the indoor energy demands to minimize the energy needs. Such a heating system should work more efficiently and economically, also taking into account comfort for the inhabitants, with minimal human intervention. This paper presents an analytical model that mathematically interprets a situation to provide insight in how to use energy more efficiently with an air to water heat pump. The results of the model have been compared with another approach which has shown more confidence in the model. This model can be integrated in a thermostat to obtain more autonomous behavior for indoor heating demands against the changes in outside air temperature over the time.

Keywords: temperature maintain energy demand; temperature increase energy demand; indoor temperature; outdoor temperature; heat pump

1. Introduction

For smart energy management it is a challenge to improve the efficiency of energy usage for given demands. Heating and cooling remain processes that consume a lot of energy and it is predicted that this will increase exponentially (for the European Union statistics, cf. [1]). According to [1], from the total energy consumption 50%

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is for heating and from that 43% is for domestic heating. Therefore, the impact of improving domestic heating energy usage in a house even by a relative small quantity will save billions of euro per year as a country and at a global scale. Also this is a major contributor to carbon dioxide (CO₂) emissions and having a proper energy management method will naturally contribute to a healthy environment and a sustainable development [2]. In practically all European countries heating is a necessary need at the last part of the year. More and more domestic heating systems are considered that allow the use of renewable energy, in contrast to gas-based and oil-based heating systems that fully depend on non-renewable energy. As an important alternative, often heat pumps are suggested, which take most of their energy (e.g., from 50% to 80%) from the heat available in the ambient air, water or soil, and for the rest use electrical energy to drive them, which also can be based on renewable production of, for example, solar and wind energy. Also for heat pumps it is important to make smart decisions about their use. This paper will focus on how to analyse in a mathematical manner the energy usage for heating based on an air-to-water heat pump. The proposed analytical model will provide the information to minimize the energy consumption. Thermostats were introduced to control the temperature in houses more easily and productively [3]. Since the invention of a basic thermostat by Cornelius van Drebbel [3], it has been extended with many features and by now it has come with features like fully programmable and smart behaviors [4]. Nevertheless, as put forward in [4] programmable thermostats have not shown the expected results and even in the USA the given certification for energy saving was discontinued by the EnergyStar™ in 2009. Therefore, it is a challenge to explore how to make thermostats smarter with limited human intervention. An important issue may be continuous parameter adjustment for optimal benefits and facilitation of the real-time sensitivity to environmental changes (both indoor and outdoor).

As one step to address the above mentioned goals, in this paper the focus is to develop a mathematical analysis that can be integrated in a thermostat allowing it to autonomously improve the heating energy usage. The temperature of the environment changes in a certain pattern which can be used as a heuristic knowledge. Making a real-time estimation of this is a basis to locally adjust the heating needs with the intuition of the current situation in the environment together with the outlook of temperature behavior in near future.

Below, in Section 2 the theoretical basis and background are described, and in Section 3 the analytical model is described. Section 4 presents some results of the use of this analytical model, and Section 5 discusses the obtained results and the future perspectives.

2. Background and theoretical basis

In this section the basic mathematical concepts will be introduced; see Table 1 for an overview. Subsequently in different subsections they will be briefly explained.

<table>
<thead>
<tr>
<th>Notation</th>
<th>Description</th>
<th>Unit</th>
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<tbody>
<tr>
<td>SPF</td>
<td>seasonal performance factor</td>
<td></td>
</tr>
<tr>
<td>T₀(t)</td>
<td>outdoor temperature at time t</td>
<td>°C</td>
</tr>
<tr>
<td>Tᵢ(t)</td>
<td>indoor temperature at time t</td>
<td>°C</td>
</tr>
<tr>
<td>Tₜ</td>
<td>water temperature of the heating system</td>
<td>°C</td>
</tr>
<tr>
<td>dd</td>
<td>degree days</td>
<td>°C day</td>
</tr>
<tr>
<td>tmmed</td>
<td>heating energy demand for maintaining a temperature</td>
<td>kWh</td>
</tr>
<tr>
<td>tied</td>
<td>heating energy demand for increasing a temperature</td>
<td>kWh</td>
</tr>
<tr>
<td>ed</td>
<td>total heating energy demand</td>
<td>kWh</td>
</tr>
<tr>
<td>eu</td>
<td>heat pump electrical energy use</td>
<td>kWh</td>
</tr>
<tr>
<td>ε</td>
<td>energy loss per degree day</td>
<td>kWh / °C day</td>
</tr>
<tr>
<td>C</td>
<td>capacity: energy needed per degree increase of temperature</td>
<td>kWh / °C</td>
</tr>
</tbody>
</table>
Degree-days
The concept degree-day has been introduced to approximate the analysis of energy consumption and energy performance of a building based on historical data (e.g., [5]). The number of degree-days is defined as the summation of individual deviations in each time step for a time interval from \( t_1 \) to \( t_2 \) in which the outdoor temperature \( (T_{oa}) \) is below a given indoor temperature \( (T_{id}) \) [6]. This can be expressed mathematically as:

