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SELF-SIMILAR SOLUTIONS OF BARENBLATT'S MODEL FOR TURBULENCE*

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Abstract. In this paper, we consider Barenblatt's k–ϵ model for turbulence. For the case of equal diffusion coefficients α and β, Barenblatt found explicit compactly supported self-similar solutions. From these, we obtain compactly supported solutions for $\alpha \neq \beta$ by transforming the equations into a four-dimensional quadratic system and verifying a transversality condition for a saddle-point connection. This involves the Poincaré transformation as well as classical properties of the hypergeometric equation and its solutions.

Key words. turbulence, compactly supported similarity solutions, quadratic systems, critical points at infinity, Poincaré transformation, saddle-point connections, transversality

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Introduction. In this paper, we consider the system

$$\text{(KE)} \quad \begin{cases} k_t = \alpha\Big(\dfrac{k^2}{\varepsilon}k_x\Big)_x - \varepsilon, \\ \varepsilon_t = \beta\Big(\dfrac{k^2}{\varepsilon}\varepsilon_x\Big)_x - \gamma\dfrac{\varepsilon^2}{k}. \end{cases}$$

Here α, β, and γ are positive parameters and k and ε are unknown nonnegative functions of x (space) and t (time). This system is called the k–ε model and describes the evolution of turbulent bursts [B] (see also [LS], [HP], and [KV]); k stands for the turbulent energy density and ε is the dissipation rate of turbulent energy. In applications, α and β are usually different [LMRS, HL]. The model is also refered to in the literature as the b–ε model, which is, in fact, the original notation due to Kolmogorov ($k = b$) [K, P, MY]. We note that (KE) is a coupled system of two quasilinear diffusion-absorption equations. The diffusion coefficients may, depending on k and ε, become degenerate (very small) or singular (very large), and the second absorption term is also singular.

The only results that have been rigorously established so far are for the case where $\alpha = \beta$: for $\gamma > 3/2$, a family of explicit self-similar compactly supported "source-type" solutions was found by Barenblatt et al. [BGL], and for $\gamma > 1$, an existence result for solutions to the Cauchy problem was proved by Bertsch, Dal Passo, and Kersner [BdPK1, BdPK2], who also showed that for $\gamma > 3/2$, the self-similar solutions describe the intermediate asymptotics of these solutions.

This paper is concerned with the existence of compactly supported self-similar solutions when $\alpha \neq \beta$. Let us recall that the Barenblatt solutions are obtained by

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substituting

$$k = \frac{A^2}{t^{2\mu}} f(\zeta), \qquad \varepsilon = \frac{A^2}{t^{2\mu+1}} g(\zeta), \qquad \zeta = \frac{x}{At^{1-\mu}}, \tag{0.1}$$

where $A > 0$ is a free-scaling parameter and where we restrict our attention to the case where $0 < \mu < 1$. Thus we look at profiles which decay and spread out as time evolves.

The equations for f and g are

$$\begin{cases} \alpha\Big(\frac{f^2}{g} f'\Big)' + (1-\mu)\zeta f' + 2\mu f - g = 0; & (0.2) \\ \beta\Big(\frac{f^2}{g} g'\Big)' + (1-\mu)\zeta g' + (1+2\mu) g - \gamma \frac{g^2}{f} = 0. & (0.3) \end{cases}$$

If we assume that

$$g(\zeta) = \kappa f(\zeta), \tag{0.4}$$

equations (0.2) and (0.3) can be reduced to one single equation if and only if

$$\alpha = \beta, \qquad \kappa = \frac{1}{\gamma - 1}, \tag{0.5}$$

the resulting equation for f being

$$\frac{\alpha}{\kappa}(ff')' + (1-\mu)\zeta f' + (2\mu - \kappa) f = 0. \tag{0.6}$$

Finally, if also

$$\mu = \frac{\kappa+1}{3}, \quad 0 < \mu < 1, \tag{0.7}$$

equation (0.6) can be written as

$$\frac{3\alpha}{\kappa(2-\kappa)}(ff')' + (\zeta f)' = 0, \tag{0.8}$$

which has compactly supported nonnegative solutions

$$f(\zeta) = \left(C - \frac{\kappa(2-\kappa)}{6\alpha}\zeta^2\right)_+, \quad C > 0, \tag{0.9}$$

if and only if

$$0 < \kappa < 2. \tag{0.10}$$

Note that (0.5), (0.7), and (0.10) imply that $\gamma > \frac{3}{2}$.

We observe that (0.9) corresponds to the well-known Barenblatt profile for the porous-medium equation (denoted by (PME)) $u_t = (u^m)_{xx}$ with $m = 2$. In fact, substitution of $\varepsilon = \kappa k$ together with (0.5) reduces the full system (KE) to (PME).

Just as in the case of the (PME) (see, e.g., [A]), we see that at the boundary of the support of the solutions, the fluxes vanish, i.e.,

$$\frac{f^2}{g} f' \to 0 \quad \text{and} \quad \frac{f^2}{g} g' \to 0. \tag{0.11}$$

The main result of this paper is a perturbation of the explicit family of compactly supported similarity solutions above, yielding a similar family of solutions for $\gamma > 3/2$ and α close to β. This is an important and strong indication that the PDE results mentioned above for $\alpha = \beta$ are not isolated but really a first step towards a full theory for (KE).

THEOREM. *There exists an open neighborhood $\mathcal{O}$ of the set*

$$\left\{(\alpha, \beta, \gamma) : \ \alpha = \beta > 0, \ \gamma > \frac{3}{2}\right\}$$

such that for every $(\alpha, \beta, \gamma) \in \mathcal{O}$, there is precisely one $0 < \mu < 1$ for which equations (0.2) *and* (0.3) *have a solution pair (f, g) with f and g symmetric and positive on* $(-1, 1)$ *and*

$$f(\zeta) \to 0, \quad g(\zeta) \to 0, \quad \frac{f(\zeta)}{g(\zeta)} f'(\zeta) \to -\alpha(1-\mu), \quad \frac{f(\zeta)^2}{g(\zeta)^2} g'(\zeta) \to -\beta(1-\mu) \tag{0.12}$$

as $\zeta \uparrow 1$. Moreover, if we write

$$\kappa = \frac{g(0)}{f(0)}, \qquad \lambda = \frac{\alpha}{\beta}, \tag{0.13}$$

then in $\lambda = 1$,

$$\kappa = \frac{1}{\gamma - 1}, \qquad \frac{d\mu}{d\lambda} = 0, \qquad \frac{d\kappa}{d\lambda} = \frac{\kappa(2-\kappa)}{\kappa+1}\left(\kappa - 1 + \frac{2}{B_\kappa}\right). \tag{0.14}$$

Here B_κ is defined by

$$B_\kappa = \frac{\Gamma(\frac{1}{2})}{\Gamma(a)\Gamma(b)}, \qquad a + b = \frac{3}{2}, \qquad ab = \frac{3}{2(2-\kappa)}. \tag{0.15}$$

