Answers – Lecture 2
Stochastic Processes and Markov Chains, Part2

Question 1
Question 1a
Solve system of equations:

$$\begin{aligned}(1-a)\varphi_1 + b\varphi_2 &= \varphi_1 \\ \varphi_1 + \varphi_2 &= 1.\end{aligned}$$

From the second equation we obtain $\varphi_2 = 1 - \varphi_1$. Substitition in the first yields

$$(1-a)\varphi_1 + b(1-\varphi_1) = \varphi_1.$$

Or:

$$-(a+b)\varphi_1 = -b.$$

Thus: $\varphi_1 = b/(a+b)$ and $\varphi_2 = a/(a+b)$.

Question 1b
Reversibility is assessed through the evaluation of the detailed balance equations. Hence, it (the detailed balance equations) needs checking whether

$$\varphi_i(\mathbf{P})_{i,j} = \varphi_j(\mathbf{P})_{j,i}$$

holds for every i and j. Hereto use the answer to 1a. Then, e.g.:

$$\varphi_1(\mathbf{P})_{1,2} = \frac{b}{a+b}a = \frac{ab}{a+b} = \frac{ab}{a+b} = \frac{a}{a+b}b = \varphi_2(\mathbf{P})_{2,1}.$$

Equation also holds for other choices of i and j. Hence, reversibility holds for all a and b.

Question 1c
Either if $a = 0 = b$, or $a = 1 = b$.

Question 2
Question 2a
The state space $\mathcal{S}$ comprises states 0, 1, 2, en 3. The transition matrix is:

$$\mathbf{P} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \lambda & 1-\lambda-\mu & \mu & 0 \\ 0 & \lambda & 1-\lambda-\mu & \mu \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where $\lambda \geq 0$, $\mu \geq 0$ and $\lambda + \mu \leq 1$

Question 2b
No, the process has two absorption states and is therefor not irreducible. This is a prerequisite for a Markov process to have a stationary distribution.

Question 2c
The transition matrix now is:

$$\mathbf{P} \quad = \quad \begin{pmatrix} 1 & 0 & 0 \\ \lambda & 1-\lambda-\mu & \mu \\ 0 & 0 & 1 \end{pmatrix}.$$

Thus:

$$\begin{aligned} P(X_t = 0 \quad \text{for any } t) &= \lambda + \lambda\,(1-\lambda-\mu) + \lambda\,(1-\lambda-\mu)^2 + \ldots \\ &= \lambda \sum_{t=0}^{\infty} (1-\lambda-\mu)^t \end{aligned}$$

Similarly, for $X_t = 2$. When $\mu \geq \lambda$, $P(X_t = 2 \quad \text{for any } t) \geq P(X_t = 0 \quad \text{for any } t)$.

Question 2d
Using Question 2c, then:

$$\lambda \sum_{t=0}^{\infty} (1-\lambda-\mu)^t \quad > \quad \mu \sum_{t=0}^{\infty} (1-\lambda-\mu)^t$$

when $\lambda > \mu$.

Question 2e
The transition matrix now (after substitution of $\lambda = \mu$) is:

$$\mathbf{P} \quad = \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ \lambda & 1-2\lambda & \lambda & 0 \\ 0 & \lambda & 1-2\lambda & \lambda \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

From this we conclude that

$$\begin{pmatrix} P(X_{t+1}=0) \\ P(X_{t+1}=1) \\ P(X_{t+1}=2) \\ P(X_{t+1}=3) \end{pmatrix} \quad = \quad \begin{pmatrix} P(X_t=0) \\ P(X_t=1) \\ P(X_t=2) \\ P(X_t=3) \end{pmatrix}^{\mathrm{T}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ \lambda & 1-2\lambda & \lambda & 0 \\ 0 & \lambda & 1-2\lambda & \lambda \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Focus on the probabilities of $X_{t+1} = 1$ and $X_{t+1} = 2$ one obtains:

$$\begin{pmatrix} P(X_{t+1}=1) \\ P(X_{t+1}=2) \end{pmatrix} \quad = \quad \begin{pmatrix} P(X_t=1) \\ P(X_t=2) \end{pmatrix}^{\mathrm{T}} \begin{pmatrix} 1-2\lambda & \lambda \\ \lambda & 1-2\lambda \end{pmatrix}.$$

Iterative application of the result above yields the following reformulation in terms of the initial probabilities:

$$\begin{aligned} \begin{pmatrix} P(X_{t+1}=1) \\ P(X_{t+1}=2) \end{pmatrix} &= \begin{pmatrix} P(X_t=1) \\ P(X_t=2) \end{pmatrix}^{\mathrm{T}} \begin{pmatrix} 1-2\lambda & \lambda \\ \lambda & 1-2\lambda \end{pmatrix}^t \\ &= \begin{pmatrix} P(X_1=1) \\ P(X_1=2) \end{pmatrix}^{\mathrm{T}} \begin{pmatrix} a_t(\lambda) & b_t(\lambda) \\ b_t(\lambda) & a_t(\lambda) \end{pmatrix}, \end{aligned}$$

where $a_t(\lambda)$ and $b_t(\lambda)$ are both positive for $\lambda = 0.1$. In particular, then $a_t(\lambda) \geq b_t(\lambda)$ for all t.

From the above we obtain (and $P(X_1 = 1) = 1$):

$$
\begin{aligned}
P(X_{t+2} = 0) &= \lambda \sum_{s=1}^{t+1} P(X_s = 1) \\
&= \lambda \sum_{s=0}^{t} a_s(\lambda) P(X_1 = 1) + b_s(\lambda) P(X_1 = 2) \\
&= \lambda \sum_{s=1}^{t} a_s(\lambda) \\
P(X_{t+2} = 3) &= \lambda \sum_{s=1}^{t} b_s(\lambda),
\end{aligned}
$$

where $\alpha_0(\lambda) = 1$. Thus: $P(X_{t+2} = 0) \geq P(X_{t+2} = 3)$ for all t.