\[
dd(t_1, t_2) = \int_{t_1}^{t_2} (T_{id} - T_{oa}(t)) \, dt \quad \text{when } T_{id} > T_{oa}(t) \text{ for all } t \text{ with } t_1 \leq t \leq t_2
\]

Daytime and nighttime outdoor air temperature
The outdoor air temperature typically shows a 24h periodic behavior (e.g., [7, 8]) and therefore also the energy usage to maintain a constant indoor temperature will vary over the time of a day. There are a few analytical models available to describe the dynamics of the outdoor air temperature. The most common ones are sine-exponential and sinusoidal models, based on four parameter values: sunrise \( (t_{sunrise}) \) and sunset \( (t_{sunset}) \) times and maximum \( (T_{max}) \) and minimum \( (T_{min}) \) temperature values (cf. [7]). Parton and Logan [8] have shown that a sine-exponential model provides more realistic predictions because of the use of both a sunrise and sunset time parameter value, whereas a sinusoidal model includes only the sunset time. Equation (2) presents the daytime outdoor temperature variation \( dot(t) \) and equation (3) presents the nighttime variation \( not(t) \) (adopted from [9]). The values for these parameters are not relative to the clock time but relative to midnight. Here in the evening (before midnight) \( T_{min} \) refers to the minimum temperature ahead in time and in the early morning (after midnight) it is of the day itself; similarly in the early morning (after midnight) \( T_{sunset} \) refers to the temperature at sunset of the previous day, but in the evening (before midnight) it refers to the temperature at sunset on the day itself.

\[
dot(t) = b + a \times \sin \left( t - 15 + \frac{t_{sunset} - t_{sunrise}}{2} \left( \frac{\pi}{t_{sunset} - t_{sunrise}} \right) \right)
\]

where \( a = (T_{max} - T_{min}/(1 - \sin \left( t_{sunrise} - 9 \right) \frac{\pi}{12} )) \) and \( b = (T_{max} - a) \)

\[
not(t) = (T_{min} - d) + (T_{sunset} - T_{min} + d) e^{-a(t-t_{sunset})}
\]

Indoor temperature when cooling down takes place
When the heating system in a house is off, and the outdoor temperature is lower, the indoor temperature decreases. The rate of change of the temperature of an object is proportional to the temperature difference between it and the ambient temperature [10]. When the ambient temperature is not a constant, modeling the indoor temperature under the cooling down process is more challenging. To approximate the time taken to cool down overnight from a given indoor temperature to another temperature, it is crucial to know at each point in time the rate of change of the indoor temperature. The following section adopts Newton’s law of cooling down for this; equation (4) presents the decay of indoor temperature under natural cooling down with a varying ambient temperature. Here \( k \) is the heat transfer coefficient; it consist with energy loss per degree day \( \varepsilon \) and the energy needed per degree increase of temperature (capacity \( C) \).

\[
\frac{dT_{id}(t)}{dt} = -k (T_{id}(t) - T_{oa}(t)) \quad \text{where } k = \frac{\varepsilon}{24C}
\]

From equation (3) the model for \( T_{oa}(t) \) (focusing on cooling in the night) can be substituted:

\[
\frac{dT_{id}(t)}{dt} = -k \left( T_{id}(t) - \left( P + Q e^{-a(t-t_1-t_{sunset})} \right) \right)
\]

where \( P = (T_{min} - d), Q = (T_{sunset} - T_{min} + d), \) and \( t_1 \) is the starting time of the cooling down process

\[
\frac{dT_{id}(t)}{dt} = kP - kT_{id}(t) + kQ e^{a(t_1+t_{sunset})} e^{-at} = A + BT_{id}(t) + De^{-at}; \text{where } A = kP, B = -k, \text{ and } D = kQ e^{a(t_1+t_{sunset})}
\]

Note that when
To estimate from this the time taken in the cooling down process for a given loss of temperature $\Delta T$ it is not possible to find the roots by solving the non-linear equation in (4). An adequate approach for this is to use a numerical method which is able to approximate roots with a sufficient accuracy. Newton’s method \cite{11} is a good choice for this due to its simplicity and good speed (cf. \cite{11}).