In order to perform the perturbation argument, we adapt the methods in [H] and introduce

$$t = \log\zeta, \quad x = \frac{\zeta f'}{f}, \quad y = \frac{\zeta g'}{g}, \quad z = \zeta^2 \frac{g}{\alpha f^2}, \quad u = \frac{g}{f}, \tag{0.16}$$

which transforms the two coupled nonautonomous second-order equations (0.2) and (0.3) into the four-dimensional first-order quadratic autonomous system

$$\text{(Q)} \quad \begin{cases} \dfrac{dx}{dt} = x(1 - 3x + y) - z(x(1-\mu) + 2\mu - u); \\ \dfrac{dy}{dt} = y(1 - 2x) - \lambda z(y(1-\mu) + 2\mu + 1 - \gamma u); \\ \dfrac{dz}{dt} = z(2 + y - 2x); \\ \dfrac{du}{dt} = u(y - x). \end{cases}$$

In section 1, we investigate this system. We find that symmetric profiles (f, g) correspond to the two-dimensional "fast" unstable manifold $\mathcal{F}$ of the positive u-axis and that the profiles satisfying the so-called interface condition as $0 < \zeta \uparrow \zeta^* < \infty$ are contained in the two-dimensional stable manifold $\mathcal{S}$ of a critical point at infinity on the line with direction vector

$$\begin{pmatrix} -(1-\mu) \\ -\lambda(1-\mu) \\ 1 \\ 0 \end{pmatrix}. \tag{0.17}$$

This involves the Poincaré transformation of (Q) and is carried out with the help of Maple. As a byproduct here, we find that it is necessary to assume that

$$\alpha \leq 2\beta \tag{0.18}$$

because otherwise $\mathcal{S}$ is one dimensional and contained in "infinity."

It follows from the analysis in section 1 that the compactly supported profiles we are looking for correspond to intersections of $\mathcal{F}$ and $\mathcal{S}$. In particular, and just as in [H], the explicit solutions above correspond to an orbit which is simply the straight line

$$x = y = -(1-\mu)z, \quad u = \kappa. \tag{0.19}$$

In section 2, we show that in the full $(x, y, z, u, \alpha, \beta, \gamma, \mu)$-space, the intersection of $\mathcal{F}$ and $\mathcal{S}$ is transversal at (0.19), thus obtaining our perturbation result. The dynamical-systems methods we use here were applied earlier in [AV] and [HV] to two-dimensional systems that come from scalar diffusion equations. However, in our case, the computations in which the hypergeometric function, the Gauss formula, and the Kummer relations appear [L] are much more involved, and again it is thanks to the help of Maple that we were able to pull through.

1. The quadratic system. In this section, we examine system (Q) in relation to the boundary conditions imposed on f and g. We note that every solution of (0.2)–(0.3) is mapped into an orbit of (Q) and that scaling with the parameter A in (0.1) corresponds to a shift in t.

By standard ODE theory [CL], there exists for every $p, q > 0$ a unique local solution (f, g) of (0.2)–(0.3) satisfying the initial conditions

$$f(0) = p, \quad g(0) = q, \quad f'(0) = 0, \quad g'(0) = 0. \tag{1.1}$$

This provides us with a two-parameter family of local solutions of (0.2)–(0.3). For the corresponding solution curve $S(t) = (x(t), y(t), z(t), u(t))$, we find

$$\lim_{t\downarrow-\infty} x(t)e^{-2t} = \lim_{\zeta\downarrow 0} \frac{f'(\zeta)}{\zeta f(\zeta)} = \frac{f''(0)}{f(0)} = \frac{1}{\alpha}(q - 2\mu p)\frac{q}{p^3}. \tag{1.2}$$

Here we have used (0.2) to compute $f''(0)$. Similarly, we find

$$\lim_{t\downarrow-\infty} y(t)e^{-2t} = \frac{1}{\beta}(\gamma q - (2\mu+1)p)\frac{q}{p^3}, \qquad \lim_{t\downarrow-\infty} z(t)e^{-2t} = \lim_{\zeta\downarrow 0} \frac{g(\zeta)}{f(\zeta)^2} = \frac{\alpha q}{p^2}, \tag{1.3}$$

and, using l'Hôpital's rule,

$$\begin{aligned}(1.4)\quad &\lim_{t\downarrow-\infty}\left(u(t)-\frac{q}{p}\right)e^{-2t}=\lim_{\zeta\downarrow 0}\frac{pg(\zeta)-qf(\zeta)}{p\zeta^2 f(\zeta)}=\lim_{\zeta\downarrow 0}\frac{pg'(\zeta)-qf'(\zeta)}{2p\zeta f(\zeta)+p\zeta^2 f'(\zeta)}\\ &=\lim_{\zeta\downarrow 0}\frac{pg''(\zeta)-qf''(\zeta)}{2pf(\zeta)+4p\zeta f'(\zeta)+p\zeta^2 f''(\zeta)}=\frac{q^2}{2p^4}\left(\frac{1}{\beta}(\gamma q-(2\mu+1)p)-\frac{1}{\alpha}(q-2\mu p)\right).\end{aligned}$$

Thus $S(t)$ comes out of the point $(x,y,z,u)=(0,0,0,q/p)$ on the positive u-axis into the (invariant) open "quadrant" $O^+=\{z>0,u>0\}$ along an eigenvector corresponding to the eigenvalue 2 of the linearization of (Q) around $(0,0,0,q/p)$, which is

$$(1.5)\qquad \begin{pmatrix}1 & 0 & -(2\mu-\kappa) & 0\\ 0 & 1 & -\lambda(2\mu+1-\gamma\kappa) & 0\\ 0 & 0 & 2 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}.$$

Here

$$(1.6)\qquad \kappa=\frac{q}{p},\qquad \lambda=\frac{\alpha}{\beta}.$$

Clearly, the positive symmetric solution pairs (f,g) are mapped into the "fast unstable manifold" of the u-axis, the sheet of integral curves tangent to the eigenvector of 2. Note that the ratio κ determines the orbit.

Next, we consider solutions of (0.2)–(0.3) with $f(\zeta)\to 0$ and $g(\zeta)\to 0$ and satisfying the no-flux condition (0.11) as $\zeta\uparrow 1$. This cannot be viewed as an initial- (or final-) boundary value problem in such a straightforward manner as above, and therefore we turn to the quadratic system. Any such solution with both components decreasing to zero as $\zeta\uparrow 1$ is mapped into an orbit which escapes to infinity in finite time. Indeed, all of the other orbits contain solutions $S(t)=(x(t),y(t),z(t),u(t))$ which persist as $t\uparrow\infty$, and it is easy to see that the corresponding solutions (f,g) are positive in $\zeta=1$. Thus we look for orbits escaping to infinity in finite time with $x<0$, $y<0$, $z>0$, and $u>0$. This means that x and y cannot both be bounded.