Question 3

Question 3a

```
> # define the matrix
> P <- matrix(c(0.1500, 0.3500, 0.3500, 0.1500, 0.1660, 0.3340, 0.3340,
+ 0.1660, 0.1875, 0.3125, 0.3125, 0.1875, 0.2000, 0.3000,
+ 0.3000, 0.2000), ncol=4, byrow=TRUE)
>
> # calculate its stationary distribution analytically
> statDist <- solve(t(P[1:3,1:3]) - t(matrix(P[4,1:3], ncol=3, nrow=3,
+ byrow=TRUE)) - diag(3), -P[4,1:3])
> statDist <- c(statDist, 1-sum(statDistP2))
> statDist
>
> # calculate its stationary distribution numerically
> # define function for raising matrix to a power
> matrixPower <- function(X, power){
+ Xpower <- X
+ if (power >= 2){
+ for (i in 2:power){
+ Xpower <- X % * % Xpower
+ }
+ }
+ return(Xpower)
+ }
> matrixPower(P, 1000)[1,]
```

Question 3b

```
> GCcontent <- sum(statDist[2:3])
> GCcontent
```

Question 4

Question 4a

The detailed balance equations are given by:

$$p_i j \varphi_i = p_j i \varphi_j.$$

The irreducibility and aperiodic properties imply that all $\varphi_j > 0$. Co-occurrence of this and a nonsymmetric off-diagonal zero ($p_{ij} = 0$ while $p_{ji} \neq 0$ for $i \neq j$, or vice versa) violates the detailed balance equations.

Question 4b

Symmetry of the transition matrix implies a uniform stationary distribution. This uniformity and the symmetry together imply that the detailed balance equations hold.

Question 5

Question 5a

Insert picture.

Question 5b

No, once you end up in states II or III, you will never be able to get to states I and IV.

Question 5c

A limiting distribution with $\lim_{t\to\infty} P(X_t = \mathtt{I}) = 0 = \lim_{t\to\infty} P(X_t = \mathtt{IV})$.

Question 5d

```
> # define the matrix
> P <- matrix(c(0.40, 0.10, 0.30, 0.20, 0.00, 0.20, 0.80,
+ 0.00, 0.00, 0.60, 0.40, 0.00, 0.30, 0.30,
+ 0.10, 0.30), ncol=4, byrow=TRUE)
>
> # perform eigen decomposition
> eigen(P)
```

Question 5e

```
> V <- eigen(P)$vectors
> Vinv <- solve(eigen(P)$vectors)
> D <- diag(eigen(P)$values)
> V % * % D % * % Vinv
```

Question 5f

The second largest eigenvalue equals 0.6. Hence, the influences of initial values wanes at the rate 0.6^n.

Question 6

Question 6a

See lecture notes.

Question 6b
Now the state space $\mathcal{S} = \{\mathtt{A}, \mathtt{G}, \mathtt{C}, \mathtt{T}\}$ is employed. That, the order of the nucleotides is not alphabetic, but order by their chemical properties: first the purines then the pyrimidines. The transition matrix is then:

$$\mathbf{P} = \left(\begin{array}{cccc} 0.35 & 0.15 & 0.25 & 0.25 \\ 0.15 & 0.35 & 0.25 & 0.25 \\ 0.20 & 0.20 & 0.25 & 0.35 \\ 0.20 & 0.20 & 0.35 & 0.25 \end{array} \right).$$

and its stationary distribution $\boldsymbol{\varphi}^T = (0.222, 0.222, 0.278, 0.278)$.

Question 6c
The transition matrix (using the same state space as with Question 6b):

$$\mathbf{P} = \left(\begin{array}{cccc} 0.40 & 0.20 & 0.20 & 0.20 \\ 0.20 & 0.40 & 0.20 & 0.20 \\ 0.20 & 0.20 & 0.40 & 0.20 \\ 0.20 & 0.20 & 0.20 & 0.40 \end{array} \right).$$

and its stationary distribution $\boldsymbol{\varphi}^T = (0.25, 0.25, 0.25, 0.25)$.

Question 6d
$\boldsymbol{\varphi}^T = (0.2360.2360.2640.264)$.

Question 6e
$\boldsymbol{\varphi}^T = (0.23320.23320.26680.2668)$.

Question 7
Question 7a
$P(\mathtt{TAGA}) = 0$.

Question 7b
```
> # define vector with nucleotide frequencies
> nuclFreq <- c(186, 242, 265, 307)
> nuclProb <- nuclFreq / sum(nuclFreq)
>
> # define the transition matrix with ML estimates
> dimerFreq <- matrix(c(58, 60, 0, 68, 39, 77, 83, 43, 43, 0, 87,
+ 134, 46, 105, 95, 61), ncol=4, byrow=TRUE)
> dimerProb <- dimerFreq / rowSums(dimerFreq)
>
> # calculates ingredients for test statistic
> motifProb <- nuclProb[4] * dimerProb[4,3] *
+ dimerProb[3,4] * dimerProb[4,2]
> motifExpFreq <- (500 - 4 + 1) * motifProb
> motifVarFreq <- (500 - 4 + 1) * motifProb * (1 - motifProb)
>
```

```
> # calculate test statistic and significance
> testStat <- (17 - motifExpFreq) / sqrt(motifVarFreq)
> pvalue <- pnorm(testStat, lower.tail=FALSE)
```