**Performance of a heat pump**

Heat pumps use renewable heat sources such as the ambient air of a building \cite{12}. However, there are some drawbacks associated with heat pumps: decrease of the heating capacity and efficiency for lower ambient temperatures and accumulating frost on outdoor coils when the ambient temperature is close to 0 (cf. \cite{13}). According to Zogou and Stamatelos \cite{14} the fuel saving gained by using a heat pump in those days was 50%. The efficiency of a heat pump is closely related to the difference between the ambient air temperature (heat source) and the output temperature of the heat pump \cite{15}, in addition to many other factors (cf. \cite{14}). The performance of a heat pump is indicated by its Seasonal Performance Factor (SPF) in equation (5): the ratio of the heat delivered by the heat pump (energy output: $e_o$) and the electrical energy supplied to it (energy input: $e_i$). For air to water heat pumps SPF usually varies between 2 and 4 and according to Omar and Bo \cite{13} the most efficient pumps may even show a value of 5.9. Being a dynamic property over the outdoor temperature to approximate its value equation (6) can be used which was adopted from \cite{9, 16}.

$$SPF = \frac{e_o}{e_i}$$

$$SPF(T_{od}) = 7.5 - 0.1(T_w - T_{od})$$

where $T_w$ is the heating system water temperature and $T_{od}$ is the outdoor temperature

3. The analytical model

Knowing the key factors to analyse the energy usage the next step is to mathematically construct a model that facilitates the insight of this behavior for certain scenarios. The energy demand ($e_d$) for heating is an essential factor in the analysis of energy usage and mainly depends on maintaining a particular temperature (thermal comfort) given the natural loss of heat and to increasing the indoor temperature from a low value to a high value whenever wanted. The temperature maintenance energy demand ($t_{med}$) depends on the energy loss for a given pair of indoor and outdoor temperatures (where outdoor temperature < indoor temperature): it indicates the same amount of energy through the heating system to compensate for this loss and thus maintain the given indoor temperature. The degree-days concept explained in Section 2 expresses this energy loss sufficiently \cite{6}; the energy loss per degree-day is assumed to be $e$; this is different for each house/building and depends on the isolation of the border between indoor and outdoor with walls, windows, floor, roof, ventilation, etc.). For a given time interval $t_{med}$ can be expressed as in equation (7). Furthermore, temperature increase energy demand ($t_{ied}$) depends on the heat energetical capacity $C$ of the house: this indicates how much energy is needed to raise the temperature by 1 degree. Therefore $t_{ied}$ is proportional to the temperature difference $\Delta T_{id}$ made and relates to the notion of capacity $C$ of the house as in the equation (8). Finally the total energy demand can be expressed as in equation (9).
For a small time interval with length $\Delta t$ the energy usage $eu$ is proportional to the energy demand and relates to the seasonal performance factor SPF of the heat pump; similarly the energy cost $ec$ is proportional to the energy usage and relates to the price $\pi_{el}(t)$ of electricity at time $t$ (see [9]) as expressed in equations (10) and (11) subsequently.

\[
eu(t, t + \Delta t) = \frac{ed(t, t + \Delta t)}{SPF(T_{od}(t))} \tag{10}
\]
\[
ec(t, t + \Delta t) = eu(t, t + \Delta t)\pi_{el}(t) \tag{11}
\]

3.1. Temperature maintenance energy usage

Using equation (7) it is possible to determine $tmed$ and to approximate the value when the time difference is very small. This section provides the simplification for the temperature maintenance energy usage ($tmueu$) as in equation (12), irrespective of the size of $\delta t$ for a night goal temperature ($T_{ng}$).

\[
tmueu(t_1, t_2) = \int_{t_1}^{t_2} \frac{e}{24} \left( T_{ng} - T_{out}(t) \right) / SPF(T_{od}(t)) \, dt \tag{12}
\]