For the study of the unbounded orbits, we use the Poincaré transformation to determine the critical points at infinity. Rewriting (Q) as

$$(\mathrm{Q})\quad \begin{cases}\dot{x}_1=P_1(x_1,x_2,x_3,x_4);\\ \dot{x}_2=P_2(x_1,x_2,x_3,x_4);\\ \dot{x}_3=P_3(x_1,x_2,x_3,x_4);\\ \dot{x}_4=P_4(x_1,x_2,x_3,x_4),\end{cases}$$

where $(x_1,x_2,x_3,x_4)=(x,y,z,u)$ and dots denote differentiation with respect to t, we introduce the new coordinates X_1,X_2,X_3,X_4, and V as follows:

$$(1.7)\qquad x_i=\frac{X_i}{V}\quad (i=1,2,3,4),\qquad X_1^2+X_2^2+X_3^2+X_4^2+V^2=1.$$

This transforms (Q) into an autonomous polynomial system of five first-order differential equations for X_1,X_2,X_3,X_4, and V, which leaves the 4-sphere $S^4=\{X_1^2+X_2^2+X_3^2+X_4^2+V^2=1\}$ invariant.

Differentiating (1.7), we have

$$V\dot{V} + \sum_{j=1}^{4} X_j \dot{X}_j = 0 \tag{1.8}$$

and

$$\dot{X}_i V - X_i \dot{V} = P_i^*, \tag{1.9}$$

where

$$P_i^*(X_1, X_2, X_3, X_4, V) = V^2 P_i(x_1, x_2, x_3, x_4). \tag{1.10}$$

Thus the P_i^*'s are homogeneous polynomials of degree 2. Combining (1.8) and (1.9), we obtain

$$V\left(V^2 + \sum_{j=1}^{4} X_j^2\right)\dot{V} = -V\sum_{j=1}^{4} X_j P_j^* \tag{1.11}$$

and, with (1.8) again,

$$\begin{aligned} V\left(V^2 + \sum_{j=1}^{4} X_j^2\right)\dot{X}_i &= \left(V^2 + \sum_{j=1}^{4} X_j^2\right)P_i^* + X_i\left(V^2 + \sum_{j=1}^{4} X_j^2\right)\dot{V} \\ &= V^2 P_i^* + \sum_{j=1}^{4} X_j (X_j P_i^* - X_i P_j^*). \end{aligned} \tag{1.12}$$

Thus integral curves of (Q) correspond to integral curves with $V > 0$ on S^4 of the system

$$(\tilde{\mathrm{Q}}) \quad \begin{cases} X_i' = V^2 P_i^* + \sum_{j=1}^{4} X_j (X_j P_i^* - X_i P_j^*) \quad (i = 1, 2, 3, 4); \\ V' = -V \sum_{j=1}^{4} X_j P_j^*. \end{cases}$$

Here we have absorbed the factor

$$V\left(V^2 + \sum_{j=1}^{4} X_j^2\right)$$

in the derivative.

Unbounded solutions of (Q) correspond to solutions of ($\tilde{\mathrm{Q}}$) which approach the invariant set $S^4 \cap \{V = 0\}$. The critical points "at infinity" of (Q) are by definition the critical points of ($\tilde{\mathrm{Q}}$) on $S^4 \cap \{V = 0\}$, which in turn are the solutions of

$$\begin{cases} \sum_{j=1}^{4} X_j (X_j P_i^* - X_i P_j^*) = 0 \quad (i = 1, 2, 3, 4); & (1.13) \\ X_1^2 + X_2^2 + X_3^2 + X_4^2 + V^2 = 1. & (1.14) \end{cases}$$

Note that we have five equations for four unknowns. It is implicit in the Poincaré transformation that these equations are dependent.

Using Maple and again writing X, Y, Z, and U for X_1, X_2, X_3, and X_4, we find that (1.13) is equivalent to

$$\begin{cases} \big(Y^3 + (\lambda-1)(1-\mu)Y^2Z - \gamma\lambda YZU - (1-\mu Z^3 - (1-\mu)ZU^2\big)X \\ \qquad\qquad + ZU(Z^2+Y^2+U^2) + (Y^2+2U^2+Z^2)X^2 = 0, \\ \big(XY - Y^2 - (1-\mu)(\lambda-1)YZ + \gamma\lambda ZU\big)X^2 - (U^2+ZU)XY \\ \qquad - (U^2+Z^2)Y^2 - \lambda(1-\mu)(Z^2+U^2)YZ + \gamma\lambda ZU(U^2+Z^2) = 0, \\ Z\big(X^3 + (1-\mu)ZX^2 - (ZU+U^2)X + Y^3 + \lambda(1-\mu)Y^2Z - \gamma\lambda YZU\big) = 0, \\ U\big(2X^3 + (1-\mu)X^2Z + (Y^2+Z^2-ZU)X + Y^3 + \lambda(1-\mu)Y^2Z - \gamma\lambda YZU\big) = 0, \end{cases}$$

which at first sight looks too complicated to evaluate. However, if we multiply the third equation by Z and the fourth equation by U, subtraction gives

$$XZU(X^2+Y^2+Z^2+U^2) = XZU = 0, \tag{1.15}$$

which reduces the system. Also, if we substitute $X = 0$ in the first equation, we obtain

$$ZU(Y^2+Z^2+U^2) = ZU = 0. \tag{1.16}$$

Thus all the solutions of (1.13)–(1.14) have either $Z = 0$ or $U = 0$ or have both. This allows us to solve (1.13)–(1.14) explicitly, either by hand or by again using Maple. The solutions (X, Y, Z, U) with $Z \geq 0$ and $U \geq 0$ are

$$(\pm1,0,0,0), \quad (0,\pm1,0,0), \quad (0,0,1,0), \quad (0,0,0,1), \quad \left(\pm\sqrt{\frac{1}{2}}, \pm\sqrt{\frac{1}{2}}, 0, 0\right),$$

$$\left(\frac{-(1-\mu)}{\sqrt{1+(1-\mu)^2}}, 0, \frac{1}{\sqrt{1+(1-\mu)^2}}, 0\right), \quad \left(0, \frac{-\lambda(1-\mu)}{\sqrt{1+\lambda^2(1-\mu)^2}}, \frac{1}{\sqrt{1+\lambda^2(1-\mu)^2}}, 0\right),$$

and, last but not least,

$$P = \left(\frac{-(1-\mu)}{\sqrt{1+(1+\lambda^2)(1-\mu)^2}}, \frac{-\lambda(1-\mu)}{\sqrt{1+(1+\lambda^2)(1-\mu)^2}}, \frac{1}{\sqrt{1+(1+\lambda^2)(1-\mu)^2}}, 0\right). \tag{1.17}$$

Solution curves of ($\tilde{\mathrm{Q}}$) going into P from $S^4 \cap \{V > 0\}$ correspond to solution curves of (Q) with

$$\frac{x}{z} \to -(1-\mu), \qquad \frac{y}{z} \to -\lambda(1-\mu), \qquad \frac{u}{z} \to 0 \tag{1.18}$$

so that in view of the equation for z,

$$\frac{d}{dt}\frac{1}{z(t)} \to (\lambda-2)(1-\mu). \tag{1.19}$$

Thus if $\lambda < 2$, these orbits reach infinity in a finite time t^* with none of the functions $x(t)$, $y(t)$, and $z(t)$ integrable near t^*. (Note that $\lambda > 2$ is impossible, as the linearization of ($\tilde{\mathrm{Q}}$) around P will confirm.) Since

$$\int x(t)dt = \int \frac{f'(\zeta)}{f(\zeta)}d\zeta, \qquad \int y(t)dt = \int \frac{g'(\zeta)}{g(\zeta)}d\zeta, \tag{1.20}$$

it follows that

$$f(\zeta^*) = g(\zeta^*) = 0, \quad \zeta^* = e^{t^*}. \tag{1.21}$$

From (1.18), we also have

(1.22)
$$\frac{f(\zeta)}{g(\zeta)} f'(\zeta) = \frac{\alpha x(t)}{z(t)} e^t \to -\alpha(1-\mu)\zeta^*, \qquad \frac{f(\zeta)^2}{g(\zeta)^2} g'(\zeta) = \frac{\alpha y(t)}{z(t)} e^t \to -\beta(1-\mu)\zeta^*$$

so that condition (0.11) is satisfied at ζ^*. We shall call (1.22) the interface conditions for f and g.

The linearization of ($\tilde{\mathrm{Q}}$) around P has eigenvalues

$$-\frac{1-\mu}{\sqrt{1+(1+\lambda^2)(1-\mu)^2}}, \qquad -\frac{(1-\mu)(2-\lambda)}{\sqrt{1+(1+\lambda^2)(1-\mu)^2}}, \qquad 0,$$
$$\frac{1-\mu}{\sqrt{1+(1+\lambda^2)(1-\mu)^2}}, \qquad \frac{\lambda(1-\mu)}{\sqrt{1+(1+\lambda^2)(1-\mu)^2}},$$

i.e., up to a (positive if $\mu < 1$) multiple, simply

$$-1, \quad -(2-\lambda), \quad 0, \quad 1, \quad \lambda.$$

We note that zero is always an eigenvalue with eigenvector perpendicular to S^4. Since we only consider the flow on S^4, this eigenvector is irrelevant.

The only eigenvector with a nonzero V-component is the one corresponding to the eigenvalue which changes sign when λ crosses the value 2. Consequently, we may distinguish between two cases.

$0 < \lambda < 2$: The stable and unstable manifolds both have dimension two. The stable manifold contains a one-parameter family of solutions satisfying the interface conditions.

$\lambda > 2$: The stable manifold has dimension one and the unstable manifold dimension three. The stable manifold is contained in $\{V = 0\}$, implying that there are no orbits going into P coming from $\{V > 0\}$.

2. Transversality of the connection. In this section, we show that the explicit compactly supported solution which exists for

$$\alpha = \beta, \qquad \mu = \frac{\gamma}{3(\gamma-1)}, \qquad \gamma > \frac{3}{2}, \tag{2.1}$$

can be used to obtain a compactly supported solution for $\alpha \neq \beta$. Throughout this section, the value of $\gamma > 3/2$ is fixed. Condition (2.1) follows from (0.5) and (0.7).

The orbit of (Q) corresponding to the exact solutions in the introduction is the straight line (0.19), and it belongs to an analytic family of solution curves of the form

$$x = X(z;\kappa,\mu,\lambda), \qquad y = Y(z;\kappa,\mu,\lambda), \qquad u = U(z;\kappa,\mu,\lambda), \tag{2.2}$$

which are defined as the images under (0.16) of the symmetric solutions to (0.2)–(0.3), and together form the "fast unstable manifold" $\mathcal{F}$ of the u-axis. In particular, we have

$$X(0;\kappa,\mu,\lambda) = 0, \qquad Y(0;\kappa,\mu,\lambda) = 0, \qquad U(0;\kappa,\mu,\lambda) = \kappa = \frac{1}{\gamma-1}. \tag{2.3}$$

Thus we can use z and κ as a coordinate system on $\mathcal{F}$. Note that the analyticity of (2.2) excludes the other "slow" orbits coming out of the u-axis.

On the other hand, we have that at infinity the orbit (0.19) goes into the critical point P given by (1.17). Thus (0.19) also belongs to the stable manifold $\mathcal{S}$ of P. In the previous section, we have seen that $\mathcal{S}$ contains the similarity profiles satisfying the interface conditions and that its dimension is two. It can be written as a family of solutions of the form

$$x = X^*(z; c, \mu, \lambda), \qquad y = Y^*(z; c, \mu, \lambda), \qquad u = U^*(z; c, \mu, \lambda). \tag{2.4}$$

Here z and c are the parameters which can be used as a coordinate system on $\mathcal{S}$. We note that c is really given by the proof of the stable-manifold theorem and corresponds to a suitable smooth curve in the linearized stable manifold [Pe].

Both $\mathcal{F}$ and $\mathcal{S}$ are two dimensional. The straight line (0.19) lies in the intersection of $\mathcal{F}$ and $\mathcal{S}$. Since we are working in a four-dimensional space, the set of parameters for which this intersection is a curve should generically be a set of codimension one. To show that this is really the case in the vincinity of the exact solution above, we apply the implicit-function theorem to the following set of equations:

$$X(z; \kappa, \mu, \lambda) - X^*(z; c, \mu, \lambda) = 0; \tag{2.5}$$

$$Y(z; \kappa, \mu, \lambda) - Y^*(z; c, \mu, \lambda) = 0; \tag{2.6}$$

$$U(z; \kappa, \mu, \lambda) - U^*(z; c, \mu, \lambda) = 0. \tag{2.7}$$

Here the value of z can be taken fixed because the flow leaves $\mathcal{F}$ and $\mathcal{S}$ invariant.

In order to conclude that the solution set of (2.5)–(2.7) is of the form

$$\kappa = \kappa(\lambda), \qquad \mu = \mu(\lambda), \qquad c = c(\lambda), \tag{2.8}$$

we have to show that the matrix containing the partial derivatives of the left-hand sides with respect to κ, μ, and c has a nonzero determinant.

The functions $X(z)$, $Y(z)$, $U(z)$, $X^*(z)$, $Y^*(z)$, and $U^*(z)$ are solutions of the three-dimensional nonautonomous system obtained from (Q) by taking z as a new independent variable:

$$(\mathrm{Q}^*) \quad \begin{cases} \dfrac{dx}{dz} = \dfrac{x(1-3x+y) - z(x(1-\mu)+2\mu-u)}{z(2+y-2x)}; \\ \dfrac{dy}{dz} = \dfrac{y(1-2x) - \lambda z(y(1-\mu)+2\mu+1-\gamma u)}{z(2+y-2x)}; \\ \dfrac{du}{dz} = \dfrac{u(y-x)}{z(2+y-2x)}. \end{cases}$$

It follows from the proof of the stable-manifold theorem that we can compute the derivatives of these functions by differentiating (Q*) with respect to the parameters and solving the resulting equations under the appropiate boundary conditons.