From (3) and (6) it follows:

\[
tmueu(t_1, t_2) = \int_{t_1}^{t_2} \frac{e}{24} \left( T_{ng} - (T_{min} - d) + (T_{sun} - T_{min} + d) e^{-a(t-t_{sun})} \right) \, dt
\]

\[
tmueu(t_1, t_2) = \frac{e}{24} \left[ a_1 + a_2 e^{-a(t-t_{sun})} \right] / b_1 + b_2 e^{-a(t-t_{sun})} \, dt
\]

where $a_1 = T_{ng} - T_{min} + d$, and

$a_2 = -(T_{sun} - T_{min} + d)$, $b_1 = 7.5 - 0.1 T_w + 0.1 (T_{min} - d)$, & $b_2 = 0.1 (T_{sun} - T_{min} + d) = -0.1 a_2$

From (3) it follows $e^{-a(t-t_{sun})} = -((T_{od}(t) - T_{min} + d) / a_2)$ and therefore

\[
tmueu(t_1, t_2) = \left[ \frac{ea_1}{24} \ln \left( b_2 + b_1 e^{a(t-t_{sun})} \right) \right]_{t_1}^{t_2} - \left[ \frac{ea_2}{24} \ln \left( b_1 + b_2 e^{a(t-t_{sun})} \right) \right]_{t_1}^{t_2}
\]

3.2. Temperature increase energy usage

Temperature increase energy usage ($tieu$) can be determined by equation (13); for the sake of simplicity it is assumed that the temperature increase will happen instantly without a time gap.

\[
tieu(t_1, t_2) = \frac{C(T_2 - T_1)}{SPF(T_{od})} \tag{13}
\]
4. Using the model in heating scenarios

The analytical model can be used to analyse and improve the energy usage under dynamic environmental changes too. It represents useful knowledge to take actions to minimize the energy usage (at night on heating) against the the environmental conditions over the time. In this section the use of the model for certain heating scenarios is discussed and the results are compared to results from an alternative approach based on the simulation model presented in [16].

4.1. Analytical vs simulation comparison

In [16] it was the energy usage with the same factors explained in this paper are analysed based on simulation; therefore results from the analytical model introduced here can be compared to those of the simulation model. For this purpose a scenario was considered for both simulation and analytical approaches.

The simulation models in [16] use equations (14) and (15) for simulation through discrete time steps (Δt) of half an hour, whereas as in the analytical model introduced in this paper time is taken continuous. To improve the accuracy of the simulation model the step size was further reduced to 6mins. The behavior was analyzed specifically for data available from indoor temperature 20°C at time 21:00hrs February 1, 2012 to 06:00hrs February 2, 2012. It is assumed that the heating program is not using energy until the temperature reaches to the night goal temperature T_ng (autonomous cooling down takes place). Once the indoor temperature becomes T_ng that temperature is maintained until 06:00hrs 2nd February 2012, and the indoor temperature is increased from T_ng to 20°C at the 06:00hrs. Throughout this time interval, the outdoor temperature is assumed to be behaving as in the equation (3) and SPF is calculated as in the equation (6). According to the collected actual data in [16] for this period of time, the minimum temperature T_min=8.8°C, the temperature at sunset T_sunset=−2.52°C, the time of the sunset t_sunset=17:00, and the outdoor temperature at 21:00hrs T_0d(21:00hrs)=−6.6°C. Furthermore, for the remaining parameters the values were: C=4.6, ε=4, α=0.25, d=0.1, T_n=50°C, and time step size Δt = 6min was for the simulation.

\[
T_{id}(t) = \max \left( T_{id}(t-\Delta t) - \left( \frac{\varepsilon}{24C} T_{id}(t-\Delta t) - T_{od}(t)\Delta t ,T_{ng} \right) \right) \tag{14}
\]

\[
ed = tmed + tied = \varepsilon (T_{id} - T_{od})\Delta t + C(T_{id}(t) - T_{id}(t-\Delta t)) \tag{15}
\]

For this comparison different night goal temperatures were selected: 14°C, 14.5°C, 15°C, 15.5°C, 16°C, 16.5°C, 17°C, 17.5°C, 18°C, 18.5°C, and 19°C. The results are shown in Fig. 1(a). According to Fig. 1(a) the analytical model gives a smooth curve, as expected when increasing the night goal temperature (changing the night goal temperature from 15 to 19), will increase the energy usage from 22.12kWh to 24.12kWh) whereas the simulation

Fig. 1. (a) total energy usage for different night goal temperatures; (b) individual energy usage (i.e. tmeu & tieu) for different night goal temperatures.
model also provides a similar pattern but with a slight zig-zag behavior specially when the night goal temperature is high (changing the night goal temperature from 15 to 19, will increase the energy usage from 22.21kWh to 24.23kWh, so just slightly higher than for the analytical model). The main reason assumed for this change is due to the fact that discrete vs continuous time steps in the approaches. In simulation, in each time step, the energy usage is determined according to the $T_{id}$ and $T_{ad}$ at the beginning of the time interval, and the changes during the time interval are neglected. In the simulation approach, being discretized it is not required to pre-calculate the time taken for cooling down from 20°C to $T_{ng}$. Nevertheless, calculating this value is a must in the analytical approach and for that it is required to find the roots of the equation (4). Due to the nature of the equation (4) it is impossible to find the roots mathematically and therefore as suggested earlier the Newton’s method [11] was used to approximate this value.