Writing (Q*) as

$$\frac{d\xi}{dz} = H(\xi) = H(\xi; \mu, \lambda), \quad \xi(z) = (x(z), y(z), u(z)), \tag{2.9}$$

we have for the variation $d\xi(z) = (dx(z), dy(z), du(z))$ of ξ the equation

$$\frac{d}{dz}d\xi - \frac{\partial H}{\partial \xi}d\xi = dH = \frac{\partial H}{\partial \mu}d\mu + \frac{\partial H}{\partial \lambda}d\lambda. \tag{2.10}$$

In (2.10), the derivatives of H have to be evaluated at

$$x = y = -(1-\mu)z, \quad u = \kappa, \quad \mu = \frac{\kappa+1}{3}, \quad \gamma = \frac{\kappa+1}{\kappa}, \quad \lambda = 1. \tag{2.11}$$

Using Maple again, we find

$$\frac{\partial H}{\partial \xi} = \begin{pmatrix} \frac{1}{2z} + \frac{3}{2}\frac{2-\kappa}{6+(2-\kappa)z} & 0 & \frac{3}{6+(2-\kappa)z} \\ 0 & \frac{1}{2z} + \frac{3}{2}\frac{2-\kappa}{6+(2-\kappa)z} & \frac{3(\kappa+1)}{\kappa(6+(2-\kappa)z)} \\ \frac{-\kappa}{2z} + \frac{2-\kappa}{2}\frac{\kappa}{6+(2-\kappa)z} & \frac{\kappa}{2z} - \frac{2-\kappa}{2}\frac{\kappa}{6+(2-\kappa)z} & 0 \end{pmatrix}, \tag{2.12}$$

while

$$\frac{\partial H}{\partial \mu} = \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix}, \qquad \frac{\partial H}{\partial \lambda} = \begin{pmatrix} 0 \\ \frac{2-\kappa}{3} - \frac{3(2-\kappa)}{6+(2-\kappa)z} \\ 0 \end{pmatrix}. \tag{2.13}$$

In what follows, we shall compute the general solution of (2.9)–(2.13) explicitly in terms of hypergeometric functions. To do so we transform (2.9)–(2.13) by

$$w = \frac{(2-\kappa)z}{6+(2-\kappa)z}, \qquad z = \frac{6w}{(2-\kappa)(1-w)},$$

$$G(w) = dx(z), \qquad J(w) = dx(z) - dy(z), \qquad F(w) = du(z) \tag{2.14}$$

into

$$\begin{pmatrix} G'(w) \\ J'(w) \\ F'(w) \end{pmatrix} = \begin{pmatrix} \frac{1}{2w} + \frac{2}{1-w} & 0 & \frac{3}{2-\kappa}\frac{1}{1-w} \\ 0 & \frac{1}{2w} + \frac{2}{1-w} & -\frac{1}{\kappa}\frac{3}{2-\kappa}\frac{1}{1-w} \\ 0 & -\frac{\kappa}{2w} & 0 \end{pmatrix} \begin{pmatrix} G(w) \\ J(w) \\ F(w) \end{pmatrix}$$

$$- \frac{6}{2-\kappa}\frac{1}{(1-w)^2}\begin{pmatrix} d\mu \\ 0 \\ 0 \end{pmatrix} + \frac{1-3w}{(1-w)^2}\begin{pmatrix} 0 \\ d\lambda \\ 0 \end{pmatrix}. \tag{2.15}$$

For $F(w)$, this yields

$$w(1-w)\frac{d^2}{dw^2}F(w) + \left(\frac{1}{2} - \frac{5}{2}w\right)\frac{d}{dw}F(w) - \frac{3}{2(2-\kappa)}F(w) = \frac{\kappa(3w-1)}{2(1-w)}d\lambda, \tag{2.16}$$

which has an explicit particular solution, namely,

$$\kappa(2-\kappa)\left(1 - \frac{2}{\kappa+1}\frac{1}{1-w}\right)d\lambda. \tag{2.17}$$

The homogeneous part of (2.16) is the standard hypergeometric equation

$$w(1-w)f''(w) + (c-(1+a+b)w)f'(w) - abf(w) = 0 \tag{2.18}$$

with parameters a, b, and c given by

$$a + b = \frac{3}{2}, \qquad ab = \frac{3}{2(2-\kappa)}, \qquad \text{and} \quad c = \frac{1}{2}. \tag{2.19}$$

The general solution of (2.18) is given by

$$f(w) = C_1 F_1(w) + C_2 F_2(w), \tag{2.20}$$

where

$$F_1(w) = F\left(a, b; \frac{1}{2}; w\right) = \frac{1}{1-w} F\left(\frac{1}{2} - a, \frac{1}{2} - b; \frac{1}{2}; w\right) \tag{2.21}$$

and

$$F_2(w) = w^{\frac{1}{2}} F\left(\frac{1}{2} + a, \frac{1}{2} + b; \frac{3}{2}; w\right) = \frac{w^{\frac{1}{2}}}{1-w} F\left(1 - a, 1 - b; \frac{3}{2}; w\right). \tag{2.22}$$

Consequently, the general solution of the homogeneous part of (2.15) is given by

$$\begin{pmatrix} G_{\text{hom}}(w) \\ J_{\text{hom}}(w) \\ F_{\text{hom}}(w) \end{pmatrix} = C_1 \begin{pmatrix} 2wF_1'(w) \\ -\frac{2w}{\kappa} F_1'(w) \\ F_1(w) \end{pmatrix} + C_2 \begin{pmatrix} 2wF_2'(w) \\ -\frac{2w}{\kappa} F_2'(w) \\ F_2(w) \end{pmatrix} + C_3 \begin{pmatrix} \frac{w^{\frac{1}{2}}}{(1-w)^2} \\ 0 \\ 0 \end{pmatrix}. \tag{2.23}$$

The hypergeometric part in (2.23) can be derived from the special form of the matrix in (2.15).

A particular solution of (2.15) is

$$\begin{pmatrix} G_p(w) \\ J_p(w) \\ F_p(w) \end{pmatrix} = \begin{pmatrix} \frac{2\kappa w}{(1-w)^2}(3\frac{\kappa-1}{\kappa+1} - w) \\ \frac{4(2-\kappa)}{\kappa+1}\frac{w}{(1-w)^2} \\ \kappa(2-\kappa)(1 - \frac{2}{\kappa+1}\frac{1}{1-w}) \end{pmatrix} d\lambda + \begin{pmatrix} \frac{-12}{2-\kappa}\frac{w}{(1-w)^2} \\ 0 \\ 0 \end{pmatrix} d\mu. \tag{2.24}$$

Thus the general solution of (2.15) is the sum of (2.23) and (2.24).

We can now write the partial derivatives of (2.2) for (2.11). The analyticity near $z = 0$ combined with (2.3) implies that we have to take

$$C_2 = C_3 = 0, \quad C_1 + \kappa(2-\kappa)\frac{\kappa-1}{\kappa+1} d\lambda = d\kappa \tag{2.25}$$

so that

$$\begin{pmatrix} \frac{\partial X}{\partial \kappa} & \frac{\partial X}{\partial \mu} & \frac{\partial X}{\partial \lambda} \\ \frac{\partial (X-Y)}{\partial \kappa} & \frac{\partial (X-Y)}{\partial \mu} & \frac{\partial (X-Y)}{\partial \lambda} \\ \frac{\partial U}{\partial \kappa} & \frac{\partial U}{\partial \mu} & \frac{\partial U}{\partial \lambda} \end{pmatrix} = \tag{2.26}$$

$$\begin{pmatrix} 2wF'(a,b;\frac{1}{2};w) & \frac{-12}{2-\kappa}\frac{w}{(1-w)^2} & \frac{2\kappa w}{(1-w)^2}(3\frac{\kappa-1}{\kappa+1} - w) + 2\kappa(2-\kappa)\frac{1-\kappa}{\kappa+1} wF'(a,b;\frac{1}{2};w) \\ -\frac{2w}{\kappa}F'(a,b;\frac{1}{2};w) & 0 & \frac{4(2-\kappa)}{\kappa+1}\frac{w}{(1-w)^2} - 2(2-\kappa)\frac{1-\kappa}{\kappa+1} wF'(a,b;\frac{1}{2};w) \\ F(a,b;\frac{1}{2};w) & 0 & \kappa(2-\kappa)(1 - \frac{2}{\kappa+1}\frac{1}{1-w}) + \kappa(2-\kappa)\frac{1-\kappa}{\kappa+1}F(a,b;\frac{1}{2};w) \end{pmatrix}.$$

Next, we compute the partial derivatives of (2.4). The boundary conditions are now at $z=\infty$ and follow from (1.17), which implies that

$$\frac{dx(z)}{z}\to d\mu,\quad \frac{dy(z)}{z}\to d\mu-\frac{2-\kappa}{3}d\lambda,\quad \frac{du(z)}{z}\to 0\quad\text{as } z\to\infty, \tag{2.27}$$

equivalent (recall (2.14)) to

$$\lim_{w\uparrow 1}(1-w)G(w)=\frac{6}{2-\kappa}d\mu,\qquad \lim_{w\uparrow 1}(1-w)J(w)=2d\lambda,\qquad \lim_{w\uparrow 1}(1-w)F(w)=0. \tag{2.28}$$

In order to choose the constants C_1, C_2, and C_3 accordingly, we need the asymptotic expansions of (2.23)–(2.24) as $w\uparrow 1$. At first glance, the reader may want to skip these calculations and proceed directly to (2.47).

We note that Gauss's formula implies that

$$\lim_{w\uparrow 1}(1-w)F_1(w)=B_\kappa=\frac{\Gamma(\frac{1}{2})}{\Gamma(a)\Gamma(b)},\qquad \lim_{w\uparrow 1}(1-w)F_2(w)=A_\kappa=\frac{\Gamma(\frac{3}{2})}{\Gamma(\frac{1}{2}+a)\Gamma(\frac{1}{2}+b)}, \tag{2.29}$$

i.e.,

$$F_1(w)=\frac{B_\kappa}{1-w}+o\left(\frac{1}{1-w}\right),\qquad F_2(w)=\frac{A_\kappa}{1-w}+o\left(\frac{1}{1-w}\right)\quad\text{as } w\uparrow 1. \tag{2.30}$$

For the corresponding first components of the homogeneous solution, we find

$$\begin{aligned}2wF_1'(w)&=2w\frac{ab}{\frac{1}{2}}F\left(a+1,b+1;\frac{3}{2};w\right)=\frac{6}{2-\kappa}\frac{w}{(1-w)^2}F\left(\frac{1}{2}-a,\frac{1}{2}-b;\frac{3}{2};w\right)\\&=\frac{6}{2-\kappa}\frac{w}{(1-w)^2}\\&\quad\times\left(\frac{\Gamma(\frac{3}{2})\Gamma(2)}{\Gamma(a+1)\Gamma(b+1)}-\frac{(\frac{1}{2}-a)(\frac{1}{2}-b)}{\frac{3}{2}}\frac{\Gamma(\frac{5}{2})\Gamma(1)}{\Gamma(a+1)\Gamma(b+1)}(1-w)\right.\\&\qquad\left.+o(1-w)\right)\\&=2B_\kappa\left(\frac{1}{(1-w)^2}-\frac{w}{(1-w)^2}\right)\left(1+\left(\frac{1}{2}-\frac{3}{2(2-\kappa)}\right)(1-w)+o(1-w)\right)\\&=B_\kappa\left(\frac{2}{(1-w)^2}-\left(1+\frac{3}{2-\kappa}\right)\frac{1}{1-w}+o(\frac{1}{1-w})\right)\quad\text{as } w\uparrow 1.\end{aligned} \tag{2.31}$$

In this computation, we have used the Gauss relation for both $F(1/2-a, 1/2-b;3/2;w)$ and its derivative. Similarly, we have

$$\begin{aligned}2wF_2'(w)&=w^{\frac{1}{2}}F\left(\frac{1}{2}+a,\frac{1}{2}+b;\frac{3}{2};w\right)+2w^{\frac{3}{2}}\frac{(\frac{1}{2}+a)(\frac{1}{2}+b)}{\frac{3}{2}}F\left(\frac{3}{2}+a,\frac{3}{2}+b;\frac{5}{2};w\right)\\&=F_2(w)+2w^{\frac{3}{2}}\frac{(\frac{1}{2}+a)(\frac{1}{2}+b)}{\frac{3}{2}}\frac{1}{(1-w)^2}F\left(1-a,1-b;\frac{5}{2};w\right)\\&=F_2(w)+\frac{2w^{\frac{3}{2}}}{(1-w)^2}\end{aligned} \tag{2.32}$$

$$\times \frac{(\frac{1}{2}+a)(\frac{1}{2}+b)}{\frac{3}{2}} \left(\frac{\Gamma(\frac{5}{2})\Gamma(2)}{\Gamma(\frac{3}{2}+a)\Gamma(\frac{3}{2}+b)} - \frac{(1-a)(1-b)}{\frac{5}{2}} \frac{\Gamma(\frac{7}{2})\Gamma(1)}{\Gamma(\frac{3}{2}+a)\Gamma(\frac{3}{2}+b)}(1-w) + o(1-w) \right)$$

$$= F_2(w) + \frac{2w^{\frac{3}{2}}}{(1-w)^2} A_\kappa \left(1 + \left(\frac{1}{2} - \frac{3}{2(2-\kappa)}\right)(1-w) + o(1-w)\right)$$

$$= \frac{A_\kappa}{1-w} + o\left(\frac{1}{1-w}\right)$$

$$+ \frac{2}{(1-w)^2}\left(1 - \frac{3}{2}(1-w) + o(1-w)\right) A_\kappa \left(1 + \left(\frac{1}{2} - \frac{3}{2(2-\kappa)}\right)(1-w) + o(1-w)\right)$$

$$= A_\kappa \left(\frac{2}{(1-w)^2} - \left(1 + \frac{3}{2-\kappa}\right)\frac{1}{1-w} + o\left(\frac{1}{1-w}\right)\right) \quad \text{as } w \uparrow 1.$$

For the first component corresponding to C_3, we have

$$\frac{w^{\frac{1}{2}}}{(1-w)^2} = \frac{1}{(1-w)^2} - \frac{1}{2}\frac{1}{1-w} + o\left(\frac{1}{1-w}\right) \quad \text{as } w \uparrow 1. \tag{2.33}$$