Fig. 1(b) presents the changes of temperature maintenance energy usage and temperature increase energy usage for both approaches. As per the Fig. 1 (b), $tieu$ by the analytical model and by the simulation model almost has the same value for each night goal temperature (when it comes to the $T_{ng}$ 19°C the simulation data has shown a slightly higher value); nevertheless $tmeu$ also has somewhat equal results for both, but the simulation model has clearly shown a slight zig-zag behavior (which has contributed to have a noticeable zig-zag pattern in Fig. 1(a) being total energy demand as in the equation (9)).

4.2. Energy usage optimization in real time

In Section 4.1 the comparison of the model with a simulation model was discussed and some confidence in the results was obtained. Nevertheless, in real time usage of this model it may not be possible to provide some parameter values. As a result of that it is important to guess/predict those values and necessary to adjust those values over the time. By knowing this practical limitation it is necessary to do the calculation of this model with certain intervals to achieve the most economical energy usage. An effective way to reduce the domestic energy usage is to let the home temperature lower when they are asleep. Actually, it is common among many families, to let the home temperature be a few degrees lower in the night. But, a question arises, what is the appropriate temperature during the night? On the one hand, the family members are not comfortable if the home is too cold during the night, though it seems that, the lower indoor temperature during the night, the lower energy usage. A comfort range for a house is given as $T_{ng1}$ to $T_{ng2}$ (where $T_{ng1} > T_{ng2}$) and we need to maintain the room temperature within this range as in Fig. 2. From $T_{ng1}$ to $T_{ng2}$ temperature can automatically drop due to the natural cooling. Lets consider a time point $t_2$ when the indoor temperature is $T_2$. At this time point we have 3 options to consider as in the Fig. 2 and in each option it is required to increase the temperature back to the $T_{ng1}$ at the time $t_4$:

- **A**: To increase the temperature by $\Delta T$ ($< T_{ng1}$) and maintain that value until time $t_4$.
  \[ e_{u_{optionA}} = tieu_{A}(t_2) + tmeu_{A}(t_2, t_4) + tieu_{A}(t_4) \]  \[ (16) \]

- **B**: To maintain the temperature $T_2$ until time $t_4$.
  \[ e_{u_{optionB}} = tmeu_{B}(t_2, t_4) + tieu_{B}(t_4) \]  \[ (17) \]

- **C**: To let the temperature $T_2$ drop until $T_{ng2}$ and then from $t_3$ to $t_4$ maintain the temperature $T_{ng2}$.
  \[ e_{u_{optionC}} = tmeu_{C}(t_3, t_4) + tieu_{C}(t_4) \]  \[ (18) \]

![Fig. 2. Temperature maintains options for a night time heating program](image-url)
As in the information in Section 2 and 3 it is possible to solve equations (16), (17), and (18) with the selected parameter values and can take the option which consumes the lowest energy. This particular process can be executed in defined time intervals and in each interval it is possible to select the best option. Therefore, due to the nature of the environment, it provides the most economical solution according to the parameter values available or predicted.

5. Conclusion and future work

In this paper, the domestic energy usage by using an air-to-water heat pump was investigated through an analytical mathematical model. The proposed model calculates the energy used to heat a house according a particular heating program. The proposed model was compared with a simulation. The results were almost the same, and small differences are because of time discretization in simulation. The smaller timesteps in simulation, the smaller difference in results. The implementation results show that there is an advantage of 7.2% in energy usage if let the night temperature comes down to 15 instead of 19.

As we saw, in the mentioned heating program, the house temperature is increased around the sunrise. At this time the outdoor temperature is close to its minimum; as a result, the pump’s performance is low. So, it seems possible to save more energy by starting the heating process a little later. So, an important question which can be answered in future works is that, when is the best time for increasing the temperature, given requirements on comfort?

References