For the particular solution corresponding to $d\lambda$, the third component is

$$\kappa(2-\kappa)\left(1 - \frac{2}{\kappa+1}\frac{1}{1-w}\right) = -\frac{2\kappa(2-\kappa)}{\kappa+1}\frac{1}{1-w} + o\left(\frac{1}{1-w}\right) \quad \text{as } w \uparrow 1, \tag{2.34}$$

the second component is

$$\frac{4(2-\kappa)}{\kappa+1}\frac{w}{(1-w)^2} = \frac{4(2-\kappa)}{\kappa+1}\frac{1}{(1-w)^2} - \frac{4(2-\kappa)}{\kappa+1}\frac{1}{1-w} + o\left(\frac{1}{1-w}\right) \quad \text{as } w \uparrow 1, \tag{2.35}$$

and the first component is

$$\frac{2\kappa w}{(1-w)^2}\left(3\frac{\kappa-1}{\kappa+1} - w\right) = \frac{4\kappa(\kappa-2)}{\kappa+1}\frac{1}{(1-w)^2} + \frac{2\kappa(5-\kappa)}{\kappa+1}\frac{1}{1-w} + o\left(\frac{1}{1-w}\right) \quad \text{as } w \uparrow 1. \tag{2.36}$$

Finally, for the first component of the particular solution for $d\mu$,

$$-\frac{12}{2-\kappa}\frac{w}{(1-w)^2} = -\frac{12}{2-\kappa}\frac{1}{(1-w)^2} + \frac{12}{2-\kappa}\frac{1}{1-w} + o\left(\frac{1}{1-w}\right) \quad \text{as } w \uparrow 1. \tag{2.37}$$

Now that we have all of the asymptotic expansions as $w \uparrow 1$, we have to choose the constants in such a way that (2.28) is satisfied. First, we look at the third component,

$$\begin{aligned} F(w) &= C_1F_1(w) + C_2F_2(w) + \kappa(2-\kappa)\left(1 - \frac{2}{\kappa+1}\frac{1}{1-w}\right)d\lambda \\ &= \left(C_1B_\kappa + C_2A_\kappa - \frac{2\kappa(2-\kappa)}{\kappa+1}d\lambda\right)\frac{1}{1-w} + o\left(\frac{1}{1-w}\right) \quad \text{as } w \uparrow 1, \end{aligned} \tag{2.38}$$

which forces us to take

$$C_1 B_\kappa + C_2 A_\kappa - \frac{2\kappa(2-\kappa)}{\kappa+1} d\lambda = 0. \tag{2.39}$$

Then by the Kummer relation,

$$2A_\kappa F_1(w) - 2B_\kappa F_2(w) = F(a,b;2;w), \tag{2.40}$$

$$F(w) = \kappa(2-\kappa)\left(\frac{1}{B_\kappa}\frac{2}{\kappa+1}F\left(a,b;\frac{1}{2};w\right) - \frac{2}{\kappa+1}\frac{1}{1-w} + 1\right)d\lambda + CF(a,b;2;w), \tag{2.41}$$

where $C = -C_2/(2B_\kappa)$.

For the second component, we then obviously have that the terms with $(1-w)^{-2}$ disappear and that

$$\begin{aligned} J(w) &= -\frac{2w}{\kappa}F'(w) = -C_1\frac{2w}{\kappa}F_1'(w) - C_2\frac{2w}{\kappa}F_2'(w)\frac{4(2-\kappa)}{\kappa+1}\frac{w}{(1-w)^2}d\lambda \\ &= \left(\frac{1}{\kappa}(C_1B_\kappa + C_2A_\kappa)\left(1+\frac{3}{2-\kappa}\right) - \frac{4(2-\kappa)}{\kappa+1}d\lambda\right)\frac{1}{1-w} + o\left(\frac{1}{1-w}\right) \\ &= \frac{2}{1-w}d\lambda + o\left(\frac{1}{1-w}\right) \quad \text{as } w \uparrow 1, \end{aligned} \tag{2.42}$$

which agrees with (2.28) and therefore gives no further restriction on the constants C_1, C_2, and C_3. Thus

$$J(w) = \frac{4(2-\kappa)w}{\kappa+1}\left(-\frac{1}{B_\kappa}F'\left(a,b;\frac{1}{2};w\right) + \frac{1}{(1-w)^2}\right)d\lambda - \frac{2Cw}{\kappa}F'(a,b;2;w), \tag{2.43}$$

Finally, for the first component, using (2.39) again,

$$\begin{aligned} G(w) &= 2wC_1F_1'(w) + 2wC_2F_2'(w) + C_3\frac{w^{\frac{1}{2}}}{(1-w)^2} \\ &\qquad + \frac{2\kappa w}{(1-w)^2}\left(3\frac{\kappa-1}{\kappa+1} - w\right)d\lambda - \frac{12}{2-\kappa}\frac{w}{(1-w)^2}d\mu \\ &= \left(C_3 - \frac{12}{2-\kappa}d\mu\right)\frac{1}{(1-w)^2} + \left(\frac{12}{2-\kappa}d\mu - \frac{1}{2}C_3\right)\frac{1}{1-w} + o\left(\frac{1}{1-w}\right) \quad \text{as } w \uparrow 1 \end{aligned} \tag{2.44}$$

so that

$$C_3 = \frac{12}{2-\kappa}d\mu \tag{2.45}$$

ensures that (2.28) holds. Thus

$$\begin{aligned} G(w) = &\left(\frac{4\kappa(2-\kappa)w}{(\kappa+1)B_\kappa}F'\left(a,b;\frac{1}{2};w\right) + \frac{2\kappa w}{(1-w)^2}\left(3\frac{\kappa-1}{\kappa+1} - w\right)\right)d\lambda \\ &+ 2CwF'(a,b;2;w) + \frac{12}{2-\kappa}\frac{w^{\frac{1}{2}} - w}{(1-w)^2}d\mu. \end{aligned} \tag{2.46}$$

From (2.41), (2.43), and (2.46), we then have for an appropiate choice of the coordinate c in (2.4) that

$$\begin{pmatrix} \frac{\partial X^*}{\partial c} & \frac{\partial X^*}{\partial \mu} & \frac{\partial X^*}{\partial \lambda} \\ \frac{\partial (X^*-Y^*)}{\partial c} & \frac{\partial (X^*-Y^*)}{\partial \mu} & \frac{\partial (X^*-Y^*)}{\partial \lambda} \\ \frac{\partial U^*}{\partial c} & \frac{\partial U^*}{\partial \mu} & \frac{\partial U^*}{\partial \lambda} \end{pmatrix} \tag{2.47}$$

$$= \begin{pmatrix} 2wF'(a,b;2;w) & \frac{12}{2-\kappa}\frac{w^{\frac{1}{2}}-w}{(1-w)^2} & \frac{2\kappa w}{(1-w)^2}(3\frac{\kappa-1}{\kappa+1}-w)+\frac{4\kappa(2-\kappa)w}{(\kappa+1)B_\kappa}F'(a,b;\frac{1}{2};w) \\ -\frac{2w}{\kappa}F'(a,b;2;w) & 0 & \frac{4(2-\kappa)}{\kappa+1}\frac{w}{(1-w)^2}-\frac{4(2-\kappa)w}{(\kappa+1)B_\kappa}F'(a,b;\frac{1}{2};w) \\ F(a,b;2;w) & 0 & \kappa(2-\kappa)(1-\frac{2}{\kappa+1}\frac{1}{1-w})+\frac{2\kappa(2-\kappa)}{(\kappa+1)B_\kappa}F(a,b;\frac{1}{2};w) \end{pmatrix}.$$

Writing (2.5)–(2.7) as

$$\mathcal{F}(z;c,\kappa,\mu,\lambda) = \begin{pmatrix} X-X^* \\ X-X^*-Y+Y^* \\ U-U^* \end{pmatrix} = 0,$$

it follows that $\partial\mathcal{F}/\partial(c,\kappa,\mu,\lambda) =$

$$\begin{pmatrix} -2wF'(a,b;2;w) & 2wF'(a,b;\frac{1}{2};w) & -\frac{12}{2-\kappa}\frac{w^{\frac{1}{2}}}{(1-w)^2} & \frac{2\kappa(2-\kappa)}{\kappa+1}\beta_\kappa wF'(a,b;\frac{1}{2};w) \\ \frac{2w}{\kappa}F'(a,b;2;w) & -\frac{2w}{\kappa}F'(a,b;\frac{1}{2};w) & 0 & -\frac{2(2-\kappa)}{\kappa+1}\beta_\kappa wF'(a,b;\frac{1}{2};w) \\ -F(a,b;2;w) & F(a,b;\frac{1}{2};w) & 0 & \frac{\kappa(2-\kappa)}{\kappa+1}\beta_\kappa F(a,b;\frac{1}{2};w) \end{pmatrix}, \tag{2.48}$$

where

$$\beta_\kappa = 1-\kappa-\frac{2}{B_\kappa}. \tag{2.49}$$

Clearly, the first three columns in this matrix have maximal rank for any $0 < w < 1$ because the Wronskian of the two hypergeometric functions $F(a,b;1/2;w)$ and $F(a,b;2;w)$ is nonzero. It follows that we can write the solution set of (2.5-7) in the form (2.8) with

$$\begin{pmatrix} -2wF'(a,b;2;w) & 2wF'(a,b;\frac{1}{2};w) & -\frac{12}{2-\kappa}\frac{w^{\frac{1}{2}}}{(1-w)^2} \\ \frac{2w}{\kappa}F'(a,b;2;w) & -\frac{2w}{\kappa}F'(a,b;\frac{1}{2};w) & 0 \\ -F(a,b;2;w) & F(a,b;\frac{1}{2};w) & 0 \end{pmatrix}\begin{pmatrix} \frac{dc}{d\lambda} \\ \frac{d\kappa}{d\lambda} \\ \frac{d\mu}{d\lambda} \end{pmatrix}$$

$$= -\begin{pmatrix} \frac{2\kappa(2-\kappa)}{\kappa+1}\beta_\kappa wF'(a,b;\frac{1}{2};w) \\ -\frac{2(2-\kappa)}{\kappa+1}\beta_\kappa wF'(a,b;\frac{1}{2};w) \\ \frac{\kappa(2-\kappa)}{\kappa+1}\beta_\kappa F(a,b;\frac{1}{2};w) \end{pmatrix}, \tag{2.50}$$

whence, using Cramer's rule,

$$\frac{dc}{d\lambda} = \frac{d\mu}{d\lambda} = 0, \qquad \frac{d\kappa}{d\lambda} = \frac{\kappa(2-\kappa)}{\kappa+1}\left(\kappa-1+\frac{2}{B_\kappa}\right). \tag{2.51}$$

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REFERENCES

[A] D. G. Aronson, *The porous medium equation*, in Some Problems in Nonlinear Diffusion, A. Fasano and M. Primicerio, eds., Lecture Notes in Math. 1224, Springer-Verlag, Berlin, 1986, pp. 1–46.

[AV] D. G. Aronson and J. L. Vazquez, *Calculation of anamolous exponents in nonlinear diffusion*, Phys. Rev. Lett., 72 (1994), pp. 348–351.

[B] G. I. Barenblatt, *Self-similar turbulence propagation from an instantaneous plane source*, in Nonlinear Dynamics and Turbulence, G. I. Barenblatt, G. Iooss, and D. D. Joseph, eds., Pitman, Boston, 1983, pp. 48–60.

[BGL] G. I. Barenblatt, N. L. Galerkina, and M. V. Luneva, *Evolution of a turbulent burst*, Inzh.-Fiz. Zh., 53 (1987), pp. 773–740 (in Russian).

[BdPK1] M. Bertsch, R. Dal Passo, and R. Kersner, *Parameter dependence in the $b-\varepsilon$ model*, Differential Integral Equations, 7 (1994), pp. 1195–1214.

[BdPK2] M. Bertsch, R. Dal Passo, and R. Kersner, *The evolution of turbulent bursts: The $b-\varepsilon$ model*, European J. Appl. Math., 5 (1994), pp. 537–557.

[CL] E. A. Coddington and N. Levinson, *Theory of Ordinary Differential Equations*, Krieger Publishing Company, Melbourne, FL, 1984.

[HL] K. Hanjalic and B. E. Launder, *A Reynolds stress model of turbulence and its applications to thin shear flows*, J. Fluid. Mech., 52 (1974), pp. 609–638.

[HP] S. P. Hastings and L. A. Peletier, *On the decay of turbulent bursts*, European J. Appl. Math., 3 (1992), pp. 319–341.

[H] J. Hulshof, *Similarity solutions of the porous medium equation with sign changes*, J. Math. Anal. Appl., 157 (1991), pp. 75–111.

[HV] J. Hulshof and J. L. Vazquez, *Selfsimilar solutions of the second kind for the modified porous medium equation*, European J. Appl. Math., 5 (1994), pp. 391–403.

[KV] S. Kamin and J. L. Vazquez, *The propagation of turbulent bursts*, European J. Appl. Math., 3 (1992), pp. 263–272.

[K] A. N. Kolmogorov, *Equation of turbulent motion of incompressible fluids*, Izv. Akad. Nauk SSSR, 6 (1942), pp. 56–58.

[L] N. N. Lebedev, *Special Functions and Their Applications*, Dover, New York, 1972.

[LMRS] B. E. Launder, A. P. Morse, W. Rodi, and D. B. Spalding, *Prediction of free shear flows: A comparison of six turbulence models*, NASA SP, 321 (1972).

[LS] B. E. Launder and D. B. Spalding, *The numerical computation of turbulent flows*, Comput. Math. Appl. Mech. Engrg., 3 (1974), pp. 269–289.

[MY] A. S. Monin and A. M. Yaglom, *Statistical Fluid Mechanics*, Vols. 1 and 2, MIT Press, Cambridge, MA, 1971 and 1975.

[Pe] L. Perko, *Differential Equations and Dynamical Systems*, Springer-Verlag, Berlin, 1991.

[P] L. Prandtl, *Über ein neues Formelsystem für die ausgebildete Turbulenz*, Nachr. Akad. Wiss. Göttingen Math.-Phys., K1 (1945), pp. 6–18.